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HOMOTOPY INVARIANCE OF ALMOST FLAT BETTI NUMBERS

by Roman SAUER and Thomas SCHICK

In this note, we would like to present a question that belongs to the circle of analytical-topological problems around the Novikov conjecture.

The Novikov conjecture about the homotopy invariance of higher signatures has been a driving force for research in topology for many years. One of the approaches, which gives partial results, uses the signature operator twisted with almost flat bundles.

In this context Hilsum and Skandalis [2] prove the following result.

THEOREM 57.1. *Let M_1 and M_2 be closed, oriented Riemannian manifolds, and $f\colon M_1 \to M_2$ a homotopy equivalence. Then there is a constant $c > 0$ with the following property:*

Let (E, ∇) be a Euclidean vector bundle over M_2. Let $s_2(E)$ denote the index of the signature operator on M_2 twisted with E, and let $s_1(f^*E)$ denote the index of the signature operator on M_1 twisted with the pullback bundle f^*E.

If $\left\|\nabla^2\right\| < c$, i.e. if the curvature of the bundle (E, ∇) is sufficiently small (where the norm is the supremum of the operator norm in the unit sphere bundle of $\Lambda^2 TM_2$), then

$$s_2(E) = s_1(f^*E)\,.$$

In other words, the index of the twisted signature operator is a homotopy invariant for twisting bundles with sufficiently small curvature.

Hilsum and Skandalis prove this with a clever deformation argument (with a proof that evidently also covers the flat case). An alternative proof, which reduces the statement to the homotopy invariance for flat twisting bundles of C^*-algebra modules and bases on calculations in the K-theory of C^*-algebras, has been recently worked out by Bernhard Hanke and the second author [1].

The following question is motivated by the theorem above and arose from discussions with Paolo Piazza and Sara Azzali.

QUESTION 57.2. *Retain the situation of the Theorem stated above. The kernels of the twisted signature operators are graded by the degree of differential forms. By taking their dimensions we obtain the so-called* twisted Betti numbers *of E and f^*E, which we denote by $b_k(E)$ and $b_k(f^*E)$.*

Is it true that, for sufficiently small $c > 0$ as in the theorem, $\|\nabla^2\| < c$ implies $b_k(E) = b_k(f^*E)$?

Note that this *is* the case for the corresponding Euler characteristic:

$$\sum_k (-1)^k b_k(E) = \sum_k (-1)^k b_k(f^*E)$$

for sufficiently small curvature $\|\nabla^2\|$, since the Euler characteristic is again an index. The proof for the invariance of the twisted Euler characteristic is actually considerably easier than the one for the invariance of the twisted signature.

We find the question interesting since a positive answer would imply that the η-invariants of twisted signature operators are much better behaved than one could expect *a priori*. This could open up the way to construct new "higher" homotopy invariants of smooth manifolds. The question grew out of the work of the second author and Paolo Piazza [3], where related results concerning rho-invariants are established.

It seems that there is no "standard" approach to answer the question. Analysts tend to think the statement should not be true. However, if we twist with a flat bundle, the answer to our question is *yes* — but requires the insight of the de Rham theorem. Analysis alone can be misleading dealing with such questions, because it suggests too many deformations that do not have a topological meaning.

REFERENCES

[1] HANKE, B. and T. SCHICK. Enlargeability and index theory. *J. Differential Geom. 74* (2006), 293–320.

[2] HILSUM, M. and G. SKANDALIS. Invariance par homotopie de la signature à coefficients dans un fibré presque plat. *J. Reine Angew. Math. 423* (1992), 73–99.

[3] PIAZZA, P. and T. SCHICK. Bordism, rho-invariants and the Baum–Connes conjecture. *J. Noncommut. Geom. 1* (2007), 27–111.

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