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## 61

### ON ALGEBRAIC CHARACTERIZATIONS FOR THE FINITENESS OF THE DIMENSION OF $\underline{E}G$

by Olympia TALELLI

In [5] the following theorem was proved :

**THEOREM 61.1.** *If  $G$  is an  $\mathfrak{H}\mathfrak{F}$ -group of type  $\text{FP}_\infty$  then  $G$  admits a finite dimensional model for  $\underline{E}G$ .*

The class  $\mathfrak{H}\mathfrak{F}$  was introduced by P.H. Kropholler in [4] and it is defined as the smallest class of groups containing the class of finite groups, with the property : if a group  $G$  admits a finite dimensional contractible  $G$ -CW-complex with all cell stabilizers in  $\mathfrak{H}\mathfrak{F}$  then  $G$  is in  $\mathfrak{H}\mathfrak{F}$ .

This theorem, especially its proof, was the motivation for defining groups of type  $\Phi$  in [7] and for proposing those as the ones which admit a finite dimensional model for  $\underline{E}G$ .

**DEFINITION 61.2** ([7]). A group  $G$  is said to be of type  $\Phi$  if it has the property that for every  $\mathbf{Z}G$ -module  $M$ ,  $\text{projdim}_{\mathbf{Z}G} M < \infty$  if and only if  $\text{projdim}_{\mathbf{Z}H} M < \infty$  for every finite subgroup  $H$  of  $G$ .

**CONJECTURE 61.3.** *The following statements are equivalent for a group  $G$  :*

- (1)  $G$  admits a finite dimensional model for  $\underline{E}G$ .
- (2)  $G$  admits a finite dimensional contractible  $G$ -CW-complex with finite cell stabilizers.
- (3)  $G$  is of type  $\Phi$ .
- (4)  $\text{spli } \mathbf{Z}G < \infty$ .
- (5)  $\text{silp } \mathbf{Z}G < \infty$ .
- (6)  $\text{findim } \mathbf{Z}G < \infty$ .

The algebraic invariants  $\text{spli } \mathbf{Z}G$  and  $\text{silp } \mathbf{Z}G$  were defined in [3]:  $\text{silp } \mathbf{Z}G$  is the supremum of the injective lengths of the projective  $\mathbf{Z}G$ -modules and  $\text{spli } \mathbf{Z}G$  is the supremum of the projective lengths of the injective  $\mathbf{Z}G$ -modules. It was shown in [3] that  $\text{silp } \mathbf{Z}G \leq \text{spli } \mathbf{Z}G$ , and that if  $\text{spli } \mathbf{Z}G < \infty$  then  $\text{spli } \mathbf{Z}G = \text{silp } \mathbf{Z}G$ . The *finitistic dimension* of  $\mathbf{Z}G$ ,  $\text{findim } \mathbf{Z}G$ , is the supremum of the projective dimensions of the  $\mathbf{Z}G$ -modules of finite projective dimension.

It is not very difficult to show that (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5)  $\Rightarrow$  (6), see [7].

If  $G$  is an  $\mathfrak{H}\mathfrak{F}$ -group of type  $\text{FP}_\infty$  then there is a bound on the orders of the finite subgroups of  $G$  [4], and  $\text{projdim}_{\mathbf{Z}G} B(G, \mathbf{Z})$  is finite, where  $B(G, \mathbf{Z})$  is the  $\mathbf{Z}G$ -module of the bounded functions from  $G$  to  $\mathbf{Z}$  [1].

Theorem 61.1 is now an immediate consequence of the following theorem which was proved in [5]:

**THEOREM 61.4.** *If  $G$  is an  $\mathfrak{H}\mathfrak{F}$ -group such that there is a bound on the orders of the finite subgroups and  $\text{projdim}_{\mathbf{Z}G} B(G, \mathbf{Z})$  is finite, then  $G$  admits a finite dimensional model for  $\underline{E}G$ .*

Since if  $G$  is an  $\mathfrak{H}\mathfrak{F}$ -group,  $\text{projdim}_{\mathbf{Z}G} B(G, \mathbf{Z})$  is finite if and only if  $\text{findim } \mathbf{Z}G$  is finite [1], it follows from Theorem 61.4 that (6) $\Rightarrow$ (1) in Conjecture 61.3 if  $G$  is an  $\mathfrak{H}\mathfrak{F}$ -group with a bound on the orders of the finite subgroups. Moreover in [8] it is shown that (6)  $\Rightarrow$  (1) if  $G$  is a torsion-free elementary amenable group.

In [6] it was shown that if a group  $G$  admits a finite dimensional model for  $\underline{E}G$  then for every finite subgroup  $H$  of  $G$ ,  $W(H)$  admits a finite dimensional model for  $\underline{E}W(H)$ , where  $W(H) = N_G(H)/H$ . In [7] we show that if  $\text{spli } \mathbf{Z}G$  is finite, then  $\text{spli } \mathbf{Z}W(H)$  is finite for every finite subgroup  $H$  of  $G$ .

In support of Conjecture 61.3 is also the following characterization of finite groups, which we obtained in [2]: a group  $G$  is finite if and only if  $\text{spli } \mathbf{Z}G = 1$ .

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