Zeitschrift: L'Enseignement Mathématique
Band: 54 (2008)
Heft: 1-2

Artikel: The Whitehead conjecture and L^2-Betti-numbers

Autor: Berrick, A. Jon / Hillman, Jonathan A.

DOI: https://doi.org/10.5169/seals-109881

L'Enseignement Mathématique (2) 54 (2008), 35–37

12

THE WHITEHEAD CONJECTURE AND L^2-BETTI NUMBERS

by A. Jon BERRICK and Jonathan A. HILLMAN

In [8], J. H. C. Whitehead asked whether any subcomplex of an aspherical 2-dimensional complex must be aspherical. An affirmative answer to this question is widely known as the Whitehead Conjecture.

Failure of the Whitehead Conjecture implies either

(a) there is a finite 2-complex X with $\pi_2(X) \neq 0$ and a map $f\colon S^1 \to X$ such that $Y = X \cup_f e^2$ is contractible; or

(b) there is an infinite ascending chain $K_n \subset K_{n+1}$ of finite 2-complexes with $\pi_2(K_n) \neq 0$ for all n and such that $\bigcup_{n\geq 1} K_n$ is aspherical ([4]).

Moreover, a counterexample of the first type implies the existence of a counterexample of the second type ([7]).

Our concern here is with the finite case of the conjecture, namely the assertion that any subcomplex of a finite aspherical 2-complex is also aspherical. This assertion implies that (a) above does not hold. An interesting question is whether the negation of (a) above is actually equivalent to the finite case of the conjecture.

If X is a finite 2-complex such that $Y = X \cup_f e^2$ is contractible then $\pi = \pi_1(X)$ has a presentation of deficiency 1 (since $\chi(X) = 0$), and π is the normal closure of the element represented by the attaching map f; so π has weight 1. Conversely, the usual 2-complex of any deficiency 1 presentation of a group π of weight 1 is such a finite subcomplex X of a contractible 2-complex (where f corresponds to a normal generator of π). (Every such group π is the group of a 2-knot [5].)

We now introduce L^2-Betti numbers $\beta_i^{(2)}$ [6] into this situation. Relevant facts here include that for a finite 2-complex X,

$$\chi(X) = \chi^{(2)}(X) = \beta_0^{(2)}(X) - \beta_1^{(2)}(X) + \beta_2^{(2)}(X)$$

and $\beta_i^{(2)}(X) = \beta_i^{(2)}(\pi_1(X))$ for $i = 0, 1$. (The question of whether the equality of the two Euler characteristics still holds for finitely dominated X relates to the weak Bass conjecture for $\pi_1(X)$ [1]).

A finite 2-complex X is aspherical if $\chi(X) = 0$ and $\beta_1^{(2)}(X) = 0$ [3]. From [2], the L^2-Betti number condition is satisfied if, for instance, $\pi_1(X)$ has an infinite subgroup that is

(i) amenable and ascendant; or

(ii) finitely generated, subnormal and of infinite index.

In fact, in the presence of (i), $\chi(X) = 0$ is necessary and sufficient for asphericity. On the other hand, in the presence of (ii) instead, $\chi(X) = 0$ is not necessary for asphericity. To see this, take for X the classifying space of the group $F(2) \times F(2)$, where $F(2)$ denotes the free group of rank 2; here $\chi(X) = 1$.

If πK is the group of a tame classical knot $K \subset S^3$ then $\beta_1^{(2)}(\pi K) = 0$ (see §4.3 of [6]), and so the 2-complex associated to any deficiency 1 presentation of a classical knot group is aspherical.

On the basis of this modest evidence, we suggest that a better understanding of L^2-Betti numbers may contribute to the finite case of the Whitehead Conjecture.

QUESTION 12.1. If a group π has weight 1 and a finite presentation of deficiency 1, is $\beta_1^{(2)}(\pi) = 0$?

Neither "deficiency 1" alone nor "weight 1" alone is enough. The free product $\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ has deficiency 1, but it is also a semidirect product $\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} \cong F(2) \rtimes \mathbf{Z}/2\mathbf{Z}$ and so $\beta_1^{(2)}(\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}) = \frac{1}{2}\beta_1^{(2)}(F(2)) > 0$. The free product $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ has weight 1 (since equating generators for the free factors kills the group), but its commutator subgroup is free of rank 2 and has index 6, so $\beta_1^{(2)}(\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}) = \frac{1}{6}\beta_1^{(2)}(F(2)) > 0$. (Neither of these groups is π_1 of a finite aspherical complex.)

On the other hand, $\mathbf{Z}^2$ has deficiency 1 and $\beta_1^{(2)}(\mathbf{Z}^2) = 0$, but clearly $\mathbf{Z}^2$ has weight 2. The semidirect product $(\mathbf{Z}/3\mathbf{Z}) \rtimes_{-1} \mathbf{Z}$ has weight 1 and $\beta_1^{(2)}((\mathbf{Z}/3\mathbf{Z}) \rtimes_{-1} \mathbf{Z}) = 0$, but this group has deficiency 0. Thus neither hypothesis "deficiency 1" nor "weight 1" is implied by the conjunction of the other with the condition $\beta_1^{(2)}(\pi) = 0$.

REFERENCES

[1] BERRICK, A. J., I. CHATTERJI and G. MISLIN. Homotopy idempotents on manifolds and Bass' conjectures. In: *Proc. Internat. Conf. Algebraic Topology*, Kinosaki Japan 2003, in honor of G. Nishida, *Geom. Topol. Monogr. 10* (2006), 41–62.

[2] BERRICK, A. J. and J. A. HILLMAN. Perfect and acyclic subgroups of finitely presentable groups. *J. London Math. Soc. (2) 68* (2003), 683–698.

[3] HILLMAN, J. A. L^2-homology and asphericity. *Israel J. Math. 99* (1997), 271–283.

[4] HOWIE, J. Some remarks on a problem of J. H. C. Whitehead. *Topology 22* (1983), 475–485.

[5] KERVAIRE, M. A. Les noeuds de dimensions supérieures. *Bull. Soc. Math. France 93* (1965), 225–271.

[6] LÜCK, W. *L^2-Invariants: Theory and Applications to Geometry and K-Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, *44*, Springer-Verlag, Berlin, 2002.

[7] LUFT, E. On 2-dimensional aspherical complexes and a problem of J. H. C. Whitehead. *Math. Proc. Cambridge Philos. Soc. 119* (1996), 493–495.

[8] WHITEHEAD, J. H. C. On adding relations to homotopy groups. *Ann. of Math. (2) 42* (1941), 409–428.

A. Jon Berrick

National University of Singapore
2 Science Drive 2
Singapore 117543
e-mail: berrick@math.nus.edu.sg

Jonathan A. Hillman

The University of Sydney
NSW 2006
Australia
e-mail: jonh@maths.usyd.edu.au