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AN FP_m-CONJECTURE FOR NILPOTENT-BY-ABELIAN GROUPS

by Kai-Uwe BUX

Let G be a finitely generated *metabelian group*, i.e., we have a short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

with N and Q Abelian groups, wherein the quotient Q is finitely generated and the kernel N is finitely generated as a $\mathbf{Z}Q$-module. For any homomorphism $\chi\colon Q \to \mathbf{R}$, let $Q_\chi := \{q \in Q \mid \chi(q) \geq 0\}$ be the monoid of elements in Q that are non-negative with respect to χ. R. Bieri and R. Strebel defined the *geometric invariant* of G as

$$\Sigma_Q(N) := \{\chi \in \mathrm{Hom}(Q, \mathbf{R}) \mid N \text{ is finitely generated over } \mathbf{Z}Q_\chi\}\,.$$

Note that homomorphisms that are positive scalar multiples of one another define the same non-negative sub-monoid of Q. Thus, the geometric invariant is a conical subset of the real vector space $\mathrm{Hom}(Q, \mathrm{R})$. Also note that $Q_0 = Q$, whence the geometric invariant contains 0 since G is finitely generated.

Bieri–Strebel showed that $\Sigma_Q(N)$ determines whether G is finitely presented. However, this information is more easily extracted from the complement

$$\Sigma_Q^c(N) := \mathrm{Hom}(Q, \mathbf{R}) - \Sigma_Q(N)\,.$$

THEOREM 20.1 (Bieri–Strebel [4]). *The following are equivalent:*

(1) *G is finitely presented.*

(2) *G is of type* FP_2.

(3) *The complement $\Sigma_Q^c(N)$ does not contain two antipodal points, i.e., whenever $\chi \in \Sigma_Q^c(N)$, then $-\chi \notin \Sigma_Q^c(N)$.*

Bieri conjectured that the information about higher finiteness properties of G is also encoded in $\Sigma^c_Q(N)$. Recall that a group G is *of type* FP_m if there is a partial resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow \mathbf{Z}$$

of $\mathbf{Z}$, regarded as the trivial $\mathbf{Z}G$-module, by finitely generated projective $\mathbf{Z}G$-modules.

CONJECTURE 20.2 (Bieri). *For any $m \geq 2$, the following are equivalent:*

(1) *G is of type* FP_m.

(2) *The complement $\Sigma^c_Q(N)$ is m-tame.*

Here, we call a conical subset U of a real vector space *m-tame* if

$$0 \notin \underbrace{U + U + \cdots + U}_{m \text{ summands}}.$$

Evidence for this conjecture is mounting. It has been proved for many special cases. In particular, H. Åberg settled the case when N is virtually torsion free of finite rank [2], and the case $m = 3$ was settled by R. Bieri and J. Harlander for the case of split extensions [3].

Now, let G be *nilpotent-by-Abelian*, i.e., suppose G fits into a short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

where N is nilpotent and Q is Abelian. Again, we assume that G is finitely generated. In that case, every Abelian factor $M_i := N_i/N_{i+1}$ along the lower central series $N = N_1 > N_2 > N_3 > \ldots$ is a finitely generated $\mathbf{Z}Q$-module to which we can associate, as above, a geometric invariant $\Sigma_Q(M_i)$ and a complement denoted by $\Sigma^c_Q(M_i)$.

Note that a necessary condition for G to be of type FP_m is that the homology groups $\mathrm{H}_i(G;\mathbf{Z})$ are finitely generated in dimensions up to m. Therefore, the most optimistic and most straightforward generalization of the FP_m-conjecture to the class of nilpotent-by-Abelian groups would be that the metabelian quotient of G contains all of the relevant information needed besides the obvious homological restrictions. We thus arrive at:

CONJECTURE 20.3. *For $m \geq 2$, the following are equivalent:*

(1) *G is of type* FP_m.

(2) *The complement $\Sigma^c_Q(M_1)$ is m-tame and the homology groups $\mathrm{H}_i(N;\mathbf{Z})$ are finitely generated as $\mathbf{Z}Q$-modules for all dimensions $i \in \{1, 2, \ldots, m\}$.*

Surprisingly, this very optimistic conjecture has some support: by results of H. Abels, the conjecture holds for $m = 2$ if G is a solvable S-arithmetic group over a number field [1]. My own results on solvable S-arithmetic groups over function fields [5] are also compatible with the conjecture. However, the conjecture appears too optimistic, so a better question might be:

Is there a way to characterize the higher FP_m*-properties of a nilpotent-by-Abelian group* G *in terms of its homology and the geometric invariants of the modules* M_i *?*

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