The planar rook algebra and Pascal's triangle

Autor(en): **Flath, Daniel / Halverson, Tom / Herbig, Kathryn**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **55 (2009)**

Heft 1-2

PDF erstellt am: **09.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-110095

L'Enseignement Mathématique (2) 55 (2009), 77–92

THE PLANAR ROOK ALGEBRA AND PASCAL'S TRIANGLE

by Daniel FLATH, Tom HALVERSON and Kathryn HERBIG *)

Surely the best known recursively defined integers are the binomial coefficients $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ appearing in Pascal's triangle. They admit many interpretations, the most familiar of which are

1. Algebraic: Expansion of $(x+y)^n$ with recursion

$$(x+y)^n = (x+y)^{n-1}(x+y).$$

2. Combinatorial: Number of subsets of $[1,n] = \{1,2,\ldots,n\}$ with recursion

$$[1,n] = [1,n-1] \cup \{n\}.$$

In this paper we propose a linear algebra interpretation rooted in representation theory. We construct natural vector spaces V_k^n with $\dim V_k^n = \binom{n}{k}$ and direct sum decompositions $V_k^n = V_{k-1}^{n-1} \oplus V_k^{n-1}$. The vector spaces are natural in the sense that the V_k^n are the spaces of all the distinct irreducible representations of an algebra $\mathbf{C}P_n$, the *planar rook algebra*. The direct sums describe decomposition upon restriction arising from an embedding $\mathbf{C}P_{n-1} \to \mathbf{C}P_n$. Even the multiplicative structure of the binomial coefficients arises from the representation theory of the planar rook algebras, as we discover upon decomposition of tensor product representations.

The planar rook algebra is an example of a "diagram algebra", which for our purposes is a finite-dimensional algebra with a basis given by a collection of diagrams and multiplication described combinatorially by diagram concatenation. When the basis diagrams can be drawn without edge crossings,

*) Halverson and Herbig were supported in part by National Science Foundation Grant DMS-0100975.

we get a *planar algebra*. There is a growing theory of planar algebras initiated by V. Jones (see [Jo]) that uses a more refined definition of planarity than what we give here.

The main goal of this paper is to work out the combinatorial representation theory of the planar rook algebra $\mathbb{C}P_n$ and to show that it is governed by the theory of binomial coefficients. The following are the main results:

1. A classification of the irreducible $\mathbb{C}P_n$-modules (Theorems 2.1 and 3.2).
2. An explicit decomposition of the regular representation of $\mathbb{C}P_n$ into a direct sum of irreducibles (Theorem 3.2).
3. A computation of the Bratteli diagram for the tower of algebras $\mathbb{C}P_0 \subseteq \mathbb{C}P_1 \subseteq \mathbb{C}P_2 \subseteq \dots$ (Section 4).
4. A computation of the character table for $\mathbb{C}P_n$.

1. The planar rook monoid

Let R_n denote the set of $n \times n$ matrices with entries from $\{0,1\}$ having *at most* one 1 in each row and in each column. For example,

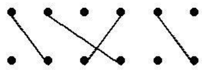
$$R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

We call these "rook matrices", since the 1s correspond to the possible placement of non-attacking rooks on an $n \times n$ chessboard. The *rank* of a rook matrix is the number of 1s in the matrix, and so to construct a rook matrix of rank k, we choose k rows and k columns in $\binom{n}{k}^2$ ways and then we place the 1s in $k!$ ways. Thus the cardinality of R_n is given by

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!\,.$$

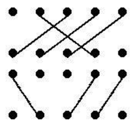
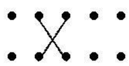
We let $R_0 = \{\varnothing\}$. There is no known closed formula for the sequence $|R_n|$, $n \geq 0$, which begins $1, 2, 7, 34, 209, 1546, 13327, \dots$ (see [OEIS, A002720]). The set R_n contains the identity matrix and is closed under matrix multiplication, so R_n is a monoid (a set with an associative binary operation and an identity, but where elements are not necessarily invertible). The invertible matrices in R_n are the permutation matrices (having rank n); they form a subgroup isomorphic to the symmetric group $S_n \subseteq R_n$.

We associate each element of R_n with a *rook diagram*, which is a graph on two rows of n vertices, such that vertex i in the top row is connected to vertex j in the bottom row if and only if the corresponding matrix has a 1 in the (i,j)-position. For example in R_6 we have



$$\leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

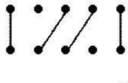
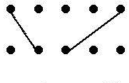
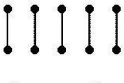
Matrix multiplication is accomplished on diagrams d_1 and d_2 by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2 (i.e., connecting the dots). For example,

$d_1 =$, $d_2 =$  $=$  $= d_1 d_2$

is the diagrammatic representation of the matrix product,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We say that an element of R_n is *planar* if its diagram can be drawn (keeping inside of the rectangle formed by its vertices) without any edge crossings. We let $P_n \subseteq R_n$ denote the set of planar elements of R_n. Our R_6 example is not planar. In the multiplication example above, the diagram d_2 is planar and d_1 is not. Below are a few more examples of elements in P_5 (of rank 4, 2, 0, and 5, respectively). The fourth diagram is the identity in $P_5 \subseteq R_5$, and the third corresponds to the matrix of all 0s.

   $= id$.

It is easy to see (by drawing diagrams) that the product of two planar rook diagrams is again planar, so P_n also forms a submonoid of R_n. The *only* invertible (rank n) planar rook diagram is the identity id.

To construct a planar rook diagram of rank k, we choose k vertices from each row. Then there is exactly one non-crossing way to connect them. Thus there are $\binom{n}{k}^2$ planar rook diagrams of rank k, and the number of planar rook diagrams is

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

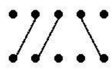
where we will let $P_0 = \{\varnothing\}$. The last equality above is a well-known binomial identity. To see it in this setting, choose any n of the $2n$ vertices in the rook diagram. Let k be the number of these chosen vertices that are in the top row (thus there are $n-k$ in the bottom row). Connect (in *the* one and only non-crossing way) the k chosen chosen vertices from the top row to the k *not* chosen vertices from the bottom row.

The algebraic properties of the rook monoid are studied in [So], [Gr], [Re], and [Ha]. The planar rook monoid appears in [Re] as the order preserving "partial permutations" of $\{1,2,\ldots,n\}$, and some combinatorics of P_n arise in [HL].

2. Planar rook diagrams acting on sets

In this section, we construct $\binom{n}{k}$-dimensional vector spaces V_k^n, for $0 \le k \le n$, and we will define a natural action (as linear transformations) of P_n on these vector spaces. In this way, we can homomorphically represent multiplication in P_n by multiplication of $\binom{n}{k} \times \binom{n}{k}$ matrices. Furthermore, we show that these matrix representations are all different, are irreducible, and include all the irreducible representations of P_n.

For a planar rook diagram d, let $\tau(d)$ and $\beta(d)$ denote the vertices in the top and bottom rows of d, respectively, that are incident to an edge. For example,

if $d =$  then $\tau(d) = \{2,3,4\}$ and $\beta(d) = \{1,2,5\}$.

The sets $\tau(d)$ and $\beta(d)$ uniquely determine d since there is only one planar way to connect the vertices by edges. We can view d as a 1-1 function with domain $\beta(d)$ and codomain $\tau(d)$. So, in our example, $d(1) = 2$, $d(2) = 3$, and $d(5) = 4$.

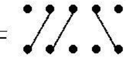
Now consider a subset $S = \{s_1,\ldots,s_k\}$ of order k chosen from the set $\{1,2,\ldots,n\}$. If $d \in P_n$ and if S is a subset of the domain $\beta(d)$ of d, then we can define an action of d on S by $d(S) = \{d(s_1), d(s_2), \ldots, d(s_k)\}$. Notice that $d(S)$ and S have the same cardinality.

There are 2^n subsets of $\{1,2,\ldots,n\}$, and we define a vector space V^n over $\mathbf{C}$ with dimension 2^n having a basis $\{\mathbf{v}_S\}$ labeled by these subsets $S \subseteq \{1,\ldots,n\}$. Thus

$$V^n = \mathbf{C}\text{-span}\left\{\mathbf{v}_S \mid S \subseteq \{1,\ldots,n\}\right\}. \tag{2.1}$$

We define an action of P_n on V^n as follows. For $d \in P_n$ and $S \subseteq \{1,\ldots,n\}$, define

$$d\mathbf{v}_S = \begin{cases} \mathbf{v}_{d(S)} & \text{if } S \subseteq \beta(d), \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

This defines an action of d on the basis of V^n which we then extend linearly to all of V^n. To illustrate with some examples, if we again let $d =$ , then $d\mathbf{v}_{\{1,2,5\}} = \mathbf{v}_{\{2,3,4\}}$, $d\mathbf{v}_{\{2,5\}} = \mathbf{v}_{\{3,4\}}$, and $d\mathbf{v}_{\{1,2,3\}} = 0$.

It follows from diagram multiplication that $(d_1d_2)\mathbf{v}_S = d_1(d_2(\mathbf{v}_S))$. This means that V^n is a "module" for P_n. The map from P_n to the set $\mathrm{End}(V^n)$ of linear transformations on V^n is an injective monoid homomorphism.

For $0 \leq k \leq n$ consider the subspace of V^n spanned by subsets of cardinality k,

$$V_k^n = \mathbf{C}\text{-span}\left\{\mathbf{v}_S \mid S \subseteq \{1,\ldots,n\} \text{ and } |S| = k\right\}. \tag{2.3}$$

Since the action of P_n preserves the size of the subset (or sends it to the zero vector) we see that the V_k^n are P_n-invariant submodules. The following theorem describes the structure of V^n as a module for P_n.

THEOREM 2.1. *For all $n \geq 0$ and $0 \leq k \leq n$, we have*

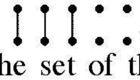
(a) *V_k^n is a P_n-module, and the V_k^n are non-isomorphic for different k.*

(b) *V_k^n is irreducible (it contains no proper, nonzero P_n-invariant subspaces).*

(c) *V^n decomposes as*

$$V^n \cong \bigoplus_{k=0}^{n} V_k^n,$$

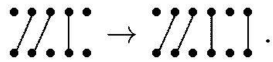
where each irreducible module appears with multiplicity 1.

Proof. (a) The fact that V_k^n is a module follows from the discussion preceding the theorem. Since the dimensions of these modules are binomial coefficients, the only possible isomorphism could occur between V_k^n and V_{n-k}^n. The element of the form $\pi_\ell =$ , which has ℓ vertical edges, acts by zero on V_k^n, with $k > \ell$. So the set of these π_ℓ, $0 \leq \ell \leq n$, will distinguish the V_k^n from one another.

(b) To show that V_k^n is irreducible, suppose that $W \subseteq V_k^n$ is a P_n-invariant subspace, and that $0 \neq w \in W$. We expand w in the basis as $w = \sum_{|S|=k} \lambda_S \mathbf{v}_S$, with $\lambda_S \in \mathbf{C}$. Since $w \neq 0$, there must be at least one $\lambda_S \neq 0$. Let $d \in P_n$ be the unique planar diagram with $\tau(d) = \beta(d) = S$. Then $dw = \lambda_S \mathbf{v}_S$, so $\mathbf{v}_S \in W$. Now let S' be any other subset of order k and let $d' \in P_n$ be the unique planar diagram with $\beta(d') = S$ and $\tau(d') = S'$. Then, $d'\mathbf{v}_S = \mathbf{v}_{d'(S)} = \mathbf{v}_{S'}$, so $\mathbf{v}_{S'} \in W$. This shows that all the basis vectors of V_k^n must be in W and so $W = V_k^n$.

(c) The fact that V^n decomposes as stated follows immediately from the fact that each $\mathbf{v}_S$ appears in exactly one of the V_k^n. In Section 3 we will prove that these are all of the irreducible representations by showing that these are the only representations that show up in the regular representation of P_n acting on itself by multiplication.

Let $P_0 = \{\varnothing\}$ and view $P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots$ by placing a vertical edge on the right of each diagram in P_{n-1}, i.e., an edge that connects the n^{th} vertex in each row. For example,



It is natural to look at the restriction of the action of P_n on V_k^n to the submonoid P_{n-1}. To this end, we construct the following subspaces of V_k^n:

$$V_{k,n}^n = \mathbf{C}\text{-span}\left\{\mathbf{v}_S \mid |S| = k,\ n \in S\right\},$$
$$\widehat{V}_{k,n}^n = \mathbf{C}\text{-span}\left\{\mathbf{v}_S \mid |S| = k,\ n \notin S\right\}.$$

If $d \in P_{n-1}$, then by the way we embed P_{n-1} into P_n, we have $n \in \tau(d)$ and $n \in \beta(d)$. Thus it is always the case that for a subset $S \subseteq \{1, \dots, n\}$, we have $n \in d(S)$ if and only if $n \in S$. This means that under the action defined in (2.2), the subspaces $V_{k,n}^n$ and $\widehat{V}_{k,n}^n$ are P_{n-1}-invariant.

From the point of view of P_{n-1}, we see that

$$V_{k,n}^n \cong V_{k-1}^{n-1} \quad \text{and} \quad \widehat{V}_{k,n}^n \cong V_k^{n-1},$$

since, in the first case, we are simply ignoring the element n, and in the second case, the basis vectors already are of the form $\mathbf{v}_S$ with $S \subseteq \{1, \dots, n-1\}$. Thus, the vector space V_k^n is irreducible under the P_n action, but it breaks up into the following direct sum of irreducible P_{n-1}-invariant subspaces,

$$V_k^n \cong V_{k-1}^{n-1} \oplus V_k^{n-1}, \tag{2.4}$$

where we drop the V_{k-1}^{n-1} if $k = 0$ and drop the V_k^{n-1} if $k = n$.

3. THE PLANAR ROOK ALGEBRA

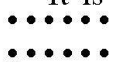
In this section we let P_n act on itself by multiplication; this is called the *regular representation* of P_n.

The Artin-Wedderburn theory of semisimple algebras (see for example [CR, Ch. IV] or [HR, Sec. 5]) states that if the regular representation of an algebra decomposes into a direct sum of irreducible modules, then (1) every irreducible module of the algebra is isomorphic to a summand of the regular representation and (2) every module of the algebra is isomorphic to a direct sum of irreducible modules. With this motivation, we construct an algebra associated to P_n, the planar rook algebra, and show explicitly that its regular representation reduces into a direct sum of modules each isomorphic to one of the V_k^n.

We define $\mathbf{C}P_n$ to be the $\mathbf{C}$-vector space with a basis given by the elements of P_n. That is,

$$\mathbf{C}P_n = \mathbf{C}\text{-span}\left\{d \mid d \in P_n\right\} = \left\{\sum_{d \in P_n} \lambda_d\, d \;\middle|\; \lambda_d \in \mathbf{C}\right\}.$$

This is the vector space of all (formal) linear combinations of planar rook diagrams, and it has dimension equal to the cardinality $|P_n| = \binom{2n}{n}$. This complex vector space $\mathbf{C}P_n$ is also equipped with a multiplication given by extending linearly the multiplication of diagrams in P_n. This makes $\mathbf{C}P_n$ an algebra over $\mathbf{C}$ which we call the *planar rook algebra.*

It is interesting to notice that the diagram associated to the zero matrix, , is a basis element in this vector space, whereas the 0 vector is the linear combination with all the $\lambda_d = 0$.

Since $\mathbf{C}P_n$ is spanned by planar rook diagrams, an element $d \in P_n$ acts naturally on the vector space $\mathbf{C}P_n$ by multiplication on the left. That is, if $d \in P_n$ and $\mathbf{v} = \sum_{b \in P_n} \lambda_b b \in \mathbf{C}P_n$ we have

$$d\mathbf{v} = d\Big(\sum_{b \in P_n} \lambda_b\, b\Big) = \sum_{b \in P_n} \lambda_b\, db\,.$$

Multiplication of planar rook diagrams has the property that rank does not go up, i.e.,

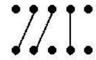
$$\operatorname{rank}(d_1 d_2) \leq \min\big(\operatorname{rank}(d_1), \operatorname{rank}(d_2)\big)\,.$$

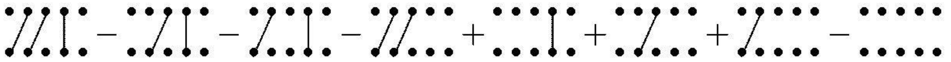
Thus if we let X_k^n be the span of the diagrams with rank less than or equal to k, we have a tower of P_n-invariant subspaces $X_0^n \subseteq X_1^n \subseteq \cdots \subseteq X_n^n$. These X_k^n are not irreducible and they do not decompose the space $\mathbf{C}P_n$ into

P_n-invariant subspaces. To accomplish such a decomposition, we first need to change to a different but closely related basis.

If $d_1, d_2 \in P_n$ we say that $d_1 \subseteq d_2$ if the edges of the diagram d_1 are a subset of the edges of the diagram d_2. If $d_1 \subseteq d_2$, we let $|d_2 \setminus d_1| = \text{rank}(d_2) - \text{rank}(d_1)$, or the number of edges in d_2 minus the number of edges in d_1. Now define

$$x_d = \sum_{d' \subseteq d} (-1)^{|d \setminus d'|} d'. \tag{3.1}$$

For example, if $d =$  then

$x_d =$ 

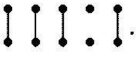
Under any ordering on the planar rook diagrams that extends the partial ordering given by rank (i.e., a comes before b if $\text{rank}(a) < \text{rank}(b)$), the transition matrix from the basis $\{d \mid d \in P_n\}$ to the set $\{x_d \mid d \in P_n\}$ is upper triangular with 1s on the diagonal. Thus $\{x_d \mid d \in P_n\}$ is also a basis for $\mathbf{C}P_n$.

This change of basis was necessary in order to obtain the conclusion "$dx_a = 0$ otherwise" in (3.2) below. Notice how close this statement is to (2.2). We are now realizing the subset action inside of $\mathbf{C}P_n$.

PROPOSITION 3.1. *Let $a, d \in P_n$. Then*

$$dx_a = \begin{cases} x_{da} & \text{if } \tau(a) \subseteq \beta(d), \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Proof. If $\tau(a) \subseteq \beta(d)$ then multiplication on the left (or top) by d on any $d' \subseteq a$ simply rearranges the top vertices of d' to their corresponding position in da, and the result follows by the definition of x_{da}.

If $\tau(a) \not\subseteq \beta(d)$, then let $i \in \tau(a)$ such that $i \notin \beta(d)$. Consider the diagram p_i, which is the same as the identity element id except that the edge connecting the i^{th} vertex in each row is removed. For example, in P_5, we have $p_4 =$ . Then $dp_i = d$, since $i \notin \beta(d)$, and

$$p_i x_a = \sum_{d' \subseteq a} (-1)^{|a \setminus d'|} p_i d' = \sum_{\substack{d' \subseteq a \\ i \in \tau(d')}} (-1)^{|a \setminus d'|} p_i d' + \sum_{\substack{d' \subseteq a \\ i \notin \tau(d')}} (-1)^{|a \setminus d'|} p_i d'.$$

Now, if $i \notin \tau(d')$ then $p_i d' = d'$, and if $i \in \tau(d')$ then $p_i d'$ is the same diagram as d' except with the edge connected to the i^{th} vertex (in the top row of d') removed. There is a bijection between $\{d' \subseteq d \mid i \in \tau(d')\}$ and

$\{d' \subseteq d \mid i \notin \tau(d')\}$ given by removing the edge connected to the i^{th} vertex (equivalently, multiplying by p_i). This bijection changes the sign $(-1)^{|a \setminus d'|}$, so the two summations displayed above cancel one another giving $p_i x_a = 0$. Thus $dx_a = dp_i x_a = 0$.

For diagrams a and d we have $\text{rank}(a) = \text{rank}(da)$ if and only if $\tau(a) \subseteq \beta(d)$. Thus from (3.2), we see that $dx_a = 0$ unless $\text{rank}(a) = \text{rank}(da)$. It follows that the subspace

$$W^{n,k} = \mathbf{C}\text{-span}\left\{x_a \mid \text{rank}(a) = k\right\}$$

is a P_n-invariant subspace of $\mathbf{C}P_n$. Notice that the action of d on x_a in (3.2) does not change the bottom row $\beta(a)$. That is, $\beta(a) = \beta(da)$ when $\tau(a) \subseteq \beta(d)$. Thus, if we let

$$W_T^{n,k} = \mathbf{C}\text{-span}\left\{x_a \mid \text{rank}(a) = k,\ \beta(a) = T\right\},$$

then for each such T, we have that $W_T^{n,k}$ is a P_n-invariant subspace of $W^{n,k}$ and for any subset U with $|U| = |T| = k$, we have

$$\text{(3.3)} \qquad W_T^{n,k} \cong W_U^{n,k} \cong V_k^n \qquad \text{as } P_n\text{-invariant subspaces of } \mathbf{C}P_n\,.$$

The last isomorphism comes from the fact that the action of $d \in P_n$ on x_a in (3.2) is the same as the action of d on $\mathbf{v}_S$ in (2.2), where $S = \tau(a)$.

For subsets S, T of $\{1, \ldots, n\}$ with $|S| = |T|$, we define

$$x_{S,T} = x_d\,, \qquad \begin{array}{l}\text{where } d \text{ is the unique planar rook diagram}\\ \text{with } \tau(d) = S \text{ and } \beta(d) = T.\end{array}$$

For example, the diagram in the example after equation (3.1) is denoted $x_{\{2,3,4\},\{1,2,4\}}$. In this notation, the isomorphism in (3.3) is given explicitly on basis elements by $x_{S,T} \leftrightarrow x_{S,U} \leftrightarrow \mathbf{v}_S$.

Inside of $W^{n,k}$ we have found $\binom{n}{k}$ copies of the P_n-invariant subspaces $W_T^{n,k}$ (one for each choice of T), and each of these is isomorphic to V_k^n. Thus we have explicitly constructed the decompositions in part (a) of the following theorem. Part (b) follows from the fact that every irreducible module must appear as a component in the regular representation.

THEOREM 3.2.

(a) *The decomposition of* $\mathbf{C}P_n$ *into* P_n*-invariant subspaces is given by*

$$\mathbf{C}P_n = \bigoplus_{k=0}^{n} W^{n,k} = \bigoplus_{k=0}^{n} \bigoplus_{|T|=k} W_T^{n,k} \cong \bigoplus_{k=0}^{n} \binom{n}{k} V_k^n\,.$$

(b) *The set* $\{V_k^n \mid 0 \le k \le n\}$ *is a complete set of irreducible* $\mathbf{C}P_n$*-modules.*

In the previous theorem, the modules $W^{n,k}$ are the "isotypic components" which consist of a sum of all of the irreducible subspaces that are isomorphic to V_k^n. Notice also that the dimension and the multiplicity of V_k^n in $\mathbf{C}P_n$ is $\binom{n}{k}$. Finally, since the irreducible modules that appear in $\mathbf{C}P_n$ are exactly the V_k^n, we know that these form a complete set of irreducible modules as claimed in Theorem 2.1.

PROPOSITION 3.3. *For subsets S, T, U, V of $\{1, \ldots, n\}$ with $|S| = |T|$ and $|U| = |V|$ we have*

$$x_{S,T} x_{U,V} = \begin{cases} x_{S,V} & \text{if } T = U, \\ 0 & \text{if } T \neq U. \end{cases}$$

Proof. If $T \neq U$, then there exists $i \in T$ with $i \notin U$ or $i \in U$ with $i \notin T$. The same argument as in Proposition 3.1 shows that $x_{S,T} x_{U,V} = 0$. If $T = U$ then let $a, b \in P_n$ be the diagrams such that $x_{S,T} = x_a$ and $x_{U,V} = x_b$. By (3.2) we have that $a x_b = x_{ab}$ and $a' x_b = 0$ for every $a' \subseteq a$ with $a' \neq a$. So by the definition of x_a we see that $x_a x_b = x_{ab}$.

Proposition 3.3 tells us that the $x_{S,T}$ behave just like the matrix units $E_{i,j}$ which have a 1 in row i and column j and 0 everywhere else. This correspondence reveals the structure of the planar rook algebra, given in the following corollary.

COROLLARY 3.4. $\mathbf{C}P_n \cong \bigoplus_{k=0}^{n} \mathrm{Mat}\left(\binom{n}{k}, \binom{n}{k}\right)$, *where* $\mathrm{Mat}(m, m)$ *is the algebra of all* $m \times m$ *complex matrices.*

4. THE BRATTELI DIAGRAM IS PASCAL'S TRIANGLE

The binomial coefficients have appeared in a very natural way throughout the representation theory of P_n. For example, by comparing dimensions on both sides of the decomposition of V^n in Theorem 2.1, we get

$$2^n = \sum_{k=0}^{n} \binom{n}{k}. \tag{4.1}$$

By computing dimensions on both sides of the decomposition of $\mathbf{C}P_n$ in Theorem 3.2, we have

(4.2) $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

By comparing dimensions of the decomposition of V_k^n into irreducible modules for P_{n-1} in (2.4), we get

(4.3) $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

These are well-known binomial identities. For a beautiful discussion of combinatorial proofs of binomial identities such as these, see [BQ]. We can view the work in this article as representation-theoretic interpretations of these binomial identities.

Pascal's triangle itself arises naturally through the representation theory of P_n. The *Bratteli diagram* (see for example [GHJ]) for the tower $P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots$ is the infinite rooted graph whose vertices are the irreducible representations V_k^n and whose edges correspond to the restriction rules from P_n to P_{n-1}. Specifically, there is an edge from V_k^n to V_ℓ^{n-1} if and only if V_ℓ^{n-1} appears as a summand when V_k^n is viewed as a module for P_{n-1}. According to the rules in (2.4) we get the Bratteli diagram shown in Figure 1. The dimensions of these modules give Pascal's triangle.

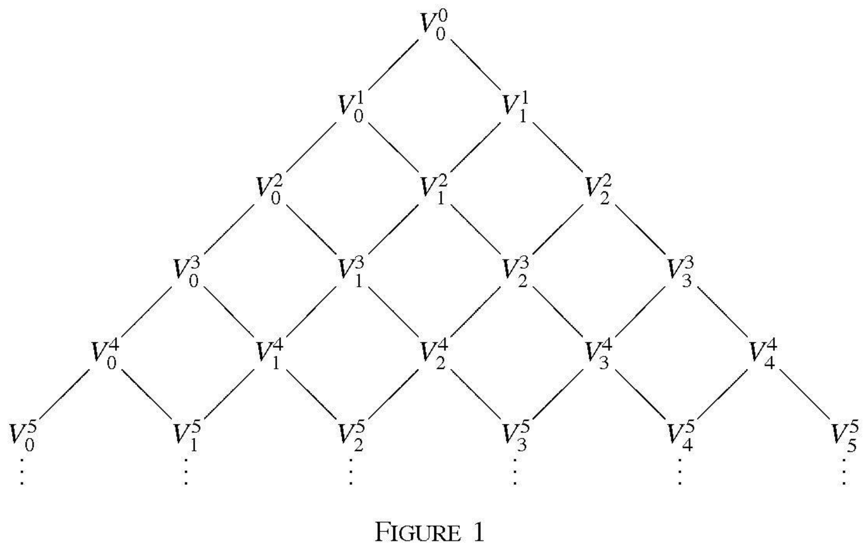


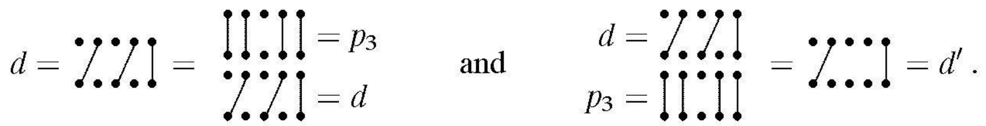
FIGURE 1

The Bratteli diagram for the tower of containments of planar rook monoids $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$; counting dimensions gives Pascal's triangle

5. The character table is Pascal's triangle

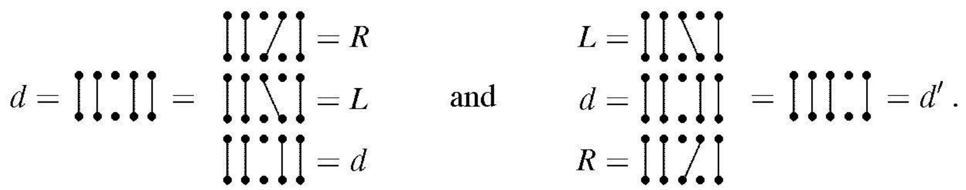
For each irreducible representation V_k^n, its character χ_k^n is the $\mathbf{C}$-valued function that gives the trace of the $d \in P_n$ as a linear transformation on V_k^n. In this section, we show that the table of character values for P_n is given by the first n rows of Pascal's triangle. The characters are linearly independent functions (over $\mathbf{C}$), so from the character of any finite representation you can identify the isomorphism class of the representation.

For a planar rook diagram $d \in P_n$ we say that an edge in d is *vertical* if it connects the i^{th} vertex in the top row to the i^{th} vertex in the bottom row for some $1 \leq i \leq n$. We also say that a vertex that is not incident to an edge in d is an *isolated* vertex. Suppose that d is a diagram such that its i^{th} vertex in the top row is isolated. As in Section 3, let p_i be the diagram obtained from the identity id by deleting the i^{th} edge. Then $p_i d = d$ and $d' = dp_i$ has the property that the i^{th} vertex in both the top and bottom rows is isolated. For example,

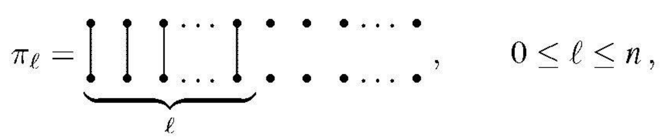


Now, the point of all this is that for any matrix trace Tr we have $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$, so in this case $\mathrm{Tr}(d) = \mathrm{Tr}(p_3 d) = \mathrm{Tr}(dp_3) = \mathrm{Tr}(d')$. So by iterating this process, we see that for any matrix trace we have $\mathrm{Tr}(d) = \mathrm{Tr}(d')$, where d' is the diagram d with all of its non-vertical edges removed.

Furthermore, we can use the following trick to move all of the vertical edges to the left of the diagram. In the pictures below, we see that $d = RLd$ and $d' = LdR$ has the vertical edge moved one position to the left. Furthermore $\mathrm{Tr}(d) = \mathrm{Tr}(RLd) = \mathrm{Tr}(LdR) = \mathrm{Tr}(d')$.



By iterating this process, we see that the character value of d will be the same as the character value on one of the diagrams,



where ℓ is the number of vertical edges in d. The set of diagrams $\{\pi_\ell \mid 0 \leq i \leq n\}$ is analogous to a set of conjugacy class representatives in a group in the sense that any trace is completely determined by its value on one of these diagrams.

In the next theorem, we show that the trace of $d \in P_n$ on the representation V_k^n is a binomial coefficient. The proof is to count subsets fixed by d.

THEOREM 5.1. *For $0 \leq k \leq n$ and $d \in P_n$, the value of the irreducible character is given by*

$$\chi_k^n(d) = \begin{cases} \binom{\ell}{k} & \text{if } k \leq \ell, \\ 0 & \text{if } k > \ell, \end{cases}$$

where ℓ is the number of vertical edges in d.

Proof. The elements $d \in P_n$ permute (or send to 0) the vectors $\mathbf{v}_S$ which span V_k^n. The $\mathbf{v}_S$-$\mathbf{v}_S$ entry of the matrix of d will be 1 if $d(S) = S$ and 0 otherwise. This tells us that the character $\chi_k^n(d)$ gives the number of fixed points of d. By our discussion above it suffices to let $d = \pi_\ell$. Now, π_ℓ will fix S if and only if $S \subseteq \{1, \ldots, \ell\}$. And for $\mathbf{v}_S$ to be a basis element of V_k^n, S must be a subset of $\{1, \ldots, n\}$ with cardinality k. Thus, the trace is the number of subsets of $\{1, \ldots, \ell\}$ of cardinality k, or $\binom{\ell}{k}$, as desired.

6. FURTHER THOUGHTS

Here are a few more observations that make fun exercises.

1. Let $\psi^n(d)$ denote the trace of $d \in P_n$ on the regular representation $\mathbf{C}P_n$. Then by a counting argument $\psi^n(d) = \binom{n+\ell}{\ell}$, where ℓ is the number of vertical edges in d. Using the decomposition of the regular representation into irreducibles we arrive at the binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} \chi_k^n(d) = \sum_{k=0}^{\ell} \binom{n}{k}\binom{\ell}{k} = \binom{n+\ell}{\ell}.$$

2. In the x_d basis, the irreducible character values are

$$\chi_k^n(x_d) = \begin{cases} 1 & \text{if } d \text{ has exactly } k \text{ vertical edges and no other edges}, \\ 0 & \text{otherwise}. \end{cases}$$

3. The center of $\mathbf{C}P_n$ has a basis given by the elements

$$z_\ell = \sum_a x_a\,, \qquad 0 \le \ell \le n\,,$$

where the sum is over all diagrams a with exactly ℓ vertical edges and no other edges.

4. The character ring A_n of $\mathbf{C}P_n$ is the algebra with basis given by the $n+1$ functions $\chi_k^n(d) = \binom{\ell}{k}$ (where d has ℓ vertical edges), for $0 \le k \le n$. For A_n these functions have domain $0 \le \ell \le n$. Since the irreducible characters form a basis of this algebra, we can re-express the product of any two characters in terms of the basis. The structure constants for the character ring come from the multinomial coefficients in the following polynomial identity (see [Ri, § 1.4]):

$$\binom{x}{i}\binom{x}{j} = \sum_{k=\max(i,j)}^{i+j} \binom{k}{i+j-k,\ k-i,\ k-j}\binom{x}{k}.$$

In the application to A_n, the upper limit in the sum is taken to be $k = \max(i+j, n)$ since $\binom{x}{k} = 0$ for $k > n \ge x = \ell$. This gives the corresponding decomposition of the tensor product (see [CR, § 11] for an explanation of tensor products) $V_i^n \otimes V_j^n = \bigoplus_k \binom{k}{i+j-k, k-i, k-j} V_k^n$. This illustrates that even the multiplicative structure of the binomial coefficients is captured in the representation theory of P_n.

5. Some of the first examples of diagram algebras are the group algebra of the symmetric group, with a basis of permutation diagrams, and the Artin braid group, with a basis of braid diagrams. The Brauer algebra was defined in the 1930s, and its planar version, the Temperley-Lieb algebra, is important in statistical mechanics. The papers [CFS], [HR], [GL], and the references therein, give definitions and examples of these and other diagram algebras. The planar rook algebra was constructed to be a diagram algebra whose Bratteli diagram is Pascal's triangle. A good project is to find algebras whose Bratteli diagrams match the lattice of other recursively defined integers such as the Stirling numbers or the trinomial numbers.

6. The planar rook algebra also has representation theoretic importance as the centralizer algebra of the general linear group $G = GL_1(\mathbf{C}) = \mathbf{C}\setminus\{0\}$. Let $V = V_0 \oplus V_1$ such that V_0 and V_1 are the 1-dimensional G-modules where $z(v_0 + v_1) = v_0 + zv_1$ for $v_i \in V_i$ and $z \in G$. Then $\mathbf{C}P_n \cong \mathrm{End}_G(V^{\otimes n})$, which is the algebra of all endomorphisms of the tensor product $V^{\otimes n}$ that commute with G (i.e., $\mathbf{C}P_n$ is the centralizer of G on $V^{\otimes n}$). This is analogous to classical Schur-Weyl duality, where the group algebra of the symmetric group

$\mathbf{C}S_n$ is the centralizer of $GL_k(\mathbf{C})$ on $W^{\otimes n}$, for $k \geq n$, where $W = \mathbf{C}^k$ is the representation of $GL_k(\mathbf{C})$ by matrix multiplication on column vectors. If we replace a simple tensor with the subset indexed by the binary string in its subscripts — for example $v_1 \otimes v_0 \otimes v_1 \otimes v_1 \otimes v_0 \Leftrightarrow 10110 \Leftrightarrow \{1,3,4\}$ — then the action on simple tensors is the same as the action of P_n on subsets in Section 2.

ACKNOWLEDGEMENTS. We are grateful to Arun Ram who, at the 2004 AMS sectional meeting in Evanston, helped the second author discover the planar rook monoid as an example of an algebra whose Bratteli diagram is Pascal's triangle. We also thank Lex Renner for other important suggestions at that same meeting.

REFERENCES

[BQ] BENJAMIN, A. T. and J. J. QUINN. *Proofs that Really Count: the Art of Combinatorial Proof.* The Dolciani Mathematical Expositions *27*. Mathematical Association of America, Washington DC, 2003.

[CFS] CARTER, J. S., D. FLATH and M. SAITO. *The Classical and Quantum 6j-Symbols.* Mathematical Notes *43*. Princeton University Press, Princeton, 1995.

[CR] CURTIS, C. W. and I. REINER. *Representation Theory of Finite Groups and Associative Algebras.* Pure and Applied Mathematics *11*. Wiley, New York, 1962.

[GHJ] GOODMAN, F. M., P. DE LA HARPE and V. F. R. JONES. *Coxeter Graphs and Towers of Algebras.* Mathematical Sciences Research Institute Publications *14*. Springer-Verlag, New York, 1989.

[GL] GRAHAM, J. J. and G. I. LEHRER. Cellular algebras and diagram algebras in representation theory. In: *Representation Theory of Algebraic Groups and Quantum Groups,* 141–173. Advanced Studies in Pure Mathematics *40*. Math. Soc. Japan, Tokyo, 2004.

[Gr] GROOD, C. A Specht module analog for the rook monoid. *Electron. J. Combin. 9* (2002), R2.

[Ha] HALVERSON, T. Representations of the q-rook monoid. *J. Algebra 273* (2004), 227–251.

[HL] HALVERSON, T. and T. LEWANDOWSKI. RSK insertion for set partitions and diagram algebras. *Electron. J. Combin. 11* (2004/06), R24.

[HR] HALVERSON, T. and A. RAM. Partition algebras. *European J. Combin. 26* (2005), 869–921.

[Jo] JONES, V. F. R. Planar algebras, I. Preprint arXiv:math.QA/9909027 (1999).

[Re] RENNER, L. E. *Linear Algebraic Monoids.* Encyclopaedia of Mathematical Sciences *134*. Invariant Theory and Algebraic Transformation Groups, V. Springer-Verlag, Berlin, 2005.

[Ri] RIORDAN, J. *Combinatorial Identities*. John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[OEIS] SLOANE, N. J. A. The On-Line Encyclopedia of Integer Sequences (2006), http://www.research.att.com/~njas/sequences/A002720.

[So] SOLOMON, L. Representations of the rook monoid. *J. Algebra 256* (2002), 309–342.

(Reçu le 22 août 2007; version révisée reçue le 18 février 2008)

Daniel E. Flath
Tom Halverson

Department of Mathematics and Computer Science
Macalester College
1600 Grand Avenue
Saint Paul, MN 55105
U. S. A.
e-mail: flath@macalester.edu, halverson@macalester.edu

Kathryn Herbig
e-mail: kherbig@gmail.com