# A contact geometric proof of the Whitney-Graustein theorem

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# A CONTACT GEOMETRIC PROOF OF THE WHITNEY–GRAUSTEIN THEOREM

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ABSTRACT. The Whitney–Graustein theorem states that regular closed curves in the 2-plane are classified, up to regular homotopy, by their rotation number. Here we give a simple proof based on contact geometry.

## 1. INTRODUCTION

A *regular closed curve* in the 2-plane is a continuously differentiable map $\overline{\gamma}\colon [0,2\pi] \to \mathbf{R}^2$ with the following properties:

$$\begin{aligned} &\text{(i)} \quad \overline{\gamma}(0) = \overline{\gamma}(2\pi)\,, \quad \overline{\gamma}'(0) = \overline{\gamma}'(2\pi)\,,\\ &\text{(ii)} \quad \overline{\gamma}'(s) \neq \mathbf{0} \quad \text{for all } s \in [0,2\pi]\,. \end{aligned}$$

If we identify the circle $S^1$ with $\mathbf{R}/2\pi\mathbf{Z}$, we may think of $\overline{\gamma}$ as a continuously differentiable map $S^1 \to \mathbf{R}^2$.

The *rotation number* $\operatorname{rot}(\overline{\gamma})$ of $\overline{\gamma}$ is the degree of the map

$$\begin{aligned} S^1 &\longrightarrow \mathbf{R}^2 \setminus \{\mathbf{0}\}\,,\\ s &\longmapsto \overline{\gamma}'(s)\,. \end{aligned}$$

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In other words, $\text{rot}(\overline{\gamma})$ is simply a signed count of the number of complete turns of the velocity vector $\overline{\gamma}'$ as we once traverse the closed curve $\overline{\gamma}$, see Figure 1.

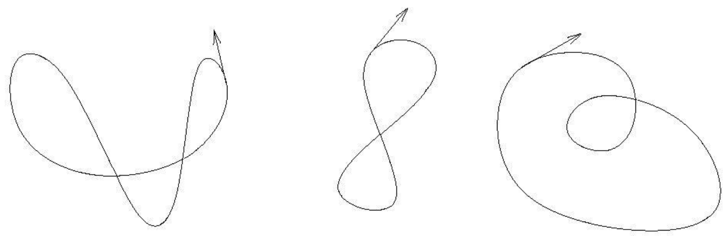

FIGURE 1
Regular closed curves $\overline{\gamma}$ with $\text{rot}(\overline{\gamma})$ equal to $1,\ 0,\ -2$, respectively

A *regular homotopy* between two such regular closed curves $\overline{\gamma}_0, \overline{\gamma}_1$ is a continuously differentiable homotopy via regular closed curves $\overline{\gamma}_t \colon S^1 \to \mathbf{R}^2$, $t \in [0,1]$. The rotation number clearly stays invariant under regular homotopies. The following theorem is commonly known as the Whitney–Graustein theorem. It was first proved in a paper by H. Whitney [5], who writes: "This theorem, together with its proof, was suggested to me by W.C. Graustein." For alternative presentations see [1, Chapter 0] or [3, p. 47 *et seq.*].

THEOREM 1. *Regular homotopy classes of regular closed curves* $\overline{\gamma} \colon S^1 \to \mathbf{R}^2$ *are in one-to-one correspondence with the integers, the correspondence being given by* $[\overline{\gamma}] \mapsto \text{rot}(\overline{\gamma})$.

Whitney's proof is elementary, but not without intricacies. Here we want to present a non-elementary proof — based on contact geometry — where the geometric ideas are actually quite simple.

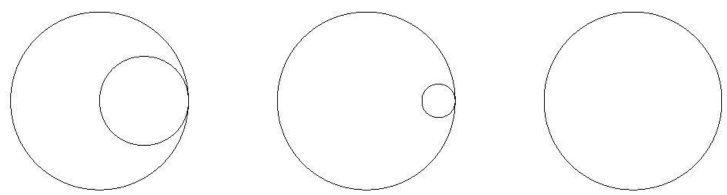

FIGURE 2
A homotopy through regular closed curves with non-invariant rot

REMARK. The modern terminology 'regular homotopy' describes what Whitney called a 'deformation' of regular closed curves. He seems to suggest, erroneously, that it is enough to require that $\gamma_t(s)$ be continuous in $s$ and $t$ and a regular closed curve for each fixed $t$, but in the course of his argument it becomes clear that he also wants $\gamma_t'(s)$ to depend continuously on $t$. Figure 2 shows a homotopy of regular closed curves (first traverse the big circle counter-clockwise, then the small circle) with $\mathrm{rot}(\gamma_t) = 2$ for $t \in [0,1)$, but $\mathrm{rot}(\gamma_1) = 1$.

## 2. LEGENDRIAN CURVES

The *standard contact structure* $\xi$ on $\mathbf{R}^3$, see Figure 3 (produced by Stephan Schönenberger), is the 2-plane field $\xi = \ker(dz + x\,dy)$. For a brief introduction to contact geometry see [2]. No knowledge of contact geometry beyond the concepts that we shall introduce explicitly will be required for the argument that follows.

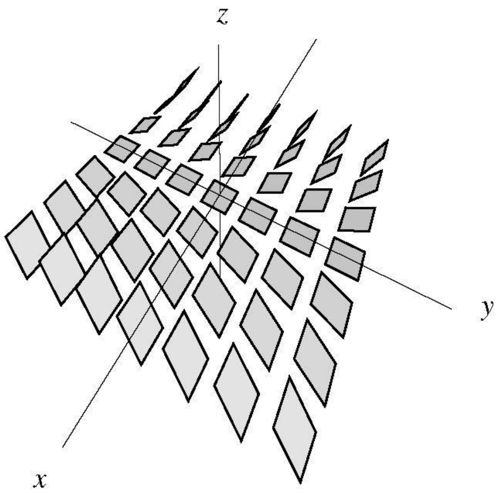


FIGURE 3
The contact structure $\xi = \ker(dz + x\,dy)$

A regular closed, continuously differentiable curve $\gamma\colon S^1 \to (\mathbf{R}^3, \xi)$ is called *Legendrian* if it is everywhere tangent to $\xi$, that is, $\gamma'(s) \in \xi_{\gamma(s)}$ for all $s \in S^1$. When we write $\gamma$ in terms of coordinate functions as $\gamma(s) = \big(x(s), y(s), z(s)\big)$, the condition for $\gamma$ to be Legendrian becomes $z' + xy' \equiv 0$. The *front projection* of $\gamma$ is the planar curve

$$\gamma_{\mathrm{F}}(s) = \big(y(s), z(s)\big)\;;$$

its *Lagrangian projection*, the curve

$$\gamma_{\mathrm{L}}(s) = \big(x(s),\, y(s)\big)\,.$$

Figure 4 shows the front and Lagrangian projection of a Legendrian unknot in $\mathbf{R}^3$.

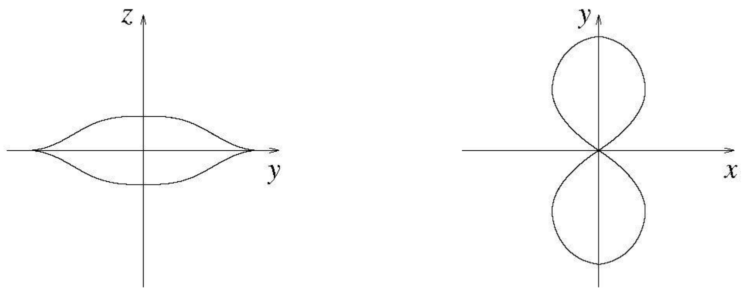


FIGURE 4
A Legendrian unknot

Notice that a Legendrian curve $\gamma$ can be recovered from its front projection $\gamma_{\mathrm{F}}$, since

$$x(s) = -\frac{z'(s)}{y'(s)} = -\frac{dz}{dy}$$

is simply the negative slope of the front projection. (Of course this only makes sense for $y'(s) \neq 0$. Generically, the zeros of the function $y'(s)$ are isolated, corresponding to isolated cusp points where $\gamma_{\mathrm{F}}$ still has a well-defined slope.) Since $x(s)$ is always finite, $\gamma_{\mathrm{F}}$ does not have any vertical tangencies, and we can sensibly speak of left and right cusps. These cusps are 'semi-cubical'; a model is given by $\big(x(s), y(s), z(s)\big) = (s, s^2/2, -s^3/3)$.

Likewise, $\gamma$ can be recovered from its Lagrangian projection $\gamma_{\mathrm{L}}$ (unique up to translation in the $z$-direction), for the missing coordinate $z$ is given by

$$z(s_1) = z(s_0) - \int_{s_0}^{s_1} x(s)y'(s)\,ds\,.$$

Observe that the integral $\int xy'\,ds = \int x\,dy$, when integrating over a closed curve, measures the oriented area enclosed by that curve. Moreover, the Lagrangian projection $\gamma_{\mathrm{L}}$ of a regular Legendrian curve $\gamma$ is always regular: if $y'(s) = 0$, the Legendrian condition forces $z'(s) = 0$, and then the regularity of $\gamma$ gives $x'(s) \neq 0$.

The idea for the proof of Theorem 1 is now the following. Given a (regular closed) Legendrian curve $\gamma$ in $(\mathbf{R}^3, \xi)$, one can assign to it an invariant (under Legendrian regular homotopies, i.e. regular homotopies via Legendrian curves). This invariant is likewise called 'rotation number'. In fact, the rotation number of $\gamma$ will be seen to equal the rotation number of its Lagrangian projection $\gamma_{\mathrm{L}}$. Alternatively, the rotation number of $\gamma$ can be computed from its front projection $\gamma_{\mathrm{F}}$, where it becomes a simple combinatorial quantity (a count of cusps). Now, given two regular closed curves $\overline{\gamma}_0, \overline{\gamma}_1$ in the plane with equal rotation number, we can consider their lifts to Legendrian curves $\gamma_0, \gamma_1$ (still with equal rotation number), and in the front projection we can now 'see', in a combinatorial way, a Legendrian regular homotopy between them. The Lagrangian projection of this Legendrian regular homotopy will give us the regular homotopy between $\overline{\gamma}_0$ and $\overline{\gamma}_1$.

## 3. The rotation number

The plane field $\xi$ is spanned by the globally defined vector fields $e_1 = \partial_x$ and $e_2 = \partial_y - x\,\partial_z$. In terms of the trivialisation of $\xi$ defined by these vector fields, we may regard the map $\gamma'$ (coming from a regular closed Legendrian curve $\gamma$) as a map

$$\begin{aligned} S^1 &\longrightarrow \mathbf{R}^2 \setminus \{\mathbf{0}\}\,, \\ s &\longmapsto \gamma'(s)\,. \end{aligned}$$

The *rotation number* $\mathrm{rot}(\gamma)$ of a Legendrian curve $\gamma$ is the degree of that map. This means that $\mathrm{rot}(\gamma)$ counts the number of rotations of the velocity vector $\gamma'$ relative to the oriented basis $e_1, e_2$ of $\xi$ as we go once around $\gamma$. The rotation number is clearly an invariant of Legendrian regular homotopies.

Under the projection $(x,y,z) \mapsto (x,y)$, each 2-plane $\xi_{\gamma(s)}$ maps isomorphically onto $\mathbf{R}^2$, and the basis $e_1, e_2$ for $\xi_{\gamma(s)}$ is mapped to the standard basis $\partial_x, \partial_y$ for $\mathbf{R}^2$. So the following proposition is immediate from the definitions.

PROPOSITION 2. *The rotation number of a (regular closed) Legendrian curve in* $(\mathbf{R}^3, \xi)$ *equals the rotation number of its Lagrangian projection.* □

A little more work is required to read off $\mathrm{rot}(\gamma)$ from the front projection $\gamma_{\mathrm{F}}$. This, however, is well worth the effort, because it turns the rotation number into a simple combinatorial quantity.

PROPOSITION 3. *Let $\gamma$ be a (regular closed) Legendrian curve in $(\mathbf{R}^3, \xi)$. Write $\lambda_+$ or $\lambda_-$, respectively, for the number of left cusps of the front projection $\gamma_F$ oriented upwards or downwards; similarly we write $\rho_\pm$ for the number of right cusps with one or the other orientation. Finally, we write $c_\pm$ for the total number of cusps oriented upwards or downwards, respectively. Then the rotation number of $\gamma$ is given by*

$$\mathrm{rot}(\gamma) = \lambda_- - \rho_+ = \rho_- - \lambda_+ = \frac{1}{2}(c_- - c_+).$$

*Proof.* The rotation number $\mathrm{rot}(\gamma)$ can be computed by counting (with sign) how often the velocity vector $\gamma'$ crosses $e_1 = \partial_x$ as we travel once along $\gamma$.

Since $x(s)$ equals the negative slope of the front projection, points of $\gamma$ where the (positive) tangent vector equals $\partial_x$ are exactly the left cusps oriented downwards (see Figure 5) and the right cusps oriented upwards.

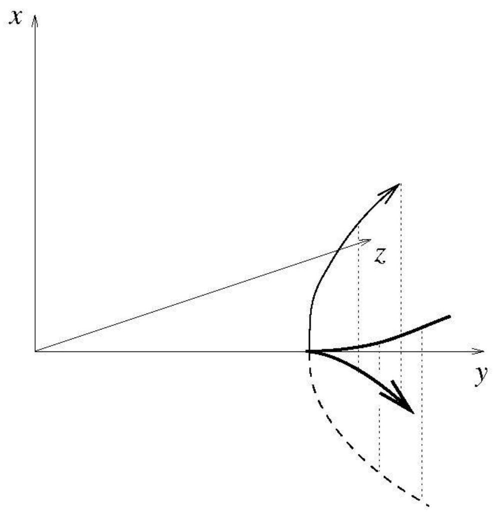


FIGURE 5
Contribution of a cusp to $\mathrm{rot}(\gamma)$

At a left cusp oriented downwards, the tangent vector to $\gamma$, expressed in terms of $e_1, e_2$, changes from having a negative component in the $e_2$-direction to a positive one, i.e. such a cusp yields a positive contribution to $\mathrm{rot}(\gamma)$. Analogously, one sees that a right cusp oriented upwards gives a negative contribution to the rotation number. This proves the formula $\mathrm{rot}(\gamma) = \lambda_- - \rho_+$. The second expression for the rotation number is obtained by counting crossings through $-e_1$ instead; the third expression is found by averaging the first two. □

## 4. Proof of the Whitney–Graustein theorem

First we give a classification of regular closed Legendrian curves up to Legendrian regular homotopy.

PROPOSITION 4. *Legendrian regular homotopy classes of regular closed Legendrian curves* $\gamma\colon S^1 \to (\mathbf{R}^3, \xi)$ *are in one-to-one correspondence with the integers, the correspondence being given by* $[\gamma] \mapsto \mathrm{rot}(\gamma)$.

*Proof.* With the help of either of the two foregoing propositions one can construct a regular closed Legendrian curve $\gamma$ with $\mathrm{rot}(\gamma)$ equal to any prescribed integer. Thus, we need only show that two regular closed Legendrian curves $S^1 \to (\mathbf{R}^3, \xi)$ with the same rotation number are Legendrian regularly homotopic.

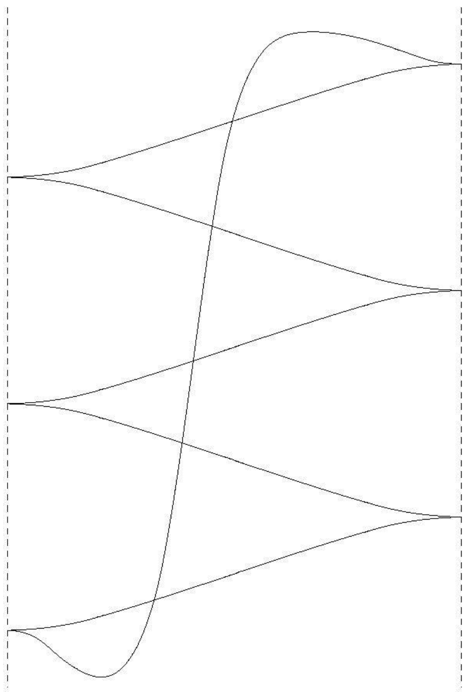

FIGURE 6
A front with cusps of one sign only

In the front projection of the Legendrian immersion $\gamma$, left and right cusps alternate. We label the up cusps with $+$ and the down cusps with $-$. The following observation will be crucial to our discussion.

Claim. *Up to Legendrian regular homotopy, $\gamma$ is completely determined by this sequence of labels, starting at a right cusp, say, and going once around $S^1$.*

This can be seen by homotoping $\gamma_F$ so that all left cusps come to lie on the line $\{y=0\}$ and all right cusps on the line $\{y=1\}$, say. The cusps on either line can be shuffled by further homotopies; in particular, they may be arranged along these lines in the same order in which they are traversed along the closed Legendrian curve. This provides a standard model for any given sequence of labels, and thus proves the claim. Figure 6 shows this standard model for a front $\gamma_F$ containing cusps of one sign only.

Continuing with the proof of the proposition, our aim now is to simplify the sequence of labels. Given a pair $+-$ in this sequence, we can cancel it (unless it constitutes the complete sequence) as follows. Arrange the adjacent vertices (by sliding them along the lines $\{y=0\}$ and $\{y=1\}$, respectively, as described before) in such a way that we have the situation on the right of Figure 7, then replace it by the situation on the left. This so-called *first Legendrian Reidemeister move* is in fact a Legendrian isotopy for that local piece of our curve, i.e. a regular homotopy not creating self-intersections. There is an analogous move with the picture rotated by $180^\circ$, which can be used to cancel any pair $-+$.

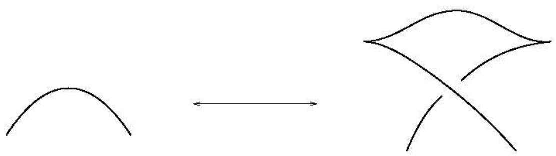

Figure 7
The first Legendrian Reidemeister move

Therefore, this sequence of labels can be reduced to a sequence containing only plus or only minus signs, or to one of the sequences $(+,-)$, $(-,+)$; see Figure 8 for an example. The formula $\mathrm{rot}(\gamma)=(c_- - c_+)/2$ shows that there are the following possibilities: if $\mathrm{rot}(\gamma)$ is positive (resp. negative), we must have a sequence of $2\,\mathrm{rot}(\gamma)$ minus (resp. plus) signs; if $\mathrm{rot}(\gamma)=0$, we must have the sequence $(+,-)$ or $(-,+)$. The proof is completed by observing that these last two sequences correspond to Legendrian isotopic knots: use a first Reidemeister move as in Figure 7, followed by the inverse of the rotated move. $\square$

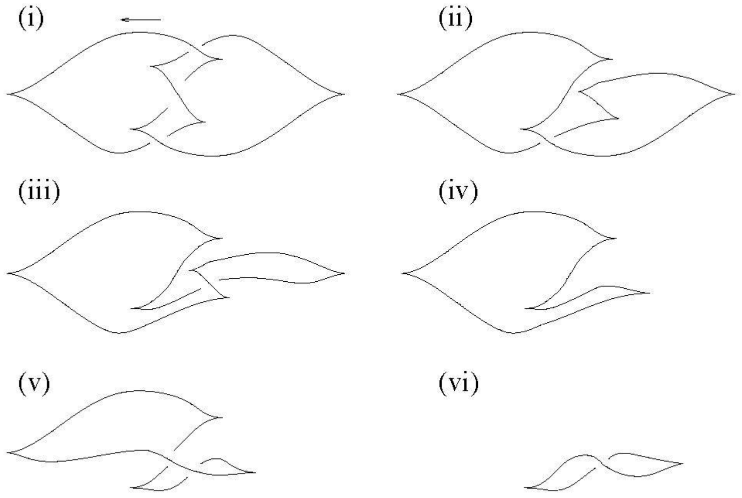


FIGURE 8
An example of a Legendrian regular homotopy

REMARK. Self-tangencies in the front projection $\gamma_F$ correspond to self-intersections of the Legendrian curve $\gamma$, since the negative slope of $\gamma_F$ gives the $x$-component of $\gamma$. Therefore, as we pass such a self-tangency in the moves of Figure 8, we effect a crossing change. With the orientation indicated in the figure, this example has $\mathrm{rot}(\gamma) = -1$.

*Proof of Theorem 1.* Again we only have to show that two regular closed curves $\overline{\gamma}_0, \overline{\gamma}_1 \colon S^1 \to \mathbf{R}^2$ (where we think of $\mathbf{R}^2$ as the $(x, y)$-plane) with $\mathrm{rot}(\overline{\gamma}_0) = \mathrm{rot}(\overline{\gamma}_1)$ are regularly homotopic.

After a regular homotopy we may assume that the $\overline{\gamma}_i$ satisfy the area condition $\oint_{\overline{\gamma}_i} x\,dy = 0$ and thus lift to regular *closed* Legendrian curves $\gamma_i \colon S^1 \to (\mathbf{R}^3, \xi)$ with, by Proposition 2, $\mathrm{rot}(\gamma_i) = \mathrm{rot}(\overline{\gamma}_i)$. By the preceding proposition, $\gamma_0$ and $\gamma_1$ are Legendrian regularly homotopic. The Lagrangian projection of this homotopy gives a regular homotopy between the curves $\overline{\gamma}_0$ and $\overline{\gamma}_1$, since — as pointed out in Section 2 — the Lagrangian projection of a regular Legendrian curve is regular. □

REMARK. See [4] for an application of the ideas in the present paper to the classification of loops tangent to the standard Engel structure on $\mathbf{R}^4$.

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