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PERIODIC p-TORSION IN THE FARRELL COHOMOLOGY OF SOME SYMPLECTIC GROUPS

by Cornelia Minette BUSCH

For an odd prime p and a nonzero integer $0 \neq n \in \mathbf{Z}$ we consider $\mathrm{Sp}\big(p-1, \mathbf{Z}[1/n]\big)$, the group of symplectic matrices with coefficients in $\mathbf{Z}[1/n]$. It is defined to be

$$\mathrm{Sp}\big(p-1, \mathbf{Z}[1/n]\big) := \left\{ Y \in \mathrm{GL}\big(p-1, \mathbf{Z}[1/n]\big) \;\middle|\; Y^{\mathrm{t}} J Y = J := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \right\},$$

where $\mathbf{1}$ denotes the identity on $\mathbf{Z}[1/n]^{(p-1)/2}$. This symplectic group has finite virtual cohomological dimension and contains elements of order p. Moreover $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ has p-periodic Farrell cohomology since each of its elementary abelian p-subgroups has rank ≤ 1. Let

$$\widehat{\mathrm{H}}^* \big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]) \,, \mathbf{Z} \big)_{(p)}$$

denote the p-primary part of the Farrell cohomology of $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ with coefficients in $\mathbf{Z}$.

QUESTION 19.1. *What is the p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^* \big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big) \,?$$

Here p is an odd prime and $0 \neq n \in \mathbf{Z}$ is any nonzero integer.

Since the group we are considering has the properties given above we get, by a result of K. S. Brown [1], the isomorphism

$$\widehat{\mathrm{H}}^* \big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big)_{(p)} \cong \prod_{P \in \mathfrak{P}} \widehat{\mathrm{H}}^* \big(N(P), \mathbf{Z}\big)_{(p)} \,.$$

Here $\mathfrak{P}$ is a set of representatives of conjugacy classes of subgroups P of order p in $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ and $N(P)$ denotes the normalizer of P.

In order to use this isomorphism, we analyze the structure of the subgroups of order p in $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$. We see in [3] that the conjugacy classes of elements of order p in $\mathrm{Sp}(p-1, \mathbf{Z}[1/n])$ are related to some ideal classes in $\mathbf{Z}[1/n][\xi]$, where ξ is a primitive pth root of unity. Therefore, if $n = 1$, the number of conjugacy classes of elements of order p in $\mathrm{Sp}(p-1, \mathbf{Z})$ depends on h^-, the relative class number of p. In [2] we get the following result.

THEOREM 19.2. *Let p be an odd prime for which the relative class number h^- is odd and let y be such that $p-1 = 2^r y$ with y odd. Then the period of $\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}\big)_{(p)}$ equals $2y$.*

The smallest prime p for which h^- is even is $p = 29$. In fact it is not known if the statement of Theorem 19.2 is true for primes with even relative class number. Since the number of ideal classes in $\mathbf{Z}[\xi]$ is finite, it is possible to choose $0 \neq n \in \mathbf{Z}$ such that $\mathbf{Z}[1/n][\xi + \xi^{-1}]$ and $\mathbf{Z}[1/n][\xi]$ are principal ideal domains and moreover the odd prime p divides n. With these assumptions, we get the following result [4].

THEOREM 19.3. *Choose p and n as above. Let y be the greatest odd divisor of $p-1$. Then $2y$ is the p-period of the Farrell cohomology ring $\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big)$ and, moreover, for any $i \in \mathbf{Z}$ we get an isomorphism*

$$\widehat{\mathrm{H}}^i\big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big) \cong \widehat{\mathrm{H}}^{i+d}\big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big)$$

with $d = y$ if and only if for each $j \mid y$ a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$, and with $d = 2y$ if for some j no such q exists.

Here f_q is the inertia degree of the prime q in $\mathbf{Z}[\xi]$. It is the order of q in the group of units of $\mathbf{F}_p$. It is a consequence of Dirichlet's theorem on primes in arithmetic progression that for every $f_q \mid p-1$ an infinite number of primes q exist with inertia degree f_q. Let us guess the answer to Question 19.1.

CONJECTURE 19.4. *Let p be an odd prime. Then the p-period of the Farrell cohomology ring $\widehat{\mathrm{H}}^*\big(\mathrm{Sp}(p-1, \mathbf{Z}[1/n]), \mathbf{Z}\big)$ equals $2y$ for all $n \neq 0$, where y is the greatest odd divisor of $p-1$.*

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