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## 10

### THE REGULARITY CONJECTURE IN THE COHOMOLOGY OF GROUPS

by Dave BENSON

Let  $k$  be a field of characteristic  $p$  and let  $G$  be a finite group.

CONJECTURE 10.1. *The Castelnuovo–Mumford regularity of the cohomology ring is equal to zero:*

$$\text{Reg } H^*(G, k) = 0.$$

This conjecture was first announced at the opening workshop of the MSRI commutative algebra year ([1]), as a refinement of a conjecture of Benson and Carlson ([4]). Subsequent work on the conjecture was reported in [2] and [3].

We begin with the definitions. Let  $H$  be a finitely generated graded commutative  $k$ -algebra, with  $H^0 = k$  and  $H^i = 0$  for  $i < 0$  (e.g.,  $H = H^*(G, k)$ ). Write  $\mathfrak{m}$  for the maximal ideal generated by the elements of positive degree. If  $M$  is a graded  $H$ -module then the local cohomology is doubly graded:  $H_{\mathfrak{m}}^{i,j} M$  denotes the part in local cohomological degree  $i$  and internal degree  $j$ . Local cohomology can either be regarded as the right derived functors of the  $\mathfrak{m}$ -torsion functor  $\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \exists n \geq 0, \mathfrak{m}^n \cdot x = 0\}$ , or as the cohomology of the stable Koszul complex (see for example Theorem 3.5.6 of Bruns and Herzog [6]).

The  $a$ -invariants of  $M$  are defined to be

$$a_{\mathfrak{m}}^i(M) = \max\{j \in \mathbf{Z} \mid H_{\mathfrak{m}}^{i,j} M \neq 0\}$$

(or  $a_{\mathfrak{m}}^i(M) = -\infty$  if  $H_{\mathfrak{m}}^i M = 0$ ).

The *Castelnuovo–Mumford regularity* of  $M$  is then defined as

$$\text{Reg } M = \max_{i \geq 0} \{a_{\mathfrak{m}}^i(M) + i\}.$$

Of particular interest is the regularity of the ring itself,  $\text{Reg } H$ .

The reason for the interest in local cohomology of group cohomology comes from the Greenlees version ([7]) of Benson–Carlson duality ([4]), in the form of a spectral sequence

$$H_m^{i,j} H^*(G, k) \Rightarrow H_{-i-j}(G, k).$$

In particular, the existence of the “last survivor” of [4] shows the following ([2]):

**THEOREM 10.2.**  $\text{Reg } H^*(G, k) \geq 0$ .

The regularity conjecture is known to hold in the following situations:

- $H^*(G, k)$  is Cohen–Macaulay; e.g., groups with abelian Sylow  $p$ -subgroups; groups with extraspecial Sylow 2-subgroups with  $p = 2$ ; groups of Lie type with characteristic coprime to  $p$  ([1]).
- Krull dimension minus depth at most two; e.g., 2-groups of order  $\leq 64$  ([2]).
- Symmetric and alternating groups in any characteristic; these are examples where Krull dimension minus depth is arbitrarily large ([3]).

There is also a corresponding conjecture for compact Lie groups. Let  $G$  be a compact Lie group of dimension  $d$ , and suppose that the adjoint action of  $G$  on  $\text{Lie}(G)$  preserves orientation. Then there is a spectral sequence (Benson–Greenlees [5])

$$H_m^{i,j} H^*(BG; k) \Rightarrow H_{-i-j-d}(BG; k).$$

**CONJECTURE 10.3.**  $\text{Reg } H^*(BG; k) = -d$ .

To explain the orientation condition, let  $G = T^3 \rtimes \mathbf{Z}/2$ , a semidirect product of a 3-torus by an involution acting through inversion, and  $k$  be a field of characteristic  $\neq 2$ . Then

$$H^*(BG; k) = H^*(BT; k)^{\mathbf{Z}/2}$$

is Cohen–Macaulay but not Gorenstein, and  $\text{Reg } H^*(BG; k) = -5$ . The appropriate modification in this situation is that if  $\varepsilon$  denotes the orientation character, then  $\text{Reg } H^*(BG; \varepsilon) = -d$ .

ADDED IN PROOF. David Green has verified the regularity conjecture for all groups of order 128. See D. J. GREEN. ‘Testing Benson’s regularity conjecture’. Preprint arXiv : math.GR/0710.2311 (2007).

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