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PROPERTY A AND EXACTNESS OF THE UNIFORM ROE ALGEBRA

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It has long been established that certain properties of groups can be described through properties of suitably chosen C^*-algebras associated with them. A model result in this direction is a theorem of Lance that a discrete group is amenable if and only if its reduced C^*-algebra $C_r^*(G)$ is nuclear.

Property A was introduced by Yu as a geometric analogue of the Følner criterion that describes amenability of a group. It has many of the same interesting consequences as amenability for a discrete group; for example, property A implies uniform embeddability in Hilbert space, which in turn gives the Coarse Baum–Connes conjecture and therefore the Novikov conjecture [8].

Property A and the uniform Roe algebra can be defined for arbitrary metric spaces. Let us recall the main definitions.

A uniformly discrete metric space (X,d) *has property* A if for all $R,\epsilon > 0$ there exists a family of finite non-empty subsets A_x of $X \times \mathbf{N}$, indexed by x in X, such that

- for all x,y with $d(x,y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$;
- there exists S such that for all x and $(y,n) \in A_x$ we have $d(x,y) \leq S$.

The *uniform Roe algebra*, $C_u^*(X)$, is the C^*-algebra completion of the algebra of bounded operators on $l^2(X)$ which have finite propagation. The details are as follows. A kernel $u\colon X \times X \to \mathbf{C}$ has *finite propagation* if there exists $R \geq 0$ such that $u(x,y) = 0$ for $d(x,y) > R$. If X is a proper discrete metric space, and $u\colon X \times X \to \mathbf{C}$ is a finite propagation kernel then for each x there are only finitely many y with $u(x,y) \neq 0$. Thus u defines a linear operator on the space of finitely supported functions on X by convolution: $(u * \xi)(x) = \sum_{y \in X} u(x,y)\xi(y)$. If additionally X has bounded geometry, then every bounded finite propagation kernel gives rise to a *bounded* operator on $l^2(X)$. The uniform Roe algebra is the completion of the algebra generated by bounded linear operators arising from bounded, finite propagation kernels.

For a discrete group G, Yu's property A is equivalent both to the nuclearity of the uniform Roe algebra $C_u^*(G)$ and to the exactness of the reduced C^*-algebra $C_r^*(G)$. This follows from the results of Anantharaman-Delaroche and Renault [1], Higson and Roe [6], Guentner and Kaminker [5], and Ozawa [7].

It is natural to state the following conjecture.

CONJECTURE 16.1. *A uniformly discrete bounded geometry metric space X has property A if and only if the uniform Roe algebra $C_u^*(X)$ is exact.*

In evidence for the conjecture we offer the following. The conjecture is true for any countable discrete group equipped with its natural coarse structure. It is then an easy exercise to show that the conjecture holds for any metric space which admits a proper co-compact action by a group of isometries.

We proved recently [2] that the conjecture also holds if the space is *sufficiently group-like* in the following sense.

From a geometric point of view, there is a subtle interplay between the left and the right action of a group on itself. By convention, the left action is by isometries while the right action has the property that each point is moved by the same distance by a given element of the group. By analogy with Euclidean geometry we call the latter *translations* even though they are not in general isometries. (It is often overlooked that it is the translation action, rather than the isometric action, that allows one to identify $C_r^*(G)$ with a subalgebra of the uniform Roe algebra $C_u^*(G)$.)

In the case of a discrete bounded geometry metric space one can at best hope to construct partial quasi-isometries (or more generally, partial coarse maps) and partial translations, owing to the lack of homogeneity of the space. Nonetheless we may abstract the crucial features of the group structure by imposing a family of partial quasi-isometries on the space, in place of the left action, and a family of partial bijections with bounded propagation, in place of the right action. This structure can be chosen to satisfy certain compatibility conditions, reflecting the relation between the left and the right action of a group on itself. Partial translations provide the space with the structure of a graph in which the edges are coloured. The lack of homogeneity of this coloured graph is measured by an invariant. In the case of the group, equipped with the left and the right action, the corresponding structure has invariant one. We thus say that a space is *sufficiently group-like* if it admits a structure with invariant one.

When the space X is sufficiently group like in this sense, the conjecture holds for X. For example, this is the case when X embeds uniformly in a group [2]. An interesting example of this situation is furnished by the Diestel–Leader graph [3], which is not quasi-isometric to the Cayley graph of a group [4], but uniformly embeds in a product of two free groups.

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