Positivity of direct image sheaves : a geometric point of view

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POSITIVITY OF DIRECT IMAGE SHEAVES
A GEOMETRIC POINT OF VIEW

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ABSTRACT. These are extended notes of talks I gave at the workshop "Rencontre positivité" in Rennes. The aim of these talks was to illustrate the interaction between the geometry of a fibration and the positivity of direct image sheaves. We give numerous examples and an introduction to the Kollár-Viehweg method for proving direct image theorems. As an application, we study the geometry of projective manifolds with nef anticanonical bundle.

1. INTRODUCTION

1.A MOTIVATION

Given a *fibration* $\varphi\colon X \to Y$, i.e. a morphism with connected fibres between projective manifolds, it is a natural and fundamental problem to try to relate positivity properties of the total space X, the base Y and the general fibre F. An important problem in this context is Iitaka's $C_{n,m}$-conjecture that predicts the subadditivity of the Kodaira dimension. More precisely, if we denote by $\kappa(.)$ the Kodaira dimension of a projective manifold, the $C_{n,m}$-conjecture states that

$$\kappa(X) \geq \kappa(Y) + \kappa(F).$$

It turns out that one of the main issues in the study of this conjecture is a discussion of the positivity of the direct image sheaves $\varphi_*(\omega_{X/Y}^{\otimes m})$, where m is a sufficiently large integer. Building up on the landmark papers due to Kawamata [Kaw81, Kaw82], this question was treated by Viehweg in a series of great articles [Vie82, Vie83].

THEOREM 1.1 ([Vie82]). *Let $\varphi\colon X \to Y$ be a fibration between projective manifolds. Then for all $m \in \mathbf{N}$, the direct image sheaf $\varphi_*(\omega_{X/Y}^{\otimes m})$ is weakly positive.*

If the general fibre F is of general type and the family has maximal variation, Kollár [Kol87] proved that the direct image sheaves $\varphi_*(\omega_{X/Y}^{\otimes m})$ are even big. By the work of Viehweg this settles the $C_{n,m}$-conjecture for fibre spaces whose general fibre is of general type.

For further applications, it is important to produce more general versions of this theorem. Given a fibration $\varphi\colon X \to Y$ and a line bundle L on X, one can ask for the positivity properties of the direct image $\varphi_*(L \otimes \omega_{X/Y})$. A moment of reflection will convince the reader that it is pointless to ask such a question for a line bundle L that is not itself positive in some sense (e.g. ample, nef, weakly positive, . . .). In this situation vanishing theorems for adjoint line bundles $L \otimes \omega_X$ replace the Hodge-theoretic approach that was the starting point of Kawamata's results. Pushing the situation one step further, we may consider fibrations between schemes that are not necessarily smooth, replace line bundles by $\mathbf{Q}$-divisors and add some ingredients from the theory of multiplier ideal sheaves. All these generalisations have been studied in the last thirty years, with applications ranging from the construction of moduli spaces [Vie95] to the proof of Shokurov's rational connectedness conjecture by Hacon and McKernan [HM07].

The goal of these lecture notes is *not* to show a certain theorem nor to give an overview of the incredibly many articles on this subject. Instead of this, our aim is to show how the positivity of a direct image sheaf and the geometry of the morphism $\varphi\colon X \to Y$ interact. We proceed in three technically independent but thematically connected steps:

- We start in Section 2 with a long series of examples that illustrates what type of direct image result one can hope to prove.
- Section 3 is devoted to explaining the basic techniques in the proof of direct image theorems. We will not aim for maximal generality, but insist on the link between the positivity of direct image sheaves and certain vanishing/extension theorems.
- In Section 4 we go back to a concrete geometric problem: the study of projective manifolds with nef anticanonical bundle. We indicate how the results proven in Section 3 give information on the structure of these varieties. Furthermore we use the direct image point of view to construct a series of examples related to the problem of boundedness of their functor of deformations (Question 4.4).

A LINGUISTIC CONVENTION. While in analytic geometry a line bundle L is defined to be positive if it admits a smooth hermitian metric with strictly positive curvature, we will use the expression "L positive" as a catch-all term whenever we do not want to make a precise statement. By contrast, "weakly positive" will be defined precisely (cf. Definition 3.4).

In the same spirit, we will call "direct image sheaf" a sheaf of the form $\varphi_*(L \otimes \omega_{X/Y})$ where $\varphi\colon X \to Y$ is a projective morphism, L a line bundle on X, and $\omega_{X/Y}$ some "relative dualising sheaf" (cf. Definition 5.22).

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1.B NOTATION

We work over the complex field $\mathbf{C}$. If not mentioned otherwise, all the topological notions refer to the Zariski topology. In particular, sheaves are defined with respect to the Zariski topology. Schemes will always be supposed to be quasi-projective over $\mathbf{C}$. A *variety* is an integral scheme of finite type over $\mathbf{C}$. A *divisor* on a normal variety is a Weil divisor. We will identify locally free sheaves and vector bundles. By a *point* on a variety we will always mean a closed point, the *fibres* of a morphism are also the fibres over a closed point of the base.

If X is a projective scheme, we denote by ω_X the canonical sheaf and by ω_X^{-1} its dual. If X is normal, we denote by K_X (resp. $-K_X$) the *canonical* (resp. *anticanonical*) divisor.

Let X be a scheme, and let $\mathcal{S}$ be a coherent sheaf on X. If $X^* \subset X$ is an open subscheme that is endowed with the canonical subscheme structure, we denote by $\mathcal{S}|_{X^*}$ the restriction to X^*. If $Z \subset X$ is a closed subscheme we denote by

$$\mathcal{S} \otimes \mathcal{O}_Z := \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$$

the restriction to Z.

Let $\varphi\colon X \to Y$ be a morphism of schemes, and let $y \in Y$ be a closed point. Then we denote by $X_y := \varphi^{-1}(y) := X \times_Y y$ the (scheme-theoretical) fibre. More generally if $Z \subset Y$ is a subscheme, we denote the fibre product $X \times_Y Z$ by X_Z.

We will say that a certain property on X *holds over a general point of* Y if there exists a non-empty Zariski open subset $Y^* \subset Y$ such that the property holds on $\varphi^{-1}(Y^*)$.

If V is a locally free sheaf, we will abbreviate $\mathrm{Sym}^d V$ by $S^d V$.

DEFINITION 1.2. A projective morphism $\varphi\colon X \to Y$ from a scheme X onto a normal variety Y is a *fibration* if it is surjective and the general fibre is irreducible.

DEFINITION 1.3. A fibration $\varphi\colon X \to Y$ is *generically smooth* if there exists a non-empty Zariski open subset $Y^* \subset Y$ such that for all $y \in Y^*$, the fibre X_y is a smooth variety.

In this case, the *φ-smooth locus* is the maximal open subset $Y^* \subset Y$ such that $\varphi^{-1}(Y^*)$ is smooth and such that for every $y \in Y^*$ the fibre X_y is smooth and of dimension $\dim X - \dim Y$. The *φ-singular locus* Δ is the complement $Y \setminus Y^*$.

REMARK. By generic smoothness it is clear that a fibration is generically smooth if and only if the singular locus of X does not dominate Y.

DEFINITION 1.4. A fibration has *generically reduced fibres in codimension* 1, if there exists a Zariski open subset $Y^* \subset Y$ such that

$$\mathrm{codim}_Y(Y \setminus Y^*) \geq 2$$

and for all $y \in Y^*$, the fibre X_y is generically reduced.

Since we will use this many times, let us recall that if $V \to Y$ is a vector bundle of rank r over a complex manifold, the *canonical bundle of its projectivisation*[1]) $\varphi\colon X := \mathbf{P}(V) \to Y$ is

$$\omega_X \simeq \varphi^*(\omega_Y \otimes \det V) \otimes \mathcal{O}_{\mathbf{P}(V)}(r), \tag{1.5}$$

where $\mathcal{O}_{\mathbf{P}(V)}(1)$ is the tautological bundle. From time to time, we will abbreviate $\mathcal{O}_{\mathbf{P}(V)}(1)$ by $\mathcal{O}_{\mathbf{P}}(1)$.

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[1]) We will always take projectivisation in the sense of Grothendieck, i.e. consider the space of hyperplanes.

2. Examples

The following direct image theorem, due to Mourougane, is an example of an ideal situation:

THEOREM 2.1 ([Mou97], Thm. 1). *Let $\varphi\colon X \to Y$ be a smooth fibration between projective manifolds, and let L be a nef and φ-big line bundle on X. Then $\varphi_*(L \otimes \omega_{X/Y})$ is locally free and nef. If L is ample, then $\varphi_*(L \otimes \omega_{X/Y})$ is ample or zero.*

The proof of this result follows a strategy developed in Kollár's landmark papers [Kol86a, Kol86b] and will be discussed in detail in Section 3.

We will be interested in situations where the fibration is not smooth and establish in Section 3 only generic results, i.e. results that are of the type: if $C \subset Y$ is a sufficiently general curve, then $\varphi_*(L \otimes \omega_{X/Y}) \otimes \mathcal{O}_C$ is nef. Such a result is of course less satisfactory as long as we don't test the optimality of the statement.

Doing these tests is the object of this section. In particular we try to indicate which properties a fibre X_y should *not* have if we want $\varphi_*(L \otimes \omega_{X/Y})$ to be positive in some neighbourhood of y. We will see in Corollary 2.6 that vanishing theorems play a crucial role in showing the positivity of direct image sheaves, so the idea behind these examples is to construct projective schemes on which these theorems do not hold.

Note that these examples are also relevant if we are only interested in morphisms $\varphi\colon X \to Y$ between projective manifolds X and Y. If $C \subset Y$ is a curve contained in the φ-singular locus, the scheme $X_C := X \times_Y C$ can be very singular. Furthermore if $\varphi_*(L \otimes \omega_{X/Y})$ commutes with arbitrary base-change (e.g. if there are no higher direct images), we have

$$\varphi_*(L \otimes \omega_{X/Y}) \otimes \mathcal{O}_C \simeq \varphi|_{X_C}(L \otimes \mathcal{O}_{X_C} \otimes \omega_{X_C/C}).$$

Before we go into the more technical statements, let us consider the following example which shows that even for a conic bundle, the positivity properties of direct image sheaves are often worse than what we might naively expect.

EXAMPLE 2.2 (Wiśniewski [Wiś91], p. 156). Let $Y := \mathbf{P}(V)$ be the projectivisation of the vector bundle

$$V := \mathcal{O}_{\mathbf{P}^3}^{\oplus 2}(2) \oplus \mathcal{O}_{\mathbf{P}^3}.$$

Set $p\colon Y \to \mathbf{P}^3$ for the projection map, and denote by $\mathcal{O}_{\mathbf{P}(V)}(1)$ the tautological bundle. Clearly $\mathcal{O}_{\mathbf{P}(V)}(1)$ is globally generated, but not ample: it is trivial on the threefold $Z \simeq \mathbf{P}^3 \subset Y$ corresponding to the quotient bundle $\mathcal{O}_{\mathbf{P}^3}^{\oplus 2}(2) \oplus \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_{\mathbf{P}^3}$. Furthermore $\omega_Y^{-1} \simeq \mathcal{O}_{\mathbf{P}(V)}(3)$, so ω_Y^{-1} is nef but not ample. Set

$$E := \mathcal{O}_{\mathbf{P}(V)}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}(V)}(1) \otimes p^*\mathcal{O}_{\mathbf{P}^3}(1),$$

and denote by $\varphi\colon X := \mathbf{P}(E) \to Y$ the projection map. The base locus of the linear system $|\mathcal{O}_{\mathbf{P}(E)}(2) \otimes \varphi^* p^* \mathcal{O}_{\mathbf{P}^3}(-2)|$ is not empty, but Wiśniewski shows by an explicit computation that it contains a smooth irreducible divisor X such that the induced morphism $\varphi|_X\colon X \to Y$ is a conic bundle. By the adjunction formula, one has

$$\omega_X^{-1} \simeq \mathcal{O}_{\mathbf{P}(E)}(1) \otimes \varphi^* p^* \mathcal{O}_{\mathbf{P}^3}(1) \otimes \mathcal{O}_X .$$

Since $E \otimes p^*\mathcal{O}_{\mathbf{P}^3}(1)$ is an ample vector bundle, this shows that ω_X^{-1} is ample. Nevertheless, the direct image sheaf

$$\omega_Y^{-1} \simeq \varphi_*(\omega_X^{-1} \otimes \omega_{X/Y})$$

is not ample.

We can push the example one step further: set $L := \mathcal{O}_{\mathbf{P}(E)}(1) \otimes \mathcal{O}_X$, then L is nef and relatively ample, but the direct image sheaf

$$\varphi_*(L \otimes \omega_{X/Y}) \simeq \omega_Y^{-1} \otimes p^*\mathcal{O}_{\mathbf{P}^3}(-1) \simeq \mathcal{O}_{\mathbf{P}(V)}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^3}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbf{P}^3}(-1))}(1)$$

is effective, but not nef: we have

$$\omega_Y^{-1} \otimes p^*\mathcal{O}_{\mathbf{P}^3}(-1) \otimes \mathcal{O}_Z \simeq \mathcal{O}_{\mathbf{P}^3}(-1).$$

An easy computation shows that the base locus of the linear system $|\omega_Y^{-1} \otimes p^*\mathcal{O}_{\mathbf{P}^3}(-1)|$ is exactly Z, so $\omega_Y^{-1} \otimes p^*\mathcal{O}_{\mathbf{P}^3}(-1)$ is nef on every curve that is not contained in Z.

While we will be able to explain very soon why the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ is negative on Z (cf. Section 2.B), it will take much more effort to see the positivity of the direct image sheaf on $Y \setminus Z$ as part of a larger framework (cf. Section 3.G).

2.A Direct images and vanishing theorems

We will now study a situation where the positivity of $\varphi_*(L \otimes \omega_{X/Y})$ is equivalent to an injectivity result for cohomology groups. This will enable us to give counterexamples to the positivity of direct image sheaves.

PROPOSITION 2.3. *Let $\varphi \colon X \to \mathbf{P}^1$ be a flat fibration, where X is a projective Cohen-Macaulay scheme (not necessarily reduced, irreducible, nor normal or with some restrictions on the singularities). Let L be a line bundle such that*

$$R^i\varphi_*(L \otimes \omega_X) = 0 \qquad \forall\ i > 0\,.$$

Then $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is locally free. It is nef if and only if

$$h^1(X, L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) = h^1(X, L \otimes \omega_{X/\mathbf{P}^1})\,. \tag{2.4}$$

Suppose $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is not zero. It is ample if and only if

$$h^1(X, L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-2)) = h^1(X, L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1))\,. \tag{2.5}$$

REMARK. If L is an ample line bundle on X, Serre vanishing implies that for k sufficiently large, the higher direct images $R^i\varphi_*(L^{\otimes k} \otimes \omega_X)$ and both cohomology groups in (2.5) vanish, so the direct image sheaf $\varphi_*(L^{\otimes k} \otimes \omega_{X/\mathbf{P}^1})$ is ample. We will see that for $k = 1$ the situation is more delicate.

COROLLARY 2.6. *Let $\varphi \colon X \to \mathbf{P}^1$ be a flat fibration, where X is a projective Cohen-Macaulay scheme such that the Kodaira vanishing theorem holds*[2]) *on X. If L is a nef and φ-ample line bundle, $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is nef. If L is ample, $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is ample or zero.*

Proof. Since L is relatively ample, we have

$$R^i\varphi_*(L \otimes \omega_X) = 0 \qquad \forall\ i > 0$$

by relative Kodaira vanishing. Since L is nef, the line bundles

$$L \otimes \varphi^*(\omega_{\mathbf{P}^1}^* \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) \quad \text{and} \quad L \otimes \varphi^*\omega_{\mathbf{P}^1}^*$$

are ample. Therefore by Kodaira vanishing both cohomology groups in (2.4) are zero, so $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is nef by Proposition 2.3.

If L is ample, the bundles

$$L \otimes \varphi^*(\omega_{\mathbf{P}^1}^* \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) \quad \text{and} \quad L \otimes \varphi^*(\omega_{\mathbf{P}^1}^* \otimes \mathcal{O}_{\mathbf{P}^1}(-2)) \simeq L$$

are ample and an analogous argument yields the second statement.

[2]) We say that *Kodaira vanishing holds on a Cohen-Macaulay scheme X* if for every ample line bundle, the cohomology $H^i(X, L \otimes \omega_X)$ vanishes for all $i > 0$. In this case also the *relative Kodaira vanishing theorem* holds: if $\varphi \colon X \to Y$ is a morphism and L a relatively ample line bundle, a Leray spectral sequence argument shows that $R^i\varphi_*(L \otimes \omega_X) = 0$ for all $i > 0$.

Proof of Proposition 2.3. Since φ is flat, the vanishing of the higher direct images implies by [Har77, III, Thm. 12.11]

$$H^i(F, L \otimes \mathcal{O}_F \otimes \omega_F) = 0 \qquad \forall\ i > 0$$

for every fibre F, so

$$h^0(F, L \otimes \mathcal{O}_F \otimes \omega_F) = \chi(F, L \otimes \mathcal{O}_F \otimes \omega_F)$$

is constant by [Har77, III, Thm. 9.9]. It follows from Grauert's *Bildgarbensatz* that $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is a locally free sheaf. Furthermore, since φ is flat, we have $\mathcal{I}_{F_y} \simeq \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)$ for any fibre $F_y = \varphi^{-1}(y)$. So we have an exact sequence

$$0 \to L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \to L \otimes \omega_{X/\mathbf{P}^1} \to L \otimes \mathcal{O}_{F_y} \otimes \omega_{F_y} \to 0\,.$$

Since $h^1(F_y, L \otimes \mathcal{O}_{F_y} \otimes \omega_{F_y}) = 0$, the associated long exact cohomology sequence shows that

$$h^1(X, L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) = h^1(X, L \otimes \omega_{X/\mathbf{P}^1})$$

if and only if the morphism

$$(*) \qquad H^0(X, L \otimes \omega_{X/\mathbf{P}^1}) \to H^0(F_y, L \otimes \mathcal{O}_{F_y} \otimes \omega_{F_y})$$

is surjective. Since

$$H^0(X, L \otimes \omega_{X/\mathbf{P}^1}) \simeq H^0(\mathbf{P}^1, \varphi_*(L \otimes \omega_{X/\mathbf{P}^1}))$$

and, by cohomology and base change,

$$(\varphi_*(L \otimes \omega_{X/\mathbf{P}^1}))_y \simeq H^0(F_y, L \otimes \mathcal{O}_{F_y} \otimes \omega_{F_y})\,,$$

the morphism $(*)$ is surjective if and only if $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is generated in y by its global sections. A vector bundle on $\mathbf{P}^1$ is nef if and only if it is generated by its global sections.

This shows the first statement. For the second statement the proof is analogous: we conclude with the observation that the sheaf $\varphi_*(L \otimes \omega_{X/\mathbf{P}^1})$ is ample if and only if

$$\varphi_*(L \otimes \omega_{X/\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1}(-1) \simeq \varphi_*(L \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes \omega_{X/\mathbf{P}^1})$$

is globally generated.

2.B Multiple fibres

We will now give an explicit example of a scheme where Equalities (2.4) and (2.5) do not hold. After the elementary but tedious proof, we will see that Wiśniewski's Example 2.2 arises from a special case of the following proposition.

PROPOSITION 2.7. *Let* $P := \mathbf{P}(V)$ *be the projectivisation of the vector bundle*

$$V := \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(d),$$

where $d \geq 1$ *and denote by* $\varphi \colon P \to \mathbf{P}^1$ *the projection map. Let* X_{red} *be the divisor defined by the quotient vector bundle* $V \to \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$ *and denote by* X *the scheme corresponding to the divisor* $2X_{red}$. *Set*

$$L := \varphi^* \mathcal{O}_{\mathbf{P}^1}(a) \otimes \mathcal{O}_P(b),$$

where $b \geq 1$ *and* $a \geq 0$. *The direct image sheaf*

$$(\varphi|_X)_*(L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1})$$

is locally free. It is nef if and only if $a - d \geq 0$. *It is ample if and only if* $a - d > 0$.

REMARK. It is straightforward to see that the line bundle L is nef and big; if $a > 0$ it is even ample. The moral of the rather technical statement is as follows: the direct image sheaf $(\varphi|_X)_*(L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1})$ takes into account the scheme-theoretic structure of X. If the normal bundle of X_{red} absorbs the positivity of L, we cannot expect to have a positive direct image sheaf.

The proof will use the log-version of the Kodaira-Akizuki-Nakano vanishing theorem, due to Norimatsu.

THEOREM 2.8 ([Nor78]). *Let* X *be a projective manifold and* D *a simple normal crossings divisor on* X. *Let* L *be an ample line bundle over* X, *then*

$$H^i(X, L \otimes \omega_X(D)) = 0 \qquad \forall\, i > 0.$$

Proof of Proposition 2.7. The statement on ampleness easily reduces to the statement on nefness, since a vector bundle $\mathcal{F}$ on $\mathbf{P}^1$ is ample if and only if $\mathcal{F}(-1)$ is nef, and $a - d > 0$ if and only if $(a-1) - d \geq 0$. Let us check when the direct image sheaf is nef.

For every $y \in \mathbf{P}^1$, the fibre X_y is isomorphic to a non-reduced conic in $\mathbf{P}^2$. Using

$$L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y} \simeq \mathcal{O}_{\mathbf{P}^2}(b-1) \otimes \mathcal{O}_{X_y}$$

and the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(b-3) \to \mathcal{O}_{\mathbf{P}^2}(b-1) \to \mathcal{O}_{\mathbf{P}^2}(b-1) \otimes \mathcal{O}_{X_y} \to 0\,,$$

we see that $H^i(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y}) = 0$ for $i > 0$. Thus by [Har77, III, Ex. 11.8], the higher direct images $R^i(\varphi|_X)_*(L \otimes \mathcal{O}_X \otimes \omega_X)$ vanish and we are in the situation of Proposition 2.3. Thus by (2.4) the statement is equivalent to

$$(*) \qquad h^1(X, L \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) = h^1(X, L \otimes \omega_{X/\mathbf{P}^1})\,.$$

STEP 1: *reduction to X_{red}*. Consider the exact sequence

$$0 \to L \otimes \omega_{P/\mathbf{P}^1} \to L(X) \otimes \omega_{P/\mathbf{P}^1} \to L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1} \to 0\,.$$

Since L is nef and φ-ample, $L \otimes \varphi^*\omega^*_{\mathbf{P}^1}$ is ample, so Kodaira vanishing on P and the associated long exact sequence imply that

$$h^i(P, L(X) \otimes \omega_{P/\mathbf{P}^1}) = h^i(X, L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1}) \qquad \forall\ i > 0\,.$$

Consider now the exact sequence

$$0 \to L(X_{\mathbf{red}}) \otimes \omega_{P/\mathbf{P}^1} \to L(X) \otimes \omega_{P/\mathbf{P}^1} \to L(X_{\mathbf{red}}) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1} \to 0\,.$$

The divisor $X_{\mathbf{red}}$ is smooth, so Norimatsu vanishing on P (cf. Theorem 2.8) and the associated long exact sequence imply that

$$h^i(P, L(X) \otimes \omega_{P/\mathbf{P}^1}) = h^i(X_{\mathbf{red}}, L(X_{\mathbf{red}}) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) \qquad \forall\ i > 0\,.$$

As an intermediate result we obtain that

$$h^i(X_{\mathbf{red}}, L(X_{\mathbf{red}}) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) = h^i(X, L \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1}) \qquad \forall\ i > 0\,.$$

Since $L \otimes \varphi^*(\omega^*_{\mathbf{P}^1} \otimes \mathcal{O}_{\mathbf{P}^1}(-1))$ is still ample we can apply the same arguments to the exact sequences tensored with $\varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)$ to get

$$\begin{aligned} h^i(X_{\mathbf{red}}, L(X_{\mathbf{red}}) \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) \\ = h^i(X, L \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1}) \end{aligned}$$

for all $i > 0$. Since $\mathcal{O}_{X_{\mathrm{red}}}(X_{\mathbf{red}}) \simeq N_{X_{\mathrm{red}}/P}$, we see that $(*)$ is equivalent to

$$\begin{aligned} (**) \quad h^1(X_{\mathbf{red}}, L \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) \\ = h^1(X_{\mathbf{red}}, L \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1})\,. \end{aligned}$$

STEP 2: *the computation on X_{red}*. The divisor $X_{\mathbf{red}}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and we will identify the restriction of φ to $X_{\mathbf{red}}$ to the projection on the first factor. Using the usual notation for line bundles on $\mathbf{P}^1 \times \mathbf{P}^1$ we see that the restriction of the tautological bundle to $X_{\mathbf{red}}$ is $\mathcal{O}_{X_{\mathrm{red}}}(0,1)$, so we have

$$L \otimes \mathcal{O}_{X_{\mathrm{red}}} \simeq \varphi^* \mathcal{O}_{\mathbf{P}^1}(a) \otimes \mathcal{O}_P(b) \otimes \mathcal{O}_{X_{\mathrm{red}}} \simeq \mathcal{O}_{X_{\mathrm{red}}}(a,b)\,.$$

Since $\omega_P \simeq \varphi^* \mathcal{O}_{\mathbf{P}^1}(d-2) \otimes \mathcal{O}_P(-3)$ and $\omega_{X_{\mathrm{red}}} \simeq \mathcal{O}_{X_{\mathrm{red}}}(-2,-2)$, we have

$$N_{X_{\mathrm{red}}/P} \simeq \omega_P^* \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes \omega_{X_{\mathrm{red}}} \simeq \mathcal{O}_{X_{\mathrm{red}}}(-d,1)\,.$$

This implies that

$$\begin{aligned} L \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1} &\simeq \mathcal{O}_{X_{\mathrm{red}}}(a,b) \otimes \mathcal{O}_{X_{\mathrm{red}}}(-d,1) \otimes \mathcal{O}_{X_{\mathrm{red}}}(0,-2) \\ &\simeq \mathcal{O}_{X_{\mathrm{red}}}(a-d,b-1)\,. \end{aligned}$$

Since $b \geq 1$, this implies that

$$(\varphi|_{X_{\mathrm{red}}})_*(L \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) \simeq \mathcal{O}_{\mathbf{P}^1}(a-d)^{\oplus b},$$

furthermore there are no higher direct images, so

$$h^1(X_{\mathbf{red}}, L \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a-d)^{\oplus b})\,.$$

By the same argument we obtain

$$h^1(X_{\mathbf{red}}, L \otimes \varphi^* \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{O}_{X_{\mathrm{red}}} \otimes N_{X_{\mathrm{red}}/P} \otimes \omega_{X_{\mathrm{red}}/\mathbf{P}^1}) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a-d-1)^{\oplus b})\,.$$

Therefore $(**)$ is equivalent to

$$h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a-d-1)^{\oplus b}) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a-d)^{\oplus b})\,,$$

which holds if and only if $a-d \geq 0$.

Let us see how Wiśniewski's Example 2.2 fits in the statement of Proposition 2.7: using the notation of the example let l be a line in $Z \simeq \mathbf{P}^3$, then

$$E \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1)\,,$$

so in the notation of Proposition 2.7 we have $d = 1$. One easily computes that

$$h^0(\mathbf{P}(E \otimes \mathcal{O}_l), \mathcal{O}_{\mathbf{P}(E)}(2) \otimes \varphi^* p^* \mathcal{O}_{\mathbf{P}^3}(-2) \otimes \mathcal{O}_{\mathbf{P}(E \otimes \mathcal{O}_l)}) = 1\,,$$

so the restriction of the linear system $|\mathcal{O}_{\mathbf{P}(E)}(2) \otimes \varphi^* p^* \mathcal{O}_{\mathbf{P}^3}(-2)|$ to $\mathbf{P}(E \otimes \mathcal{O}_l)$ has a unique element; it is a double divisor whose support corresponds to the surjection

$$E \otimes \mathcal{O}_l \to \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}\,.$$

Furthermore

$$\omega_X^{-1}|_{\mathbf{P}(E\otimes\mathcal{O}_l)} \simeq \varphi^*\mathcal{O}_{\mathbf{P}^1}(1)\otimes\mathcal{O}_{\mathbf{P}(E\otimes\mathcal{O}_l)}(1),$$

so in the notation of Proposition 2.7 we have $a=1$. Applying the proposition, we obtain (again) that since $a-d=0$, the direct image sheaf

$$\varphi_*(\omega_X^{-1}\otimes\omega_{X/Y})\otimes\mathcal{O}_l \simeq \omega_Y^{-1}\otimes\mathcal{O}_l$$

is nef but not ample. For the line bundle $L=\mathcal{O}_{\mathbf{P}(E)}(1)\otimes\mathcal{O}_X$, the same computation yields $a=0$, so $a-d=-1$. Hence the direct image sheaf

$$\varphi_*(L\otimes\omega_{X/Y})\otimes\mathcal{O}_l \simeq \omega_Y^{-1}\otimes p^*\mathcal{O}_{\mathbf{P}^3}(-1)\otimes\mathcal{O}_l$$

is not nef, although L is nef and relatively ample.

2.C Non-rational singularities

We will show in Theorem 3.30 that for a fibration $X\to Y$ and a line bundle that is nef and relatively big, the direct image sheaf is weakly positive *if the irrational locus of X does not dominate Y*. We will give an example to show that this last condition is crucial: the construction is due to [BS95, Ex. 2.2.10] where it is presented as an example of a variety for which the classical vanishing theorem fails. We follow our general philosophy to see it as an example of direct image sheaves with bad properties.

Example 2.9. Let $P:=\mathbf{P}(V)$ be the projectivisation of the vector bundle

$$V:=\mathcal{O}_{\mathbf{P}^1}(-1)\ominus\mathcal{O}_{\mathbf{P}^1}^{\oplus 3},$$

and denote by $\varphi\colon P\to\mathbf{P}^1$ the projection map. For $d\geq 4$, let X be a general member of the linear system $|\mathcal{O}_P(d)|$.

Then the restriction $\varphi|_X\colon X\to\mathbf{P}^1$ is a flat morphism of normal Gorenstein varieties. Set $L:=\mathcal{O}_P(1)\otimes\varphi^*\mathcal{O}_{\mathbf{P}^1}(1)$, then L is nef and φ-ample and the direct image sheaf $(\varphi|_X)_*(L^{\otimes k}\otimes\mathcal{O}_X\otimes\omega_{X/\mathbf{P}^1})$ is not nef for any $k\geq 1$.

Proof. Since $V\simeq\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}^{\oplus 3}$, it follows that for $d\geq 1$, the base locus of the linear system $|\mathcal{O}_P(d)|$ is supported exactly on the curve corresponding to the quotient bundle $V\to\mathcal{O}_{\mathbf{P}^1}(-1)$. By Bertini's theorem a general member X of the linear system is nonsingular in the complement of the base locus, therefore X is nonsingular in codimension 1. Since it is a divisor in P, it is Gorenstein, so it is normal by Serre's criterion 5.12. Since X has no embedded points and $\mathbf{P}^1$ is a smooth curve it is clear that $\varphi|_X$ is flat.

We want to show that the direct image sheaf $(\varphi|_X)_*(L^{\otimes k} \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1})$ is not nef. Arguing as in the proof of Proposition 2.7 we get

$$h^1(P, L^{\otimes k}(X) \otimes \omega_{P/\mathbf{P}^1}) = h^1(X, L^{\otimes k} \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1}).$$

and

$$h^1(P, L^{\otimes k}(X) \otimes \omega_{P/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) = h^1(X, L^{\otimes k} \otimes \mathcal{O}_X \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)).$$

We have $\omega_{P/\mathbf{P}^1} = \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{O}_P(-4)$, so

$$L^{\otimes k}(X) \otimes \omega_{P/\mathbf{P}^1} \simeq \varphi^*\mathcal{O}_{\mathbf{P}^1}(k-1) \otimes \mathcal{O}_P(d+k-4).$$

Since $d \geq 4$, the higher direct images of $\varphi^*\mathcal{O}_{\mathbf{P}^1}(k-1) \otimes \mathcal{O}_P(d+k-4)$ vanish, so

$$h^1(P, L^{\otimes k}(X) \otimes \omega_{P/\mathbf{P}^1}) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-1) \otimes S^{d+k-4}V)$$

and

$$h^1(P, L^{\otimes k}(X) \otimes \omega_{P/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-2) \otimes S^{d+k-4}V).$$

It is easy to see that $\mathcal{O}_{\mathbf{P}^1}(k-1) \otimes S^{d+k-4}V$ has a direct factor of the form $\mathcal{O}_{\mathbf{P}^1}(3-d)$. Since $d \geq 4$, this implies that

$$h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-2) \otimes S^{d+k-4}V) > h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-1) \otimes S^{d+k-4}V),$$

so

$$h^1(X, (L^{\otimes k} \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbf{P}^1} \otimes \varphi^*\mathcal{O}_{\mathbf{P}^1}(-1)) > h^1(X, (L^{\otimes k} \otimes \mathcal{O}_X) \otimes \omega_{X/\mathbf{P}^1})$$

for all $k \geq 1$. Since X is singular at most along a φ-section, it has at most finitely many irrational singularities in every fibre. Therefore the *relative* Kawamata-Viehweg[3]) vanishing Theorem 3.7 applies to X, so

$$R^i(\varphi|_X)_*(L^{\otimes k} \otimes \mathcal{O}_X \otimes \omega_X) = 0 \qquad \forall\, i > 0.$$

Thus we can conclude with Proposition 2.3.

2.D Effective divisors

We have seen in the preceding examples (and will see in much more generality in Section 3) that in general, we cannot expect the direct image sheaves to be globally positive, e.g. to be nef. Since we only obtain weak positivity results, it is natural to ask if we can weaken the condition on L to some weak positivity condition like the existence of global sections.

Unfortunately, the positivity of the direct image sheaf for a line bundle that is effective but not nef comes with an extra difficulty. It is not sufficient to discuss the geometry of the fibration $X \to Y$, but also its relation with

[3]) The absolute Kawamata-Viehweg/Kodaira vanishing theorem does not hold on X.

the geometry of the non-nef locus of the line bundle, represented by some multiplier ideal. In the next example we will take a line bundle L such that the stable base locus of its linear system dominates the base (cf. Section 3.H for more details).

EXAMPLE 2.10. Let $P := \mathbf{P}(V)$ be the projectivisation of the vector bundle

$$V := \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 3},$$

and denote by $\varphi \colon P \to \mathbf{P}^1$ the projection map. Set $L := \mathcal{O}_P(1)$, then for any $k \geq 4$, the linear system $|kL|$ is not empty, but the direct image sheaf $\varphi_*(L^{\otimes k} \otimes \omega_{P/\mathbf{P}^1})$ is not nef.

Proof. For $k \geq 1$, we have $\varphi_* L^{\otimes k} \simeq S^k V$, so we immediately get

$$h^0(P, L^{\otimes k}) = h^0(\mathbf{P}^1, S^k V) > 0.$$

Since $\omega_{P/\mathbf{P}^1} \simeq \mathcal{O}_P(-4) \otimes \varphi^* \mathcal{O}_{\mathbf{P}^1}(-1)$, we have

$$L^{\otimes k} \otimes \omega_{P/\mathbf{P}^1} \simeq \mathcal{O}_P(k-4) \otimes \varphi^* \mathcal{O}_{\mathbf{P}^1}(-1).$$

Hence for $k \geq 4$,

$$\varphi_*(L^{\otimes k} \otimes \omega_{P/\mathbf{P}^1}) \simeq S^{k-4} V \otimes \mathcal{O}_{\mathbf{P}^1}(-1)$$

has at least one direct factor isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)$, so it is not nef. For $k = 4$, we even have $\varphi_*(L^{\otimes 4} \otimes \omega_{P/\mathbf{P}^1}) \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$, so the direct image sheaf is antiample!

3. POSITIVITY OF DIRECT IMAGE SHEAVES

3.A THE PROBLEM

In this chapter we will explain in detail a method to prove the (weak) positivity of direct image sheaves. Technically speaking we will attack the following problem.

PROBLEM 3.1 (The direct image problem). *Let $\varphi \colon X \to Y$ be a fibration from a projective Cohen-Macaulay variety X onto a normal projective Gorenstein[4]) variety Y. Let L be a line bundle over X that satisfies certain positivity properties (e.g. nef and φ-big), and denote by $\omega_{X/Y}$ the relative dualising sheaf (cf. Definition 5.22). What can we say about the positivity properties of the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$?*

[4]) The Gorenstein condition is an important restriction which we make to simplify parts of our considerations. It is possible to weaken this condition.

We have seen in Section 2 that there are certain obstructions for the direct image sheaf to be positive, so we will not be able to give an answer in this rather general setting. Moreover we will see in this chapter that we do not get any result about the positivity $\varphi_*(L \otimes \omega_{X/Y})$ on the non-flat locus of the morphism φ. Since these loci are often very interesting this is of course a certain drawback, but is due to the nature of the problem: we will see in Section 4 that strange things can happen, even in very restricted situations. This explains the "generic" nature of the statement that is the main goal of this section:

THEOREM 3.30 ([Vie82, Kol86a]). *Let $\varphi\colon X \to Y$ be a fibration from a normal projective Cohen-Macaulay variety X onto a normal projective Gorenstein variety Y. Suppose that the locus of irrational singularities* Irr(X) *does not dominate Y. Let L be a line bundle over X that is*

- *nef and φ-big,*
- *or trivial, i.e. $L \simeq \mathcal{O}_X$.*

Then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive.

If φ is generically smooth, then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on the φ-smooth locus.

How does one prove this statement? The bulk of the work will be to prove the rather technical, but useful Lemma 3.24 that reduces the direct image problem 3.1 to the following, probably easier

PROBLEM 3.2 (The extension problem). *Let Y be a normal projective variety on which we fix a very ample line bundle H. Let $\varphi'\colon X' \to Y$ be a fibration from a projective manifold[5]) X' onto Y. Let L be a line bundle on X' that satisfies certain positivity properties (e.g. nef and φ'-big), and denote by $\omega_{X'}$ the canonical sheaf. Is the coherent sheaf $\varphi'_*(L \otimes \omega_{X'}) \otimes H^{\otimes \dim Y + 1}$ generically generated by global sections?*

We will see that this reduction step is essentially a geometric problem and independent of the line bundle L. On the other hand, the answer to the extension problem depends heavily on the positivity of L. In the case where L is nef and φ-big, we get a positive answer via the Kawamata-Viehweg vanishing theorem 3.7; when L is trivial the same is done by

[5]) The variety X' is not the same as X in Problem 3.1. In fact in order to get a positive answer to the direct image problem for X, one has to give a positive answer to the extension problem for infinitely many varieties X'.

Kollár's Theorem 3.8. Finally in Section 3.H we will indicate how analytic tools and the theory of multiplier ideals allows one to generalise Theorem 3.30.

THEOREM 3.38 ([BP07]). *Let $\varphi \colon X \to Y$ be a fibration from a projective manifold X onto a normal projective Gorenstein variety Y. Let L be a pseudo-effective line bundle over X and let h be a singular metric such that $\Theta_h(L) \geq 0$ and that the cosupport of $\mathcal{I}(h)$ does not dominate Y. Then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

Since the steps of the proof are rather technical, let us explain the general strategy in the ideal case of Mourougane's Theorem [6]) 2.1.

STEP 1: *$\varphi_*(L \otimes \omega_{X/Y})$ is locally free.* By the relative Kawamata-Viehweg theorem there are no higher direct images and we conclude via flatness.

STEP 2: *the fibre product trick* [Vie83, Kol86a]. Let $X^s := X \times_Y \ldots \times_Y X$ be the s-times fibred product and denote by $\varphi^s \colon X^s \to Y$ the induced map to Y and by $\pi^i \colon X^s \to X$ the projection on the i-th factor. Then X^s is smooth, φ^s is a smooth fibration and $L_s := \bigotimes_{i=1}^s (\pi^i)^* L$ is nef and φ^s-big. A combination of flat base change and projection formula arguments shows that

$$\varphi_*(L \otimes \omega_{X/Y})^{\otimes s} \simeq \varphi^s_*(L_s \otimes \omega_{X^s/Y}).$$

STEP 3: *an extension property for nef and relatively big line bundles.* An application of the Kawamata-Viehweg vanishing theorem combined with a Castelnuovo-Mumford regularity argument shows the following: let $\psi \colon A \to B$ be a smooth fibration between projective manifolds and let M be a nef and ψ-big line bundle on A. Let H be a very ample line bundle on B, then

$$\psi_*(M \otimes \omega_A) \otimes H^{\dim Y + 1}$$

is globally generated.

STEP 4: *the vector bundle $\varphi_*(L \otimes \omega_{X/Y})$ is nef.* Let H be a very ample line bundle on Y. Apply Step 3 to the smooth fibrations $\varphi^s \colon X^s \to Y$ to see that

$$\varphi_*(L \otimes \omega_{X/Y})^{\otimes s} \otimes \omega_Y \otimes H^{\dim Y + 1} \simeq \varphi^s_*(L_s \otimes \omega_{X^s}) \otimes H^{\dim Y + 1}$$

is globally generated for all $s > 0$. This implies that $\varphi_*(L \otimes \omega_{X/Y})$ is nef.

---

[6]) The following outline is cut and paste from Mourougane's beautifully written paper.

Here is the list of tasks that we have to undertake to adapt this proof to our situation.

• *Skip Step* 1. It is clear that for L an arbitrary line bundle, the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ is not locally free, but only torsion-free. Even if φ is flat, the direct image sheaf is only reflexive in general. For example this means that we cannot use the projection formula.

• *Try to understand the fibre products* X^s. The main setback in our situation is that the fibre products X^s can acquire some really bad singularities. If φ is flat, they are Cohen-Macaulay and under some conditions even normal. The most important point for us is to find a non-empty Zariski open subset $Y' \subset Y$, such that $(\varphi^s)^{-1}(Y') \subset X^s$ has at most rational singularities.

• *Isomorphisms become morphisms*. In our case, a formula like

$$(\varphi_*(L \otimes \omega_{X/Y}))^{\otimes s} \simeq \varphi^s_*(L_s \otimes \omega_{X^s/Y})$$

does not necessary hold. We will deal with this problem by showing these isomorphisms in codimension 1, then extend to a morphism via reflexiveness.

• *Show extension properties*. So far we haven't made use of the positivity of L. The positivity of L only enters the picture in Step 3, which we generalise for a variety of different conditions on the positivity of L.

We will carry out the first three points in Sections 3.D and 3.E; the extension of sections will then be treated in Sections 3.F and 3.H.

ADVICE FOR THE READER. For technical reasons, it will often be necessary to restrict the fibration $\varphi: X \to Y$ to some smaller locus $Y' \subset Y$ (resp. $X' \subset X$) to show a statement and then extend to the corresponding statement on the whole space. For a first reading, it might be useful to forget about these technicalities and assume that $\varphi: X \to Y$ is flat and $\varphi_*(L \otimes \omega_{X/Y})$ is locally free.

3.B WEAK POSITIVITY AND VANISHING THEOREMS

Since direct image sheaves are in general not locally free, the usual definition of ample, nef etc. are no longer valid in this context. This can be done in two different ways: try to use the same definition in the more general context or introduce a new, weaker definition for the general setting. The first way leads to the notion of ample and nef sheaves as defined in

[AT82] and [Kub70]. For our problem the notion of *weak positivity* in the sense of Viehweg is more natural.

NOTATION 3.3 ([Vie83], Def. 1.1). Let X be a normal variety and let $\mathcal{F}$ be a torsion-free sheaf on X. Let $i\colon X^* \to X$ be the largest open subvariety such that $\mathcal{F} \otimes \mathcal{O}_{X^*}$ is locally free. For all $m \in \mathbf{N}$, denote by $S^m(\mathcal{F} \otimes \mathcal{O}_{X^*})$ (resp. $(\mathcal{F} \otimes \mathcal{O}_{X^*})^{\otimes m}$) the m-th symmetric product (resp. the m-fold tensor product). Then we define $S^{[m]}\mathcal{F} := i_* S^m(\mathcal{F} \otimes \mathcal{O}_{X^*})$ and $\mathcal{F}^{[m]} := i_*(\mathcal{F} \otimes \mathcal{O}_{X^*})^{\otimes m}$; in particular these sheaves are reflexive by Lemma 5.5.

REMARK. Since $\mathcal{F}$ is supposed to be torsion-free, the set $X \setminus X^*$ has codimension at least 2. It follows from Lemma 5.5 that

$$S^{[m]}\mathcal{F} \simeq (S^m\mathcal{F})^{**} \qquad \text{and} \qquad \mathcal{F}^{[m]} \simeq (\mathcal{F}^{\otimes m})^{**}.$$

This implies that

$$(\mathcal{F}^{[a]})^{[b]} \simeq \mathcal{F}^{[ab]} \qquad \forall\ a, b \in \mathbf{N}.$$

DEFINITION 3.4 ([Vie83], Def. 1.2, Remark 1.3). Let X be a normal variety, and let $X^* \subset X$ be a non-empty open subset. We say that a torsion-free coherent sheaf $\mathcal{F}$ is *weakly positive (in the sense of Viehweg) over* X^* if for some line bundle H on X and every $\alpha \in \mathbf{N}$ there exists some $\beta \in \mathbf{N}$ such that $S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}$ is globally generated over X^*, that is the evaluation map of sections on X

$$H^0(X, S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}) \otimes \mathcal{O}_X \longrightarrow S^{[\beta\cdot\alpha]}\mathcal{F} \otimes H^{\otimes\beta}$$

is surjective over X^*. The sheaf $\mathcal{F}$ is *weakly positive* if there exists some non-empty open subset $X^* \subset X$ such that $\mathcal{F}$ is weakly positive over X^*.

The next lemma relates the notions of nefness and weak positivity. It shows that a weakly positive sheaf is nef with the exception of some proper subset that contains curves where the sheaf fails to be nef.

LEMMA 3.5 ([Vie82], Lemma 1.10). *Let X be a normal projective variety, and let $\mathcal{F}$ be a coherent sheaf over X. Suppose that $\mathcal{F}$ is weakly positive over $X^* \subset X$. Let C be a projective curve such that $C \cap X^* \neq \varnothing$ and $\mathcal{F}$ is locally free in a neighborhood of C. Then the restriction $\mathcal{F} \otimes \mathcal{O}_C$ is nef.*

Proof. Apply the Barton-Kleiman criterion [Laz04b, Prop. 6.1.18].

Vanishing theorems play a central role for the positivity of direct image sheaves. In fact every time that we can show that for a fixed very ample line bundle H on Y, and any sufficiently nice fibration $\varphi\colon M \to Y$ endowed with a positive line bundle L' on M, we have

$$(*) \qquad H^i(Y, \varphi_*(L' \otimes \omega_M) \otimes H) = 0 \qquad \forall\, i > 0\,,$$

we have a good chance of showing the weak positivity of direct image sheaves. The main reason why this works is Castelnuovo-Mumford regularity.

THEOREM 3.6 (Castelnuovo-Mumford regularity, [Laz04a], Thm. 1.8.5). *Let X be a projective variety and let H be an ample line bundle that is generated by global sections. Let $\mathcal{S}$ be a coherent sheaf on X, and let m be a natural number such that*

$$H^i(X, \mathcal{S} \otimes H^{m-i}) = 0 \qquad \forall\, i > 0\,.$$

Then $\mathcal{S} \otimes H^m$ is generated by its global sections.

There are two ways of showing a vanishing statement of type $(*)$. The first way is to show the vanishing of

$$H^i(X, L \otimes \omega_X \otimes \varphi^* H) = 0 \qquad \forall\, i > 0\,,$$

and to make a spectral sequence argument to obtain the vanishing statement $(*)$. This method can only work if L is sufficiently positive on the general fibre of the fibration, otherwise the cohomology will not vanish globally since we have cohomology along the fibres. Nevertheless if the line bundle is relatively nef and big, the relative Kawamata-Viehweg Theorem shows that we get the required vanishing.

THEOREM 3.7 (Relative Kawamata-Viehweg vanishing: [BS95], Lemma 2.2.5). *Let X be a normal quasi-projective variety and let $\varphi\colon X \to Y$ be a projective morphism to a quasi-projective variety. Let L be a φ-nef and φ-big line bundle. Then*

$$R^i\varphi_*(L \otimes \omega_X) = 0 \qquad \forall\, i > \max_{y \in \varphi(\mathrm{Irr}(X))} \dim(\varphi^{-1}(y) \cap \mathrm{Irr}(X))\,.$$

The second method is much more refined and aims at showing directly the vanishing $(*)$. This approach is based on a famous theorem due to Kollár.

THEOREM 3.8 ([Kol86a], Thm. 2.1). *Let X be a projective manifold, and let $\varphi\colon X \to Y$ be a fibration onto a normal variety Y. Then for all $j \geq 0$ the coherent sheaf $R^j\varphi_*\omega_X$ is torsion-free. Furthermore let H be an ample line bundle on Y; then for all $j \geq 0$ we have*

$$H^i(Y, R^j\varphi_*\omega_X \otimes H) = 0 \qquad \forall\ i > 0.$$

COROLLARY 3.9. *In the situation of Theorem 3.7, suppose that X has at most rational singularities and that φ is a flat Cohen-Macaulay fibration (cf. 5.14) onto a Gorenstein variety Y. Then $\varphi_*(L \otimes \omega_{X/Y})$ is locally free.*

Proof. Since φ is a flat Cohen-Macaulay fibration, the relative dualising sheaf $\omega_{X/Y}$ exists and is flat over Y by Theorem 5.18. Since Y is Gorenstein, we have

$$\omega_X \simeq \omega_{X/Y} \otimes \varphi^*\omega_Y$$

by Theorem 5.20, so ω_X is flat over Y. By the relative Kawamata-Viehweg vanishing theorem the higher direct images $R^i\varphi_*(L\otimes\omega_X)$ vanish, so $\varphi_*(L\otimes\omega_X)$ is locally free [Mum70, Cor. 2, p. 50]. Hence $\varphi_*(L\otimes\omega_{X/Y}) \simeq \varphi_*(L\otimes\omega_X)\otimes\omega_Y^{-1}$ is locally free.

3.C SIMPLIFYING THE PROBLEM – A BIT

Example 2.9 shows that if the total space has too many irrational singularities, we cannot expect to obtain a positivity result for direct image sheaves. Therefore it is necessary to start with a total space with (essentially) rational singularities.

PROPOSITION 3.10. *Let $\varphi\colon X \to Y$ be a fibration from a normal, projective Cohen-Macaulay variety onto a Gorenstein variety Y, and let L be a line bundle over X. Suppose that the locus of irrational singularities* Irr(X) *of X does not dominate Y, and let $q\colon X' \to X$ be a desingularisation of X.*

If $(\varphi \circ q)_*(q^*L \otimes \omega_{X'/Y})$ is weakly positive on some non-empty open set $Y' \subset Y$, the direct image sheaf $\varphi_*(L\otimes\omega_{X/Y})$ is weakly positive on $Y' \setminus \varphi(\mathrm{Irr}(X))$.

Proof. The desingularisation $q\colon X' \to X$ induces a morphism $q_*\omega_{X'} \to \omega_X$ that is an isomorphism in the complement of Irr(X). By the projection formula we obtain a morphism

$$(\varphi \circ q)_*(q^*L \otimes \omega_{X'/Y}) \to \varphi_*(L \otimes \omega_{X/Y})$$

that is an isomorphism in the complement of $\varphi(\mathrm{Irr}(X))$.

REMARK. The proposition tells us that we can restrict our attention to the case where the total space is smooth. This does not mean that we are obliged to do so: if we have a total space with rational singularities and some extra information on the singularities of the fibration, one might obtain more precise results if we do not desingularise.

3.D FIBRE PRODUCTS

Our goal is to give sufficient conditions such that Problem 3.1 reduces to the more accessible Problem 3.2. This goal is achieved with sufficient generality with the reduction Lemma 3.24 which is our main technical result. The idea is as follows: by definition, to show the weak positivity of $\varphi_*(L \otimes \omega_{X/Y})$ it is sufficient to show that for some very ample line bundle H and all $s \in \mathbf{N}$, the sheaves

$$[\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y \otimes H^{\otimes \dim Y+1}$$

are globally generated on some non-empty open set. This could be simpler, since we "added a lot of positivity" to the sheaf [7]). On the other hand it is difficult to say something about the tensor products $[\varphi_*(L \otimes \omega_{X/Y})]^{[s]}$. The fibre product trick will allow us to get around this problem by showing that the sheaf $\varphi_*(L \otimes \omega_{X/Y})^{[s]}$ can essentially be realised as a direct image sheaf of some associated fibration $X^s \to Y$ (Lemma 3.17). The drawback of this trick is that X^s can have irrational singularities, which is an obstruction to having positive direct image sheaves (cf. Section 2.C). For this reason, we will be rather obsessed with controlling the singularities of the varieties we construct.

For the following statements and proofs, it is very convenient to work in a relative setting, i.e. to consider projective morphisms between quasi-projective varieties.

CONSTRUCTION 3.11. Let $\varphi \colon X \to Y$ be a fibration from a quasi-projective manifold X onto a normal quasi-projective Gorenstein variety Y, and let L be a line bundle over X. The s-times fibred product

$$X \times_Y \ldots \times_Y X$$

has a unique irreducible component that dominates Y and we denote by X^s its reduction.

[7]) The proof of the invariance of plurigenera is based on this principle: add a positive line bundle so that you can extend the sections from the central fibre, then find some inductive process to get rid of the line bundle (cf. [Pău07, Siu02]).

Denote by $\pi^s : X^s \to X$ the projection on the s-th factor and by $\varphi^s : X^s \to Y$ and $\pi' : X^s \to X^{s-1}$ the induced maps. We sum this up in the commutative diagram

$$\begin{array}{ccc} X^s & \xrightarrow{\pi^s} & X \\ \downarrow{\scriptstyle \pi'} & \searrow{\scriptstyle \varphi^s} & \downarrow{\scriptstyle \varphi} \\ X^{s-1} & \xrightarrow{\varphi^{s-1}} & Y\,. \end{array}$$

Denote by $X_0 \subset X$ and $Y_0 \subset Y$ the maximal open sets such that $\varphi|_{X_0} : X_0 \to Y_0$ is a flat fibration. By Corollary 5.24 we have

$$X_0^s := (\varphi^s)^{-1}(Y_0) = X_0 \times_{Y_0} \ldots \times_{Y_0} X_0\,.$$

Moreover $\varphi^s|_{X_0^s}$ is a flat fibration and

$$\omega_{X_0^s/Y_0} \simeq (\pi^s)^* \omega_{X_0/Y_0} \otimes (\pi')^* \omega_{X_0^{s-1}/Y_0}\,.$$

By the same corollary, X_0^s is Gorenstein. We set $L_1 := L$ and define inductively for $s > 1$

$$L_s := (\pi^s)^* L \otimes (\pi')^* L_{s-1}\,.$$

LEMMA 3.12. *In the situation of Construction* 3.11, *suppose that* $\varphi|_{X_0}$ *has generically reduced fibres in codimension one (Definition* 1.4*). Then* X_0^s *is normal.*

Proof. Since X_0^s is Cohen-Macaulay, it is sufficient by Serre's criterion 5.12 to show that X_0^s is nonsingular in codimension 1. Since X_0^s is equidimensional over Y_0 and φ has generically reduced fibres in codimension 1, we may furthermore suppose that Y_0 is smooth and that all the fibres of $\varphi|_{X_0}$ are generically reduced. Let $Z \subset X$ be the smallest closed set such that

$$\Omega_{X/Y} \otimes \mathcal{O}_{X \setminus Z}$$

is locally free. The general fibre is smooth and all the fibres are generically reduced, so

$$\operatorname{codim}_X Z \geq 2$$

and $\varphi|_{X_0 \setminus Z} : X_0 \setminus Z \to Y$ is a smooth map. Since φ is equidimensional, it follows inductively that the s-times fibred product

$$(X_0 \setminus Z)^s := (X_0 \setminus Z) \times_{Y_0} \ldots \times_{Y_0} (X_0 \setminus Z)$$

is smooth and

$$\operatorname{codim}_{X_0^s}(X_0^s \setminus (X_0 \setminus Z)^s) \geq 2\,.$$

This concludes the proof.

The normality of X_0^s is in general not sufficient to ensure that a direct image sheaf is weakly positive over Y_0. The next lemma gives sufficient conditions to have rational singularities.

LEMMA 3.13 ([Kol86a], Lemma 3.4). *In the situation of Construction* 3.11, *let* $Y' \subset Y$ *and* $X' := \varphi^{-1}(Y')$ *be open sets such that* $\varphi|_{X'} : X' \to Y'$ *is flat and such that the* φ*-singular locus* Δ *satisfies the following conditions:*

(1) *the intersection* $\Delta \cap Y'$ *is a smooth divisor;*

(2) *the preimage* $\varphi^{-1}(\Delta \cap Y')$ *is a simple normal crossings divisor.*

Then $(\varphi^s)^{-1}(Y')$ *is normal with at most rational singularities.*

Proof. Conditions (1) and (2) imply that $\varphi|_{X'}$ has generically reduced fibres in codimension 1, so Lemma 3.12 shows that $(\varphi^s)^{-1}(Y')$ is normal. A local computation shows that its singularities are rational, cf. [Vie83, Lemma 3.6].

EXAMPLE 3.14. Let $\varphi : X \to Y$ be a conic bundle, where X and Y are projective manifolds. The locus $\Delta_1 \subset \Delta$ such that the fibres are double lines has codimension at least two [Sar82, Prop. 1.8.5], so the fibre products

$$X \times_Y \ldots \times_Y X$$

are normal. Furthermore if $y \in \Delta_{\text{reg}}$ is a point of the φ-singular locus such that the fibre over X_y is a union of two lines, there exists an analytic neighbourhood $y \in U \subset \Delta$ such that $\varphi^{-1}(U)$ is a union of two copies of $U \times \mathbf{P}^1$ glued along a section. Thus the above lemma applies for

$$Y' := Y \setminus (\Delta_1 \cup \Delta_{\text{sing}}) .$$

We turn now to the most crucial question of Construction 3.11: what is the relation between $\varphi_*(L \otimes \omega_{X/Y})$ and $\varphi^s_*(L_s \otimes \omega_{X^s/Y})$? The ideal relation would of course be

$$(\varphi_*(L \otimes \omega_{X/Y}))^{\otimes s} \simeq \varphi^s_*(L_s \otimes \omega_{X^s/Y}) ,$$

but this is not true in general. The first problem is already apparent in Construction 3.11: if φ is not flat, X^s is not given by a natural construction (recall that X^s is the reduction of a certain irreducible component). The sheaves $\omega_{X^s/Y}$ and $\omega_{X/Y}$ are not dualising sheaves in the sense of Definition 5.17 and there is no functorial property that relates them. We can deal with this problem by doing the computation over the flat locus where we have an easy relation between $\omega_{X_0^s/Y_0}$ and ω_{X_0/Y_0}. A second problem is that $\varphi_*(L \otimes \omega_{X/Y})$ is not locally free. We take care of this problem by working on the locus where the sheaf is locally free and extend by taking biduals.

LEMMA 3.15 (Base change lemma, locally free case). *In the situation of Construction* 3.11, *suppose that* $\varphi_*(L \otimes \omega_{X_0/Y_0})$ *is locally free. Then*

$$\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0}) \simeq [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{\otimes s}. \tag{3.16}$$

Proof. We proceed by induction on s. The case $s = 1$ is trivial. By construction, we have

$$L_s \otimes \omega_{X_0^s/Y_0} \simeq (\pi^s)^*(L \otimes \omega_{X_0/Y_0}) \otimes (\pi')^*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0}).$$

Since $\varphi|_{X_0}$ has Gorenstein fibres, ω_{X_0/Y_0} is locally free by [Kle80, p. 58]. Hence $L \otimes \omega_{X_0/Y_0}$ is locally free and the projection formula implies that

$$\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0}) \simeq \varphi_*\big(L \otimes \omega_{X_0/Y_0} \otimes \pi^s_*(\pi'^*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0}))\big).$$

Since φ is flat, we can apply flat base change to obtain

$$\pi^s_*(\pi'^*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0})) \simeq \varphi^*(\varphi^{s-1}_*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0})).$$

By the induction hypothesis

$$\varphi^{s-1}_*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0}) \simeq [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{\otimes s-1}$$

is locally free, so we can apply the projection formula a second time to see that

$$\begin{aligned}\varphi_*\big(L \otimes \omega_{X_0/Y_0} &\otimes \pi^s_*(\pi'^*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0}))\big)\\ &\simeq \varphi_*\big(L \otimes \omega_{X_0/Y_0} \otimes \varphi^*(\varphi^{s-1}_*(L_{s-1} \otimes \omega_{X_0^{s-1}/Y_0}))\big)\\ &\simeq \varphi_*(L \otimes \omega_{X_0/Y_0}) \otimes [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{\otimes s-1}.\end{aligned}$$

LEMMA 3.17 (Base change lemma). *In the situation of Construction* 3.11, *the direct image sheaf* $\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0})$ *is reflexive and*

$$\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0}) \simeq [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{[s]}. \tag{3.18}$$

Proof. Since $\varphi^s \colon X_0^s \to Y_0$ is a flat Cohen-Macaulay fibration, the sheaf $\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0})$ is reflexive by Corollary 5.26. In particular it is locally free in codimension one by Proposition 5.3. Thus Lemma 3.15 shows that there exists an open subset $Y' \subset Y_0$ such that $\operatorname{codim}_{Y_0}(Y_0 \setminus Y') \geq 2$ and

$$\varphi^s_*(L_s \otimes \omega_{X_0^s/Y_0}) \otimes \mathcal{O}_{Y'} \simeq [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{\otimes s} \otimes \mathcal{O}_{Y'}.$$

Since the bidual $[\varphi_*(L \otimes \omega_{X/Y})]^{[s]}$ is reflexive, we obtain the isomorphism by Lemma 5.5.

3.E Desingularisation

In the preceding section we have seen that the fibre product X^s is smooth over a general point of Y, but it is also clear that the fibre products are rarely normal with at most rational singularities. This is somewhat annoying since vanishing theorems tend to fail for varieties with irrational singularities. We can deal with this situation by taking a desingularisation $X^{(s)} \to X^s$, so the total space will be smooth. This comes at a certain price, since the locus where the new map $X^{(s)} \to Y$ has "nice" fibres (or is at least flat) tends to be smaller than Y_0. Therefore we get new loci where it will be hard to say something about the positivity of the direct image sheaves.

Construction 3.19. Suppose that we are in the situation of Construction 3.11, Let $\nu\colon (X^s)' \to X^s$ be the normalisation. By relative duality for finite morphisms (cf. [Har77, III, Ex. 6.10, Ex. 7.2]), there exists a natural morphism

$$\nu_* \omega_{(X^s)'} \to \omega_{X^s} .$$

Let $r\colon X^{(s)} \to (X^s)'$ be a desingularisation, then we have a morphism

$$r_* \omega_{X^{(s)}} \to \omega_{(X^s)'} .$$

Pushing the morphism down on X^s, we obtain a morphism

$$(3.20) \qquad (\nu \circ r)_* \omega_{X^{(s)}} \to \nu_* \omega_{(X^s)'} \to \omega_{X^s} ,$$

which by Definition 5.16 is an isomorphism on the locus where X^s is normal with at most rational singularities. By the projection formula, the morphism (3.20) induces a morphism $(\nu \circ r)_*(\omega_{X^{(s)}} \otimes (\nu \circ r)^* L_s) \to L_s \otimes \omega_{X^s}$. Set $\varphi^{(s)} := \varphi^s \circ \nu \circ r\colon X^{(s)} \to Y$. Pushing down via φ^s we obtain a morphism

$$(3.21) \qquad \varphi^{(s)}_*((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \to \varphi^s_*(L_s \otimes \omega_{X^s}),$$

which is an isomorphism on the largest open set $Y' \subset Y_0$ such that $(\varphi^s)^{-1}(Y')$ is normal with at most rational singularities.

Remark. Since $\varphi^{(s)}$ is not necessarily flat, the direct image sheaf $\varphi^{(s)}_*((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})$ is torsion-free, but not necessarily reflexive.

Lemma 3.22. *In the situation of Construction* 3.19, *there exists a natural map*

$$(3.23) \qquad \tau^s\colon [\varphi^{(s)}_*((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})] \to [\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y ,$$

which is an isomorphism on the largest open set $Y' \subset Y_0$ *such that* $(\varphi^s)^{-1}(Y')$ *is normal with at most rational singularities.*

REMARK. Note that in general the set Y' depends on s. Since the smooth locus of φ works for every s, we will implicitly choose Y' as the largest open set that works for all s.

Proof. We have a global morphism

$$(*) \qquad \varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \to \varphi_*^s(L_s \otimes \omega_{X^s}),$$

which is an isomorphism on the largest open subset $Y' \subset Y_0$ such that $(\varphi^s)^{-1}(Y')$ is normal with at most rational singularities.

Since Y is Gorenstein, we have

$$\varphi_*^s(L_s \otimes \omega_{X^s}) \simeq \varphi_*^s(L_s \otimes \omega_{X^s/Y}) \otimes \omega_Y .$$

Therefore Lemma 3.17 implies that

$$\varphi_*^s(L_s \otimes \omega_{X^s}) \otimes \mathcal{O}_{Y_0} \simeq [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{[s]} \otimes \omega_{Y_0} .$$

Composing the restriction of $(*)$ to Y_0 with this isomorphism, we obtain a morphism

$$\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes \mathcal{O}_{Y_0} \to [\varphi_*(L \otimes \omega_{X_0/Y_0})]^{[s]} \otimes \omega_{Y_0} .$$

Since X is Cohen-Macaulay and Y is normal, the non-flat locus has codimension at least two [Har77, III, Ex. 10.9], i.e. $\operatorname{codim}_Y(Y \setminus Y_0) \geq 2$. Since $[\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y$ is a reflexive sheaf, this morphism extends to a global morphism on Y by reflexiveness of the $\mathcal{H}om$-sheaf (Corollary 5.7). By what precedes it is clear that this extended morphism is an isomorphism on Y'.

We come to the main technical statement: it sums up the preceding considerations and explains how to reduce Problem 3.1 to Problem 3.2.

LEMMA 3.24 (Reduction lemma). *Let $\varphi\colon X \to Y$ be a fibration from a quasi-projective manifold onto a Gorenstein quasi-projective variety Y, and let L be a line bundle over X.*

Suppose furthermore that there exists a line bundle H on Y and a non-empty Zariski open subset $Y'' \subset Y$ such that for all sufficiently large $s \in \mathbf{N}$, the sheaf

$$\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y + 1}$$

is generated by global sections on Y''. Then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on $Y' \cap Y''$, where Y' is the locus defined in Lemma 3.22. In particular $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on the intersection of Y'' with the φ-smooth locus.

REMARKS 3.25. (1) The reader will have noticed that while in Section 3.D we always worked with the relative dualising sheaf $\omega_{X/Y}$ (or $\omega_{X^s/Y}$), we now work with the canonical sheaf $\omega_{X^{(s)}}$. The reason for this change is that the vanishing theorems we use are stated for the canonical sheaf.

(2) It may appear pointless to make a statement with two different sets Y' and Y'', since in the end we show a property on their intersection. This distinction is motivated by the fact that Y' is determined by the geometry of the fibration $\varphi\colon X \to Y$ and has nothing to do with the line bundle L. The set Y'' depends on the positivity of L. We will detail this in Section 3.F.

Proof. We are in the situation of Construction 3.19, so Lemma 3.22 implies the existence of a morphism

$$\tau^s\colon [\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})] \to [\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y$$

that is an isomorphism on $Y' \subset Y_0$. Since H is locally free, the induced morphism

$$\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1} \longrightarrow [\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y \otimes H^{\otimes \dim Y+1}$$

is still an isomorphism on Y'. Since the sheaf on the left hand side is generated by global sections on Y'', it follows that

$$[\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y \otimes H^{\otimes \dim Y+1}$$

is generated by global sections on $Y' \cap Y''$. By definition this implies that $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on $Y' \cap Y''$. Since Y' always contains the φ-smooth locus, the second statement is immediate.

A REMARK ON THE CASE $L = \mathcal{O}_X$. The somewhat vague task of understanding the influence of the geometry of the fibration $\varphi\colon X \to Y$ on the existence of bad loci of the direct image sheaves can now be made more precise: suppose[8]) that in the reduction lemma, we have $Y'' = Y$. Then the loci where the morphism (3.23)

$$\tau^s\colon [\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})] \to [\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y$$

is not an isomorphism contain the locus where the direct image sheaves are not weakly positive. Looking at the construction, it seems that this locus does not depend on L, but this is only part of the truth. Indeed, Lemma 3.13 does not depend on L, so the locus $Y' \subset Y_0$ does not depend on L. *But τ^s might be an isomorphism on a locus that is larger than Y'.*

---

[8]) This holds in the setting of Theorem 3.30.

To give an example, suppose that the complement $Y \setminus Y'$ has codimension at least two, so τ^s is an isomorphism in codimension one. *Thus if we can ensure that* $[\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})]$ *is reflexive, then* τ^s *is an isomorphism on* Y.

Proof. Indeed $[\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y$ is reflexive and an isomorphism in codimension one between reflexive sheaves extends to a global isomorphism by Lemma 5.5.

On the one hand since $\varphi^{(s)}$ has very little chance of being flat, in general it is hard to ensure that $[\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}})]$ is reflexive. On the other hand in the case $L = \mathcal{O}_X$, we have a fundamental result due to Kawamata:

THEOREM 3.26 ([Kaw81], Thm. 5). *Let* $\varphi\colon X \to Y$ *be a fibration between projective manifolds and let* $\Delta \subset Y$ *be the* φ*-singular locus. If* Δ *is a normal crossing divisor, the direct image sheaf* $\varphi_* \omega_X$ *is locally free.*

This theorem explains why the case $L = \mathcal{O}_X$ works slightly better than the general case:

COROLLARY 3.27 ([Kol86a], Cor. 3.7). *Let* $\varphi\colon X \to Y$ *be a fibration between projective manifolds and let* $\Delta \subset Y$ *be the* φ*-singular locus. Suppose that* Δ *is a normal crossing divisor, and that outside a codimension two set* $Z \subset Y$*, the fibres are reduced with normal crossings. Then* $\varphi_* \omega_{X/Y}$ *is locally free and nef.*

Proof. Apply Theorem 3.30, Lemma 3.13 and the preceding discussion.

3.F EXTENSION OF SECTIONS

From a geometric point of view, the main part of the work is done. We will now specify the condition on L and check that the line bundles $(\nu \circ r)^* L_s$ satisfy the same condition. Then we can apply vanishing theorems and Castelnuovo-Mumford regularity to check the condition in the reduction Lemma 3.24. Note that while so far all our considerations were local on the base Y and worked in the quasi-projective setting, we will now assume X and Y to be projective.

The weak positivity of the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ for L a nef and φ-big line bundle is the easiest case:

LEMMA 3.28. *Let Y be a normal projective variety on which we fix a very ample line bundle H. Let $\varphi\colon X \to Y$ be a fibration from a projective manifold X onto Y. Let L be a nef and φ-big line bundle on X. Then the coherent sheaf*

$$\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$$

is generated by global sections.

Proof. By Castelnuovo-Mumford regularity (Theorem 3.6) it is sufficient to show that

$$H^i(Y, \varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1-i}) = 0 \qquad \forall\ i > 0\,.$$

Fix $i \in \{1, \dots, \dim Y\}$. By the relative Kawamata-Viehweg vanishing theorem 3.7,

$$R^j\varphi_*(L \otimes \omega_X) = 0 \qquad \forall\ j > 0\,,$$

so by a degenerate case of the Leray spectral sequence

$$H^k(Y, \varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1-i}) \simeq H^k(X, L \otimes \omega_X \otimes \varphi^* H^{\otimes \dim Y+1-i})$$

for every $k \geq 0$. Since H is ample and L is nef and φ-big, the line bundle $M := \varphi^* H^{\otimes \dim Y+1-i} \otimes L$ is nef. Furthermore we have

$$c_1(M)^{\dim X} = \sum_{d=0}^{\dim Y} \binom{\dim Y}{d} c_1(\varphi^* H^{\otimes \dim Y+1-i})^d \cdot c_1(L)^{\dim X-d}\,.$$

Since L is nef and H ample on Y, every term in this sum is nonnegative. Since L is φ-big, the last term is strictly positive, so M is nef and big. Thus the standard Kawamata-Viehweg theorem yields

$$H^k(X, L \otimes \omega_X \otimes \varphi^* H^{\otimes \dim Y+1-i}) = 0 \qquad \forall\ k \geq 1\,.$$

Note the overkill of vanishing in the preceding proof. For a nef and relatively big line bundle we can vanish much more cohomology groups than we actually need, so there is some hope of obtaining similar results under weaker conditions.

LEMMA 3.29. *Let Y be a normal projective variety on which we fix a very ample line bundle H. Let $\varphi\colon X \to Y$ be a fibration from a projective manifold X onto Y. Then the coherent sheaf*

$$\varphi_*\omega_X \otimes H^{\otimes \dim Y+1}$$

is generated by global sections.

Proof. By Castelnuovo-Mumford regularity (Theorem 3.6) it is sufficient to show that

$$H^i(Y, \varphi_* \omega_X \otimes H^{\otimes \dim Y+1-i}) = 0 \qquad \forall\ i > 0\,.$$

Since H is ample, this property holds by Kollár's Theorem 3.8.

THEOREM 3.30 ([Vie82, Kol86a]). *Let $\varphi\colon X \to Y$ be a fibration from a normal projective Cohen-Macaulay variety X onto a normal projective Gorenstein variety Y. Suppose that the locus of irrational singularities* $\mathrm{Irr}(X)$ *does not dominate Y. Let L be a line bundle over X that is*

- *nef and φ-big,*
- *or trivial, i.e. $L \simeq \mathcal{O}_X$.*

Then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive.

If φ is generically smooth, then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on the φ-smooth locus.

Proof. By Proposition 3.10, we may suppose without loss of generality that X is smooth. We use the notation of Constructions 3.11 and 3.19, and fix a very ample line bundle H on Y.

First case. Since L is nef and φ-big, the bundle L_s is nef and φ^s-big. Since the restriction of $\nu \circ d\colon X^{(s)} \to X^s$ to a general $\varphi^{(s)}$-fibre is an isomorphism, the pull-back $(\nu \circ d)^* L_s$ is nef and $\varphi^{(s)}$-big. Hence the coherent sheaf

$$\varphi^{(s)}_*((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is globally generated for all $s > 0$ by Lemma 3.28, so in the notation of the reduction Lemma 3.24 we have $Y'' = Y$.

Second case. The coherent sheaf

$$\varphi^{(s)}_* \omega_{X^{(s)}} \otimes H^{\otimes \dim Y+1}$$

is globally generated for all $s > 0$ by Lemma 3.29, so in the notation of the reduction Lemma 3.24 we have $Y'' = Y$.

In both cases the reduction Lemma 3.24 shows that $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive over the maximal Zariski open subset $Y' \subset Y$ such that for all $s > 0$ the fibre product $(\varphi^s)^{-1}(Y')$ is normal with at most rational singularities. In particular if φ is generically smooth, $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on the φ-smooth locus.

3.G Return to Wiśniewski's example

Let us see how these rather abstract considerations apply to Wiśniewski's Example 2.2. In this case we consider a conic bundle $\varphi\colon X \to Y$, where X and Y are projective manifolds. The morphism φ is flat, so in the notation of Construction 3.11, we have $Y_0 = Y$ and $X_0 = X$. Furthermore if we take $L = \omega_X^{-1} \otimes \varphi^* p^* \mathcal{O}_{\mathbf{P}^3}(-1)$, the line bundle L is relatively nef and big, so by Corollary 3.9, the direct image

$$\varphi_*(L \otimes \omega_{X/Y})$$

is locally free. Therefore we can apply the base change lemma in the locally free version 3.15, so

$$\varphi^s_*(L_s \otimes \omega_{X^s/Y}) \simeq [\varphi_*(L \otimes \omega_{X/Y})]^{\otimes s}.$$

A local computation shows that the locus $\Delta_1 \subset \Delta$ such that the fibres X_y are double lines, is exactly the section Z. Moreover $\Delta \setminus Z$ is smooth, so Example 3.14 tells us that the varieties X^s are normal and have rational singularities in the complement of $(\varphi^s)^{-1}(Z)$. Thus by the reduction Lemma 3.24, the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on $Y \setminus Z$.

Let us now see that X^s does not have rational singularities over Z: for this we twist the morphism τ^s with $H^{\otimes \dim Y+1}$, where H is a very ample line bundle, to get a morphism

$$\varphi^{(s)}_*((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1} \to [\varphi_*(L \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y \otimes H^{\otimes \dim Y+1}.$$

By Lemma 3.28, the sheaf on the left hand side is globally generated for every s, so $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive on the locus where the morphism is an isomorphism. Yet we know that the direct image sheaf

$$\varphi_*(L \otimes \omega_{X/Y}) \simeq \omega_Y^{-1} \otimes p^* \mathcal{O}_{\mathbf{P}^3}(-1)$$

is antiample on the section Z. Therefore the morphism cannot be an isomorphism on this locus, in particular X^s does not have rational singularities on $(\varphi^s)^{-1}(Z)$.

The preceding arguments show that if the base of a conic bundle has dimension two, the locus over which we have irrational singularities is at most a union of points. Thus we get:

Corollary 3.31. *Let X be a smooth projective threefold that is a conic bundle $\varphi\colon X \to Y$ over a surface Y. Let L be a line bundle over X that is nef and φ-big. Then the direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ is nef.*

3.H Multiplier ideals

The preceding discussion shows that in most cases a direct image sheaf $\varphi_*(L \otimes \omega_{X/Y})$ has merely a generic positivity property like weak positivity, even if the line bundle L is positive in *every point* of X. Although it is natural to try to obtain similar results under a weaker hypothesis on L, one must be careful: Example 2.10 shows that the existence of global sections is not a sufficient condition. In this section we will explain briefly how the introduction of multiplier ideals permits to obtain significantly better results.

The idea of this section is to relate the classical point of view explained above with the recent important results by Berndtsson and Păun [BP07]. It differs from the rest of these notes in two points: we will use an analytic language and techniques, since they provide a more natural framework for extension theorems. The exposition will no longer be self-contained and we refer to [Dem96, Dem01] for basic definitions.

A line bundle L on X is *pseudo-effective* if the first Chern class $c_1(L)$ is contained in the closure of the cone of the first Chern classes of effective line bundles in the Neron-Severi group $NS(X) \otimes \mathbf{R}$. In this paragraph we will rather use the following equivalent analytic definition.

Definition 3.32. Let X be a projective manifold, and L be a line bundle over X. Then L is *pseudo-effective* if it admits a singular hermitian metric h such that its curvature current $\Theta_h(L) = -i\partial\overline{\partial}h$ is positive (in the sense of currents), which we abbreviate as $\Theta_h(L) \geq 0$.

It is well known that if $\Theta_h(L) \geq 0$, a local weight function φ of the metric defined on a small open subset $U \subset X$ is psh (plurisubharmonic).

Definition 3.33. Let φ be a psh function on an open subset $\Omega \subset \mathbf{C}^n$. Set

$$\mathcal{I}(\varphi) := \left\{ f \in \mathcal{O}_{\Omega,x} \;\middle|\; |f|^2 e^{-2\varphi} \in L^1_{loc} \;\text{ near } x \right\}.$$

Let L be a pseudo-effective line bundle and h a singular metric such that $\Theta_h(L) \geq 0$. The *multiplier ideal sheaf* $\mathcal{I}(h)$ associated to h is defined at a point $x \in X$ by $\mathcal{I}(\varphi)$, where φ is a local weight function around x. The *cosupport of* $\mathcal{I}(h)$ is the support of $\mathcal{O}_X/\mathcal{I}(h)$.

EXAMPLE 3.34. Let L be a line bundle on a projective manifold X such that $H^0(X,L) \neq 0$. Let $\sigma_1, \ldots, \sigma_k$ be a base of $H^0(X,L)$; then we can define a metric h on L with weight function

$$\varphi = \log\Bigl(\sum_{j=1}^{k} |\sigma_j(x)|^2\Bigr).$$

This metric satisfies $\Theta_h(L) \geq 0$ and is nonsingular on the complement of

$$\{x \in X \mid \sigma_1(x) = \ldots = \sigma_k(x) = 0\}.$$

In the case of Example 2.10, we have $h^0(X,L) = 3$ and all the sections vanish along the section C corresponding to

$$\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 3} \to \mathcal{O}_{\mathbf{P}^1}(-1).$$

A local computation shows that $\mathcal{I}(h) = \mathcal{I}_C$.

We will replace the Kawamata-Viehweg vanishing Theorem 3.7 by the Ohsawa-Takegoshi theorem to obtain the analogue of Lemma 3.28 in this setting.

LEMMA 3.35. *Let Y be a normal projective variety on which we fix a very ample line bundle H. Let $\varphi\colon X \to Y$ be a fibration from a projective manifold X onto Y. Let L be a pseudo-effective line bundle over X and let h be a singular metric such that $\Theta_h(L) \geq 0$ and that the cosupport of $\mathcal{I}(h)$ does not dominate Y. Then the coherent sheaf*

$$\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y + 1}$$

is generically generated by its global sections.

Proof. Since $H^0(Y, \varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}) = H^0(X, L \otimes \omega_X \otimes \varphi^* H^{\otimes \dim Y+1})$ and the fibre of $\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$ in a general point $y \in Y$ equals $H^0(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y})$, we have to show that the restriction morphism

$$H^0(X, L \otimes \omega_X \otimes \varphi^* H^{\otimes 2}) \to H^0(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y})$$

is surjective. We argue by induction on the dimension of Y.

If $\dim Y = 1$, a general element D of the linear system $|\varphi^* H|$ is a disjoint union of smooth fibres. Hence if $X_y = \varphi^{-1}(y)$ is a general fibre, we have a surjective restriction morphism

$$H^0(D, L \otimes \mathcal{O}_D \otimes \omega_D) \to H^0(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y}).$$

Fix a smooth metric h' with positive curvature on H. Since $\Theta_h(L) \geq 0$, we have

$$\Theta_{h \otimes \varphi^* h'}(L \otimes \varphi^* H) \geq \Theta_{\varphi^* h'}(\varphi^* H).$$

Thus the line bundle $L \otimes \varphi^* H^{\otimes 2}$ endowed with the metric $h \otimes \varphi^* h'^{\otimes 2}$ satisfies the conditions of the (singular) Ohsawa-Manivel-Takegoshi theorem (cf. [BP07, Thm. 5.1] and the remarks after their statement), so any section of

$$H^0(D, L \otimes \mathcal{O}_D \otimes \omega_D \otimes \mathcal{I}(h|_D))$$

extends to a section of $H^0(X, L \otimes \omega_X \otimes \mathcal{I}(h))$. Since the cosupport of $\mathcal{I}(h)$ does not dominate Y, we have $\mathcal{I}(h|_D) \simeq \mathcal{O}_D$ by the generic restriction theorem [Laz04b, Thm. 9.5.35].

If $\dim Y > 1$, a general element D of the linear system $|\varphi^* H|$ is a projective manifold such that $\varphi(D)$ is a normal variety. Furthermore $(L \otimes \mathcal{O}_D, h|_D)$ is a pseudo-effective line bundle over D such that $\Theta_{h_D}(L \otimes \mathcal{O}_D) \geq 0$ and by the generic restriction theorem the cosupport of $\mathcal{I}(h|_D)$ does not dominate $\varphi(D)$. So by induction

$$(\varphi|_D)_*(L \otimes \mathcal{O}_D \otimes \omega_D) \otimes (H \otimes \mathcal{O}_D)^{\otimes \dim Y}$$

is generically generated by its global sections. By adjunction $\omega_D \simeq \omega_X \otimes \varphi^* H \otimes \mathcal{O}_D$, so another application of [BP07, Thm. 5.1] shows that the restriction map

$$H^0(X, L \otimes \omega_X \otimes \varphi^* H^{\otimes \dim Y + 1} \otimes \mathcal{I}(h)) \to H^0(D, L \otimes \varphi^* H^{\otimes \dim Y} \otimes \mathcal{O}_D \otimes \omega_D \otimes \mathcal{I}(h|_D))$$

is surjective.

REMARK 3.36. The proof shows that we can only extend sections from smooth fibres X_y such that the multiplier ideal $\mathcal{I}(h|_{X_y})$ of the *restricted metric* is trivial. In particular we do not know whether $\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y + 1}$ is generated by its global sections on the φ-smooth locus minus the image of the cosupport of $\mathcal{I}(h)$. Nevertheless the proof shows that it is generated by its global sections on Y'', where

(3.37) $Y'' = \{y \in Y \mid X_y$ is smooth, $\mathcal{I}(h|_{X_y}) \simeq \mathcal{O}_{X_y}$, $\varphi_*(L \otimes \omega_X)$ is locally free in y and $\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y + 1} \otimes \mathcal{O}_y = H^0(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y})\}$.

This description is not very explicit, but still sufficient for what we need in the forthcoming proof.

The following statement is a reformulation of [BP07, Cor. 0.2]. While their proof uses the analytic language much more than we do, the key ingredient is the same: the Ohsawa-Manivel-Takegoshi theorem used in the proof of Lemma 3.35.

THEOREM 3.38. *Let $\varphi\colon X \to Y$ be a fibration from a projective manifold X onto a normal projective Gorenstein variety Y. Let L be a pseudo-effective line bundle over X and h a singular metric such that $\Theta_h(L) \geq 0$ and that the cosupport of $\mathcal{I}(h)$ does not dominate Y. Then $\varphi_*(L \otimes \omega_{X/Y})$ is weakly positive.*

Proof. We use the notation of Constructions 3.11 and 3.19, and fix a very ample line bundle H on Y. By the reduction Lemma 3.24, it is sufficient to find a Zariski open subset $Y'' \subset Y$ such that

$$\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$$

is globally generated on Y'' for all $s > 0$. In the case $s = 1$ it is clear that we satisfy the conditions of Lemma 3.35, so Remark 3.36 gives a set $Y'' \subset Y$ such that $\varphi_*(L \otimes \omega_X) \otimes H^{\otimes \dim Y+1}$ is generated by its global sections on Y''. We claim that the same set Y'' works for every $s > 0$: note first that since L is pseudo-effective the bundle $(\nu \circ r)^* L_s$ is pseudo-effective and we endow it inductively with the metric

$$h_s := (\nu \circ r)^*((\pi^s)^* h \otimes (\pi')^* h_{s-1}).$$

It is clear that $\Theta_{h_s}((\nu \circ r)^* L_s) \geq 0$, and we will now study[9] the multiplier ideal $\mathcal{I}(h_s)$.

Fix a point $y \in Y''$: since $(\varphi^s)^{-1}(Y'')$ is in the smooth locus of X^s, we can identify $X_y^{(s)}$ with X_y^s and

$$\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1} \otimes \mathcal{O}_y = \varphi_*^s(L_s \otimes \omega_{X^s}) \otimes H^{\otimes \dim Y+1} \otimes \mathcal{O}_y .$$

The fibre X_y is smooth and X_y^s is isomorphic to the s-fold product $X_y \times \cdots \times X_y$. Furthermore

$$L_s \otimes \mathcal{O}_{X_y^s} \simeq \bigotimes_{j=1}^{s} \pi_j^* L ,$$

where π_j is the projection on the j-th factor and

$$h_s|_{X_y^s} = \bigotimes_{j=1}^{s} \pi_j^* h|_{X_y} .$$

As $\mathcal{I}(h|_{X_y}) \simeq \mathcal{O}_{X_y}$, this implies by [Laz04b, Prop. 9.5.22] that

$$\mathcal{I}(h_s|_{X_y^s}) \simeq \bigotimes_{j=1}^{s} \pi_j^* \mathcal{I}(h|_{X_y}) \simeq \mathcal{O}_{X_y^s} .$$

[9]) In general it is not true that $\mathcal{I}(h_s) = (\nu \circ r)^{-1}((\pi^s)^{-1}(\mathcal{I}(h)) \otimes (\pi')^{-1}(\mathcal{I}(h_{s-1})))$.

Since

$$\mathcal{O}_{X_y^s} \simeq \mathcal{I}(h_s|_{X_y^s}) \subseteq \mathcal{I}(h_s) \otimes \mathcal{O}_{X_y^s} \subseteq \mathcal{O}_{X_y^s}$$

by the restriction theorem [Laz04b, Thm. 9.5.1], we see that the cosupport of $\mathcal{I}(h_s)$ does not dominate Y. Hence Lemma 3.35 applies and shows that $\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$ is generically generated by its global sections, i.e. on some open subset of Y. We will now show that it is actually generated by its global sections on Y''. Note that

$$H^0(X_y^s, L_s \otimes \mathcal{O}_{X_y^s} \otimes \omega_{X_y^s}) = \bigotimes_{j=1}^{s} \pi_j^* H^0(X_y, L \otimes \mathcal{O}_{X_y} \otimes \omega_{X_y}),$$

so the function $y \mapsto h^0(X_y^s, L_s \otimes \mathcal{O}_{X_y^s} \otimes \omega_{X_y^s})$ is constant on Y''. Thus $\varphi_*^s(L_s \otimes \omega_{X^s}) \otimes H^{\otimes \dim Y+1}$ is locally free in y and by Grauert's theorem [Har77, III, Cor. 12.9],

$$\varphi_*^s(L_s \otimes \omega_{X^s}) \otimes H^{\otimes \dim Y+1} \otimes \mathcal{O}_y = H^0(X_y^s, L_s \otimes \mathcal{O}_{X_y^s} \otimes \omega_{X_y^s}).$$

So all the conditions (3.37) hold for $\varphi^{(s)} \colon X^{(s)} \to Y$ and every point $y \in Y''$. Hence Remark 3.36 shows that $\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \omega_{X^{(s)}}) \otimes H^{\otimes \dim Y+1}$ is generated by global sections on Y''.

REMARK 3.39. A natural generalisation of Theorem 3.38 would be to prove that the direct image sheaf $\varphi_*(L \otimes \mathcal{I}(h) \otimes \omega_{X/Y})$ is weakly positive, even if the cosupport of $\mathcal{I}(h)$ dominates Y. Basically the proof should remain the same, but one has to make sure that an analogue of the morphism (3.23) still exists in this situation. More precisely we have to show the existence of a morphism

$$\tau^s \colon [\varphi_*^{(s)}((\nu \circ r)^* L_s \otimes \mathcal{I}(h_s) \otimes \omega_{X^{(s)}})] \to [\varphi_*(L \otimes \mathcal{I}(h) \otimes \omega_{X/Y})]^{[s]} \otimes \omega_Y$$

for every $s > 0$. Since the multiplier ideal sheaf $\mathcal{I}(h)$ is not locally free, the necessary base change argument would get even more technical.

4. PROJECTIVE MANIFOLDS WITH NEF ANTICANONICAL BUNDLE

We have seen in the preceding sections that the direct image of the relative dualising sheaf $\varphi_* \omega_{X/Y}$ has been intensively studied and often enjoys better properties than general direct image sheaves $\varphi_*(L \otimes \omega_{X/Y})$. It is natural to ask what happens in the case where L is the anticanonical bundle ω_X^{-1} of some

Gorenstein variety X. Indeed if $\varphi\colon X \to Y$ is a fibration between normal Gorenstein varieties X and Y, we have the trivial relation

$$\omega_Y^{-1} \simeq \varphi_*(\omega_X^{-1} \otimes \omega_{X/Y}),$$

so the positivity of ω_Y^{-1} can be studied via the direct image approach.

4.A General properties

The following example shows that even for a locally trivial fibration, this direct image is not necessarily positive.

Example 4.1 ([Zha06], p. 137). Let Y be a smooth projective curve of genus $g \geq 2$ and let M be a line bundle such that M^* has degree at least $3g-2$. Set $X := \mathbf{P}(\mathcal{O}_Y \ominus M)$ and denote by $\varphi\colon X \to Y$ the projection map. Then ω_X^{-1} has a non-zero section and is φ-ample, but

$$\omega_Y^{-1} \simeq \varphi_*(\omega_X^{-1} \otimes \omega_{X/Y})$$

is antiample.

Proof. By the canonical bundle formula we have

$$\omega_X^{-1} \simeq \varphi^*(\omega_Y^{-1} \otimes M^*) \otimes \mathcal{O}_{\mathbf{P}}(2),$$

so

$$H^0(X, \omega_X^{-1}) \simeq H^0(Y, \omega_Y^{-1} \otimes M^* \otimes S^2(\mathcal{O}_Y \oplus M)).$$

Since $\omega_Y^{-1} \otimes M^*$ is a line bundle of degree at least g on the curve Y, it has a non-zero section. Thus the right hand side is not zero.

Remark. Theorem 3.38 shows that there can only be one explanation for this "loss of positivity": the cosupport of the multiplier ideal sheaf associated to ω_X^{-1} dominates the base Y. Indeed the section corresponding to the quotient bundle

$$\mathcal{O}_Y \ominus M \to M$$

has strictly negative intersection with the anticanonical divisor $-K_X$.

The example suggests that one should start the study of the anticanonical bundle under a global assumption, like ω_X^{-1} is nef. In this case one can then refine the machinery developed in the preceding sections to obtain a good structure result:

THEOREM 4.2 ([Zha05], Main Theorem). *Let X be a projective manifold such that ω_X^{-1} is nef. Let $\varphi\colon X \dashrightarrow Y$ be a rational dominant fibration onto a smooth variety Y. Then either*

(1) *Y is uniruled (covered by rational curves); or*

(2) *the Kodaira dimension $\kappa(Y)$ is zero. Moreover in this case, the fibration has generically reduced fibres in codimension one.*

As an application, one gets the following corollary, conjectured by Demailly, Peternell and Schneider [DPS93, Conj. 2]:

COROLLARY 4.3 ([Zha05]). *Let X be a projective manifold such that ω_X^{-1} is nef. Then the Albanese morphism $\alpha\colon X \to \mathrm{Alb}(X)$ is surjective.*

Proof of the corollary. We argue by contradiction and denote by Y a desingularisation of the image of α. Since a torus does not contain any rational curve, it is clear that Y is not uniruled. Thus by Theorem 4.2 it has Kodaira dimension zero. A classical theorem by Ueno, later refined by Kawamata [Kaw81], shows that the image of α is contained in a proper subtorus of $\mathrm{Alb}(X)$. Thus $\alpha(X)$ does not generate $\mathrm{Alb}(X)$ as a group, a contradiction to the universal property of the Albanese torus.

Let us recall that a projective manifold X is *rationally connected* if two generic points x and y can be connected by a rational curve $f\colon \mathbf{P}^1 \to X$. Recall also that any uniruled projective manifold admits a fibration $\varphi\colon X' \dashrightarrow Y$ (the *rational quotient* or MRC-*fibration*, [Kol96, Thm. 5.4], [Cam04, Thm. 1.1]) such that the general fibres are rationally connected projective manifolds, moreover the base Y is not uniruled by the Graber-Harris-Starr theorem [GHS03]. Theorem 4.2 implies that the base Y has Kodaira dimension zero if $\omega_{X'}^{-1}$ is nef. Furthermore the general fibre is rationally connected with nef anticanonical bundle, so a classification theory naturally starts by looking at the rationally connected case.

4.B RATIONALLY CONNECTED MANIFOLDS

Rationally connected manifolds with nef anticanonical bundle are a natural generalisation of *Fano manifolds* (ω_X^{-1} ample), so there is some hope that some of the properties known for Fanos are still valid in this more general setting. In dimension one, the only example is $\mathbf{P}^1$ and in dimension two the Enriques classification leads to a finite, well-understood list (cf. [BP04]). This changes radically as soon as we look at projective threefolds, where a number of basic questions are still wide open. Before one can write down an explicit classification of these manifolds, one should check that there are only finitely many cases. More precisely one should try to answer the fundamental question:

QUESTION 4.4 ([JPR06], Ch. 06). *Is the functor of deformations of projective rationally connected manifolds with nef anticanonical bundle bounded, i.e. are there only finitely many deformation families in every dimension?*

If the anticanonical bundle is nef and big, it is semiample by the base-point free theorem [Kaw84, Rei83, KM98], so it is possible to study this question by looking at the *anticanonical morphism*

$$\varphi_{|-mK_X|} \colon X \to X'$$

for m sufficiently large. This allows one to prove boundedness for threefolds with nef and big anticanonical bundle, at least in dimension three [Bor01, KMMT00, McK02]. If ω_X^{-1} is nef but not big, there are examples where ω_X^{-1} is not semiample [BP04].

In this case, a natural approach to Question 4.4 comes from the minimal model program: since X is a smooth projective rationally connected manifold, there exists an *elementary Mori contraction* $\mu \colon X \to Y$, i.e. a morphism with connected fibres onto a normal variety Y such that the anticanonical bundle ω_X^{-1} is μ-ample and which satisfies the following condition: there exists a rational curve $C \subset X$ such that a curve $C' \subset X$ is contracted by μ if and only if we have an equality in $N_1(X)$

$$[C'] = \lambda\, [C] \qquad \lambda \in \mathbf{Q}^+ .$$

By $N_1(X)$ we denote the $\mathbf{Q}$-vector space of 1-cycles on X modulo numerical equivalence (cf. [Deb01, 1.3]). If the contraction μ is *of fibre type* (i.e. $\dim Y < \dim X$), Theorem 3.30 tells us that the anticanonical bundle of Y is at least weakly positive and even nef if X has dimension three (Corollary 3.31). Classification theory then allows one to show that this leads to a bounded situation [BP04].

The situation is much more tricky if the contraction μ is birational. In dimension three, Demailly, Peternell and Schneider [DPS93, Thm. 3.8] have shown that ω_Y^{-1} is nef unless μ is a birational contraction that contracts a divisor E onto a smooth rational curve $C \subset Y$ such that

$$N_{C/Y} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$$

or

$$N_{C/Y} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}.$$

In the first case, one can use a flipping construction to show that X is birational to a $\mathbf{Q}$-Fano variety (ibid.), so we get boundedness in this case. In the second case we do not have this possibility and the general hope was that these $(-2,-2)$-contractions do not exist or appear only in very special situations. We will now give some examples that destroy this hope. The basic construction is given by the following rather straightforward lemma.

LEMMA 4.5. *Let Y be a smooth projective threefold that contains a finite number of disjoint smooth rational curves $C_1, \ldots, C_r$ such that*

$$N_{C_i/Y} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}$$

for every $i \in \{1, \ldots, r\}$ and such that the base locus

$$Bs(|\omega_Y^{-1}|) = C_1 \cup \ldots \cup C_r.$$

Let $\mu\colon X \to Y$ be the blow-up of Y along the curves $C_1 \cup \ldots \cup C_r$. Then ω_X^{-1} is nef.

Proof. Denote by E_i the exceptional divisors of the blow-up, then

$$E_i \simeq \mathbf{P}(N^*_{C_i/Y}) \simeq \mathbf{P}^1 \times \mathbf{P}^1$$

and we fix the notation that the restriction of $\mu\colon X \to Y$ to E_i is the projection on the first factor of the product. With this convention and using the standard notation for line bundles on $\mathbf{P}^1 \times \mathbf{P}^1$, we have

$$\mathcal{O}_{E_i}(E_i) \simeq N_{E_i/X} \simeq \mathcal{O}_{\mathbf{P}(N^*_{C_i/Y})}(-1) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-2,-1).$$

We have

$$\omega_X^{-1} \simeq \mu^* \omega_Y^{-1} \otimes \mathcal{O}_X(-\textstyle\sum_{i=1}^r E_i),$$

so

$$H^0(X, \omega_X^{-1}) \simeq H^0(Y, \omega_Y^{-1} \otimes \mathcal{I}_{C_1 \cup \ldots \cup C_r}) \subset H^0(Y, \omega_Y^{-1}).$$

Yet $C_1 \cup \ldots \cup C_r$ is in the base locus of $|\omega_Y^{-1}|$, so we have

$$H^0(X, \omega_X^{-1}) \simeq H^0(Y, \omega_Y^{-1}).$$

In particular the base locus of $|\omega_X^{-1}|$ is contained in $\mu^{-1}(C_1 \cup \ldots \cup C_r) = \bigcup_{i=1}^r E_i$. Furthermore

$$\omega_X^{-1}|_{E_i} \simeq \mu^*(\omega_Y^{-1}|_{C_i}) \otimes \mathcal{O}_{E_i}(-E_i) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-2,0) \otimes \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,1) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(0,1),$$

so $\omega_X^{-1}|_{E_i}$ is nef. Let now D be an irreducible curve in X, then there are two cases. If D is not contained in any of the E_i, it is not in the base locus of $|\omega_X^{-1}|$, so clearly $-K_X \cdot D \geq 0$. If D is contained in some E_i, we conclude by the nefness of $\omega_X^{-1}|_{E_i}$.

The lemma reduces the question to constructing a threefold Y whose anticanonical bundle is globally generated except on a finite number of rational curves. After mailing my first example to Cinzia Casagrande, she immediately replied with the following very simple one.

EXAMPLE 4.6. Set $X := \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}\big)$ and denote by $\varphi\colon X \to \mathbf{P}^1$ the natural projection. The section $C_0 \subset X$ corresponding to the quotient bundle

$$(*) \qquad \mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbf{P}^1}(-2)$$

has normal bundle $\mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}$. Furthermore

$$\omega_X^{-1} \simeq \mathcal{O}_{\mathbf{P}}(2)\,,$$

so

$$H^0(X, \omega_X^{-1}) \simeq H^0\big(\mathbf{P}^1, S^2(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 2})\big)\,.$$

The vector bundle $S^2\big(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}\big)$ is globally generated except in the direction corresponding to $(*)$. Conclude with Lemma 4.5.

Thus we see that $(-2,-2)$-contractions exist, but this is not a problem as long as the anticanonical bundle of $X = \mathrm{Bl}_C Y$ is nef *and big*. In the case of Example 4.6, we have $(-K_Y)^3 = 54$. Adapting the formulas in [JPR05, p. 28], one sees that if X is the blow-up of Y in a $(-2,-2)$-curve, then

$$(4.7) \qquad (-K_X)^3 = (-K_Y)^3 + 2\,.$$

Thus $(-K_X)^3 = 56 > 0$ and ω_X^{-1} is nef and big. In order to get an example with $(-K_X)^3 = 0$, we need a more involved construction:

EXAMPLE 4.8. Let $p_1, \dots, p_8$ be an *almost general* configuration of not infinitely near points in $\mathbf{P}^2$, i.e. no three points except p_1, p_2, p_3 are collinear and there is no conic through more than five points. The pencil of cubics through these eight points has exactly one unassigned base point p_9 [Har77, Cor. 4.5], and we denote by S the blow-up of $\mathbf{P}^2$ in these nine points. Let D_1 be the strict transform of the line through p_1, p_2, p_3.

Set $Y := \mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(D_1))$ and denote by $\varphi\colon Y \to S$ the natural projection. Then there exist exactly two smooth rational curves $C_i \subset Y$ such that

$$N_{C_i/Y} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}\,.$$

The curves C_1 and C_2 are disjoint and the base locus of $|\omega_Y^{-1}|$ is equal to $C_1 \cup C_2$. Let $\mu\colon X \to Y$ be the blow-up of Y along the curves $C_1 \cup C_2$. Then ω_X^{-1} is nef and $(-K_X)^3 = 0$.

Proof. Note first that blowing up the points $p_1, \ldots, p_9$ resolves the indeterminacies of the map induced by the pencil, so we have a fibration $f\colon S \to \mathbf{P}^1$ such that $\omega_S^{-1} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$. The curve D_1 is a -2-curve, so it is contracted by f, and it is elementary to see that

$$f^{-1}(f(D_1)) = D_1 + D_2\,,$$

where D_2 is the strict transform of the unique conic through $p_4, \ldots, p_9$. Note that the genericity condition implies that the conic D_2 is smooth, since otherwise three of the six points would be collinear. Furthermore we have

$$D_2^2 = -2\,,\ D_1 \cdot D_2 = 2\,.$$

We denote by S_1 (resp. S_2) the two disjoint φ-sections corresponding to the quotient line bundles $\mathcal{O}_S \oplus \mathcal{O}_S(D_1) \to \mathcal{O}_S(D_1)$ (resp. $\mathcal{O}_S \oplus \mathcal{O}_S(D_1) \to \mathcal{O}_S$). For $i \in \{1, 2\}$, set $Y_{D_i} := \varphi^{-1}(D_i)$. Then Y_{D_i} is isomorphic to the second Hirzebruch surface $\mathbf{F}_2$ and the curve

$$C_i := Y_{D_i} \cap S_i$$

is the unique section of $\varphi|_{Y_{D_i}}$ such that $N_{C_i/Y_{D_i}} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)$. Since

$$N_{Y_{D_i}/Y} \simeq (\varphi|_{Y_{D_i}})^* N_{C_i/S} \simeq (\varphi|_{Y_{D_i}})^* \mathcal{O}_{\mathbf{P}^1}(-2)\,,$$

the exact sequence

$$0 \to N_{C_i/Y_{D_i}} \to N_{C_i/Y} \to N_{Y_{D_i}/Y}|_{C_i} \to 0$$

splits, hence $N_{C_i/Y} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}$. Since S_1 and S_2 are disjoint, the curves C_1 and C_2 are disjoint.

Thus we are left to prove that the base locus of $|\omega_Y^{-1}|$ is exactly $C_1 \cup C_2$. By the canonical bundle formula we have

$$\omega_Y^{-1} \simeq \varphi^* \mathcal{O}_S(D_2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(D_1))}(2)\,.$$

Denote by F a general f-fibre, then

$$\begin{aligned} \varphi_* \omega_Y^{-1} &\simeq \mathcal{O}_S(D_2) \otimes S^2(\mathcal{O}_S \oplus \mathcal{O}_S(D_1)) \\ &\simeq \mathcal{O}_S(D_2) \oplus \mathcal{O}_S(F) \oplus \mathcal{O}_S(F + D_1) \end{aligned}$$

is clearly generated by its global sections on $S \setminus (D_1 \cup D_2)$. Since ω_Y^{-1} is relatively generated by global sections the natural map

$$\varphi^* \varphi_* \omega_Y^{-1} \to \omega_Y^{-1}$$

is surjective, so ω_Y^{-1} is generated by its global sections on $Y \setminus (Y_{D_1} \cup Y_{D_2})$.

We will now show that the intersection of Y_{D_1} with the base locus of $|\omega_Y^{-1}|$ is exactly $C_1 \cup (Y_{D_1} \cap C_2)$; the corresponding statement for Y_{D_2} can be shown analogously. Together they imply our claim on the base locus. From the computation above, we get $h^0(Y, \omega_Y^{-1}) = 5$ and an analogous computation shows that

$$h^0(Y, \omega_Y^{-1} \otimes \mathcal{O}_Y(-Y_{D_1})) = h^0(S, \varphi_* \omega_Y^{-1} \otimes \mathcal{O}_S(-D_1)) = 3 .$$

Therefore the image of the restriction map

$$r \colon H^0(Y, \omega_Y^{-1}) \to H^0(Y_{D_1}, \omega_Y^{-1} \otimes \mathcal{O}_{Y_{D_1}})$$

has dimension two and we will now describe its geometry. Consider the restricted line bundle

$$\begin{aligned} \omega_Y^{-1} \otimes \mathcal{O}_{Y_{D_1}} &\simeq \varphi^* \mathcal{O}_S(D_2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(D_1))}(2) \otimes \mathcal{O}_{Y_{D_1}} \\ &\simeq (\varphi|_{Y_{D_1}})^* \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))}(2) . \end{aligned}$$

The fixed part of the linear system $|\omega_Y^{-1} \otimes \mathcal{O}_{Y_{D_1}}|$ is C_1 and

$$(\varphi|_{Y_{D_1}})^* \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))}(1)$$

is base-point free. Any section of $\omega_Y^{-1} \otimes \mathcal{O}_{Y_{D_1}}$ that extends to Y vanishes on $C_1 \cup (Y_{D_1} \cap C_2)$, thus

$$\text{(4.9)} \quad \operatorname{im} r \subseteq |H^0(Y_{D_1}, (\varphi|_{Y_{D_1}})^* \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))}(1) \otimes \mathcal{I}_{Y_{D_1} \cap C_2})| + C_1 .$$

The divisors corresponding to a general element of the linear system $|(\varphi|_{Y_{D_1}})^* \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))}(1)|$ are rational curves with normal bundle $\mathcal{O}_{\mathbf{P}^1}(2)$. Since $D_1 \cdot D_2 = 2$, the surface Y_{D_1} and the curve C_2 meet exactly in two points. These points are not on the -2-curve $C_1 \subset Y_{D_1}$, so elementary considerations on the second Hirzebruch surface show that

$$h^0(Y_{D_1}, (\varphi|_{Y_{D_1}})^* \mathcal{O}_{\mathbf{P}^1}(2) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))}(1) \otimes \mathcal{I}_{Y_{D_1} \cap C_2}) = 2$$

and the linear system has no base points except $Y_{D_1} \cap C_2$. Thus the inclusion (4.9) is an equality and the base locus of $\operatorname{im} r$ is $C_1 \cup (Y_{D_1} \cap C_2)$.

Finally let us check that X has the stated properties: by Lemma 4.5, the anticanonical bundle of X is nef. By the Chern-Wu formula for Chern classes on projectivised vector bundles [Har77, App. A, 3]

$$c_1(\mathcal{O}_{\mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(D_1))}(1))^2 - c_1(\varphi^* \mathcal{O}_S(D_1)) \cdot c_1(\mathcal{O}_{\mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(D_1))}(1)) = 0 ,$$

so an easy computation shows that $(-K_Y)^3 = -4$. Applying Formula (4.7) twice, we see that $(-K_X)^3 = 0$.

Example 4.8 shows that in order to answer Question 4.4, we must consider projective manifolds [10]) whose anticanonical bundle is *nef in codimension one*, i.e. the non-nef locus of the anticanonical bundle is a finite union of subvarieties of codimension at least two. Being very optimistic we could even ask:

Do we have boundedness of the functor of deformations of projective rationally connected manifolds whose anticanonical bundle is nef in codimension one ?

Well, the answer to this question is no, and once more the direct image point of view gives us a hint how to construct a counterexample: if $\varphi\colon X \to S$ is a $\mathbf{P}^r$-bundle over a surface such that ω_X^{-1} is nef, then ω_S^{-1} is nef (Corollary 3.31). Thus if we construct a $\mathbf{P}^r$-bundle X over a surface S such that ω_S^{-1} is not nef, we have a good candidate and all we have to ensure is that the non-nef locus does not get too big.

EXAMPLE 4.10. Let $C \subset \mathbf{P}^2$ be a smooth cubic. Let $\mu\colon S \to \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ in d points $p_1, \ldots, p_d$ lying on the cubic curve C. Denote still by $C \subset S$ the strict transform of the cubic. Since

$$\omega_S^{-1} \simeq \mu^* \omega_{\mathbf{P}^2}^{-1} \otimes \mathcal{O}_S\big(-\textstyle\sum_{i=1}^d \mu^{-1}(p_i)\big),$$

we have

$$H^0(S, \omega_S^{-1}) \simeq H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_{p_1 \cup \ldots \cup p_d}).$$

Thus

$$C \in |\omega_S^{-1}|$$

and if we blow up at least ten points, we have

$$(-K_S)^2 = 9 - d < 0,$$

so C is the only element in the anticanonical system.

Set $X = \mathbf{P}(\mathcal{O}_S^{\oplus r-1} \oplus \mathcal{O}_S(C))$, then $\varphi\colon X \to S$ is a $\mathbf{P}^{r-1}$-bundle over S and has second Betti number $d+2$. We claim that the anticanonical sheaf ω_X^{-1} is globally generated on the complement of a unique [11]) curve $C_0 \subset X$ such that $-K_X \cdot C_0 < 0$.

---

[10]) Very probably, we shall also have to admit some singularities coming from the minimal model program.

[11]) Note that the *non-nef locus* in the sense of [Tak08] is in general not the union of the curves that are $-K_X$-negative. In our case we nevertheless have $\mathrm{NNef}(-K_X) = C_0$, so $-K_X$ is nef in codimension $\dim X - 2$.

Proof of the claim. By the canonical bundle formula

$$\omega_X^{-1} \simeq \mathcal{O}_{\mathbf{P}(\mathcal{O}_S^{\oplus r-1}\oplus\mathcal{O}_S(C))}(r),$$

so it is sufficient to show the claim for $\mathcal{O}_{\mathbf{P}(\mathcal{O}_S^{\oplus r-1}\oplus\mathcal{O}_S(C))}(1)$. The vector bundle $\mathcal{O}_S^{\oplus r-1} \oplus \mathcal{O}_S(C)$ is clearly generated by its global sections on the complement of the curve C. Furthermore we have $r-1$ global sections vanishing on a divisor corresponding to a quotient bundle

$$\mathcal{O}_S^{\oplus r-1} \oplus \mathcal{O}_S(C) \to \mathcal{O}_S^{\oplus r-2} \oplus \mathcal{O}_S(C).$$

The intersection of these $r-1$ divisors is the φ-section Z corresponding to the quotient bundle

$$\mathcal{O}_S^{\oplus r-1} \oplus \mathcal{O}_S(C) \to \mathcal{O}_S(C),$$

so the base locus of $\mathcal{O}_{\mathbf{P}(\mathcal{O}_S^{\oplus r-1}\oplus\mathcal{O}_S(C))}(1)$ is contained in the intersection of $\varphi^{-1}(C)$ and Z which is an integral curve $C_0 \simeq C$. Since

$$\mathcal{O}_{\mathbf{P}(\mathcal{O}_S^{\oplus r-1}\oplus\mathcal{O}_S(C))}(1) \otimes \mathcal{O}_{C_0} \simeq \mathcal{O}_C(C) \simeq N_{C/S},$$

and $N_{C/S}$ has degree $(-K_S)^2 < 0$, the intersection number $c_1(\mathcal{O}_{\mathbf{P}}(1)) \cdot C_0$ is negative.

Example 4.10 shows that the second Betti number of a projective manifold X of dimension n such that $-K_X \cdot C \geq 0$ for all integral curves $C \subset X$ except one can be arbitrarily high. Since the second Betti number is an invariant under deformation of compact complex manifolds, this shows that the corresponding functor of deformations is not bounded.

Note that Example 4.10 does not necessarily indicate that the answer to Question 4.4 is negative. In the example, the curve C_0 such that $-K_X \cdot C_0 < 0$ is elliptic while the result of Demailly, Peternell and Schneider [DPS93, Thm. 3.8] indicates that the obstructions arising from elementary Mori contractions are rational curves. It is possible to refine the construction to get an example of a projective manifold X such that ω_X^{-1} is globally generated except on one rational curve $C_0 \subset X$ that satisfies

$$N_{C_0/X} \simeq \mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(-2d)$$

with $d \in \mathbf{N}$ arbitrary. Adapting Lemma 4.5, one might hope to get a counterexample via an iterated blow-up. Unfortunately, the normal bundle of the $-K_X$-negative curves produced in this process will not be of the form prescribed by [DPS93, Thm. 3.8], so we will not end up with a manifold whose anticanonical bundle is nef. Summa summarum, the initial problem remains open and a step towards a better understanding should be given by an answer to the following question:

QUESTION 4.11. *What is the smallest class of varieties that contains the projective rationally connected manifolds with nef anticanonical bundle and is stable under the minimal model program?*

5. A TECHNICAL APPENDIX

5.A REFLEXIVE SHEAVES

Locally free sheaves, although very convenient, are too restricted for our purposes since a direct image sheaf is not necessarily locally free, even if the fibration is smooth. We will see that *reflexive sheaves* are the appropriate framework to fill this gap. Our exposition follows closely [Har80, Ch. 1].

DEFINITION 5.1. Let X be an integral scheme and $\mathcal{S}$ a coherent sheaf on X. We define the *dual* of $\mathcal{S}$ by

$$\mathcal{S}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{S}, \mathcal{O}_X)$$

and the *bidual* of $\mathcal{S}$ by $\mathcal{S}^{**} := (\mathcal{S}^*)^*$. A sheaf $\mathcal{S}$ is *torsion-free* if the natural map

$$\mathcal{S} \to \mathcal{S}^{**}$$

is injective. It is *reflexive* if it is an isomorphism.

REMARK. Let $\operatorname{Tor}\mathcal{S} \subset \mathcal{S}$ be the *torsion subsheaf* of a coherent sheaf $\mathcal{S}$ (cf. definition in [Kob87, p. 159]). It is elementary to see that $\mathcal{S}$ is torsion-free if and only if $\operatorname{Tor}\mathcal{S} = 0$ (ibid).

By upper semicontinuity [Har77, II, Ex. 5.8] there exists a non-empty open subset $X_0 \subset X$ such that $\mathcal{S}|_{X_0}$ is locally free. It is important to give a lower bound on the codimension of $X \setminus X_0$.

NOTATION 5.2. Let X be an integral scheme, and let $\mathcal{S}$ be a coherent sheaf on X. We say that $\mathcal{S}$ is *locally free in codimension k* if there exists a closed subvariety $Z \subset X$ such that $\operatorname{codim}_X Z \geq k+1$ and $\mathcal{S}|_{X \setminus Z}$ is locally free.

REMARK. It is well known that on a smooth variety a torsion-free sheaf is locally free in codimension one (cf. [Kob87, Cor. 5.15]) and a reflexive sheaf is locally free in codimension two (cf. [Har80, Cor. 1.4]). Since a normal variety is regular in codimension one, this immediately implies

PROPOSITION 5.3. *Let X be a normal variety, and $\mathcal{S}$ a torsion-free sheaf on X. Then $\mathcal{S}$ is locally free in codimension one.*

DEFINITION 5.4. A coherent sheaf $\mathcal{S}$ on an irreducible scheme X is *normal* if for every open set $X_0 \subset X$ and every closed subset $Z \subset X_0$ such that $\operatorname{codim}_{(X_0)} Z \geq 2$, the restriction map

$$\Gamma(\mathcal{S}, X_0) \to \Gamma(\mathcal{S}, X_0 \setminus Z)$$

is bijective.

The next lemma gives a useful characterisation of reflexive sheaves on a normal variety.

LEMMA 5.5 ([Har80]). *Let $\mathcal{S}$ be a coherent sheaf on a normal variety X. The following conditions are equivalent:*

(1) *$\mathcal{S}$ is reflexive;*

(2) *$\mathcal{S}$ is torsion-free and normal;*

(3) *$\mathcal{S}$ is torsion-free, and for each open set $X_0 \subset X$ and each closed subset $Z \subset X_0$ such that $\operatorname{codim}_{(X_0)} Z \geq 2$, we have $\mathcal{S}|_{X_0} \simeq j_* \mathcal{S}|_{X_0 \setminus Z}$, where $j \colon X_0 \setminus Z \to X_0$ is the inclusion map.*

In particular if $\mathcal{S}$ and $\mathcal{F}$ are reflexive sheaves on X and there exists a closed set $Z \subset X$ such that $\operatorname{codim}_X Z \geq 2$ and $\mathcal{S}|_{X \setminus Z} \simeq \mathcal{F}|_{X \setminus Z}$, then $\mathcal{S} \simeq \mathcal{F}$.

NOTATION 5.6. In the situation above, let $\mathcal{S}$ and $\mathcal{F}$ be coherent sheaves on X such that there exists a closed set $Z \subset X$ such that $\operatorname{codim}_X Z \geq 2$ and $\mathcal{S}|_{X \setminus Z} \simeq \mathcal{F}|_{X \setminus Z}$. Then we say that $\mathcal{S}$ and $\mathcal{F}$ are *isomorphic in codimension one.*

COROLLARY 5.7. *Let X be a normal variety, and let $\mathcal{F}$ and $\mathcal{S}$ be coherent sheaves on X. If $\mathcal{S}$ is reflexive, then $\mathcal{H}om(\mathcal{F}, \mathcal{S})$ is reflexive.*

In particular if there exists a closed set $Z \subset X$ such that $\operatorname{codim}_X Z \geq 2$ and a morphism $\mathcal{F}|_{X \setminus Z} \to \mathcal{S}|_{X \setminus Z}$, the morphism extends to a unique morphism $\mathcal{F} \to \mathcal{S}$ on X.

Proof. The same statement for X smooth is [Kob87, Prop. 5.23], the proof goes through without changes.

PROPOSITION 5.8 ([Har80], Cor. 1.7, Prop. 1.8). *Let $\varphi\colon X \to Y$ be a morphism between normal varieties.*

If φ is equidimensional and dominant, and $\mathcal{S}$ is a reflexive sheaf on X, the direct image $\varphi_* \mathcal{S}$ is reflexive.

If φ is flat and $\mathcal{S}$ is a reflexive sheaf on Y, the pull-back $\varphi^* \mathcal{S}$ is reflexive.

It is a well-known and basic fact [Har77, II, Cor. 6.16] that on a smooth variety there is a bijection between linear equivalence classes of Weil divisors and isomorphism classes of invertible sheaves. On a normal variety this is no longer true since a Weil divisor is no longer necessarily Cartier. Nevertheless if we denote by X_{reg} the nonsingular locus of X, then by [Har77, II, Prop. 6.5] we can identify the divisor class groups

$$\mathrm{Cl}(X) = \mathrm{Cl}(X_{\mathrm{reg}}).$$

Thus, given a Weil divisor D, we can associate a coherent sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D) := j_* \mathcal{O}_{X_{\mathrm{reg}}}(D|_{X_{\mathrm{reg}}}),$$

where $j\colon X_{\mathrm{reg}} \to X$ is the inclusion. By Lemma 5.5 the sheaf $\mathcal{O}_X(D)$ is reflexive. On the other hand given a reflexive sheaf $\mathcal{F}$ of rank 1 on X, there exists by Proposition 5.3 an open subset $X_0 \subset X$ such that $\mathrm{codim}_X(X \setminus X_0) \geq 2$ and $\mathcal{F} \otimes \mathcal{O}_{X_0}$ is locally free. So there exists a Weil divisor D' on X_0 such that $\mathcal{F} \otimes \mathcal{O}_{X_0} = \mathcal{O}_{X_0}(D')$. Since $\mathrm{Cl}(X) = \mathrm{Cl}(X_0)$, we can see D' as a Weil divisor on X. Therefore we have $\mathcal{O}_X(D') \simeq \mathcal{F}$ by Lemma 5.5. This shows that on a normal variety we have a bijection between divisor classes and reflexive rank 1 sheaves modulo multiplication by non-vanishing functions. Unfortunately this bijection is not an isomorphism of $\mathbf{Z}$-modules, since the class of reflexive sheaves is not closed under the tensor product. In particular

$$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \simeq \mathcal{O}_X(D_1 + D_2)$$

does *not* hold in general; we have to take the bidual on the left hand side to obtain an isomorphism. We sum up these observations in the following

PROPOSITION 5.9 ([Rei80], App., Thm. 3). *Let X be a normal variety, then the correspondence*

$$\delta\colon D \mapsto \mathcal{O}_X(D),$$

where D is a Weil divisor, induces a bijection

$$\delta\colon \mathrm{Cl}(X) \to \{\textit{reflexive rank 1 sheaves}\}/H^0(X, \mathcal{O}_X^*).$$

This bijection is an isomorphism of $\mathbf{Z}$-modules if one defines the product of two reflexive sheaves $\mathcal{F}$ and $\mathcal{S}$ of rank one by $(\mathcal{F} \otimes \mathcal{S})^{**}$.

5.B Singularities

We start with an introduction to the zoology of singular varieties. All these notions are well-known to the experienced researcher, but often confusing for the young algebraic geometer. Therefore we will build up the notions as systematically as possible, but won't give any proofs.

Definition 5.10. Let X be a scheme. We denote by X_{reg} the open subscheme such that for $p \in X_{\mathrm{reg}}$, the local ring $\mathcal{O}_{X,p}$ is regular. We denote by $X_{\mathrm{sing}} = X \setminus X_{\mathrm{reg}}$ the *singular locus* of X.

If X is irreducible and $\operatorname{codim}_X X_{\mathrm{sing}} \geq k$, we say that X is *regular in codimension* $k-1$.

Definition 5.11 ([Mat89], p. 134). A scheme is *Cohen-Macaulay* if all the local rings $\mathcal{O}_{X,p}$ are Cohen-Macaulay rings in the sense of commutative algebra, that is

$$\operatorname{Depth} \mathcal{O}_{X,p} = \dim \mathcal{O}_{X,p}\,.$$

Here the *depth of a local ring* is the maximal length of a regular sequence in $\mathcal{O}_{X,p}$: a sequence $x_1, \ldots, x_r$ is *regular* if x_1 is not a zero divisor in $\mathcal{O}_{X,p}$ and for all $i \in \{2, \ldots, r\}$, the image of x_i in $\mathcal{O}_{X,p}/(x_1, \ldots, x_{i-1})$ is not a zero divisor. The *dimension* $\dim \mathcal{O}_{X,p}$ is the Krull dimension of the local ring.

Example ([Har77], II, Prop. 8.23). Any local complete intersection in a manifold is Cohen-Macaulay.

Serre's criterion gives us a convenient way to check if an integral scheme is normal.

Theorem 5.12 ([Har77], II, Thm. 8.22A). *Let X be an integral Cohen-Macaulay scheme. Then X is normal if and only if it is regular in codimension* 1.

The Cohen-Macaulay condition will be very useful for us, since it ensures the existence of a relative dualising sheaf (cf. Section 5.C). For some arguments it is nevertheless necessary to suppose the stronger Gorenstein property. We will see that for all the questions we are interested in, it makes no difference to work with smooth or the more general Gorenstein varieties.

DEFINITION 5.13. A scheme of pure dimension n is *Gorenstein* if and only if it is Cohen-Macaulay and the dualising sheaf ω_X is invertible.

EXAMPLE. Any effective Weil divisor D in a manifold X is Gorenstein. Indeed by adjunction we have $\omega_D \simeq \omega_X \otimes \mathcal{O}_D(D)$.

DEFINITION 5.14. A fibration $\varphi\colon X \to Y$ is a *flat Cohen-Macaulay* (resp. *Gorenstein) fibration* if it is flat and X is an irreducible Cohen-Macaulay (resp. Gorenstein) scheme.

The following theorem shows that the Cohen-Macaulay (resp. Gorenstein) condition is well-behaved under flat maps.

THEOREM 5.15 ([Mat89], Cor. 23.3, Thm. 23.4). *Let $\varphi\colon X \to Y$ be a flat morphism. Then X is Cohen-Macaulay (resp. Gorenstein) if and only if Y and every φ-fibre is Cohen-Macaulay (resp. Gorenstein).*

DEFINITION 5.16. A normal variety X has *at most rational singularities* if X is Cohen-Macaulay and there exists a desingularisation $r\colon X' \to X$ such that

$$r_*\omega_{X'} \simeq \omega_X .$$

REMARK. Note that if X has rational singularities, then every desingularisation satisfies the condition from the definition. By [KKMS73] and [Elk81] the definition above is equivalent to asking that there exists a desingularisation $r\colon X' \to X$ such that $r_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ and

$$R^i r_* \mathcal{O}_{X'} = 0 \qquad \forall\ i > 0 .$$

It is clear from the definition that the subset $\mathrm{Irr}(X) \subset X$ where X has non-rational singularities is closed and we will call it the *irrational locus* of X.

5.C COHERENT SHEAVES AND DUALITY THEORY

We recall the basics of duality theory, for proofs we refer to [Kle80]. The crucial result that we will apply frequently is the technical Corollary 5.24.

For the whole section, we fix the notation: let $\varphi\colon X \to Y$ and $\psi\colon Y' \to Y$ be morphisms of schemes, then we have the base change diagram for $X' := X \times_Y Y'$:

$$\begin{array}{ccc} X' & \xrightarrow{\psi'} & X \\ {\scriptstyle \varphi'}\downarrow & \searrow^{\eta} & \downarrow{\scriptstyle \varphi} \\ Y' & \xrightarrow{\psi} & Y . \end{array}$$

DEFINITION 5.17 ([Kle80], Def. 10). Let $\varphi\colon X \to Y$ be a flat projective morphism of schemes. Denote by $\mathcal{E}xt^m_\varphi$ the m-th derived functor of $\varphi_* \mathcal{H}om_X$. We say that *relative duality holds for the morphism* φ if there exists a quasi-coherent sheaf $\omega_{X/Y}$ such that the natural map

$$D^m\colon \mathcal{E}xt^m_\varphi(\mathcal{S}, \omega_{X/Y} \otimes \varphi^* \mathcal{T}) \to \mathcal{H}om_Y(R^{r-m}\varphi_* \mathcal{S}, \mathcal{T})$$

is an isomorphism for all $0 \le m \le r := \dim X - \dim Y$, and $\mathcal{S}$ a quasi-coherent sheaf on X, and $\mathcal{T}$ a quasi-coherent sheaf on Y. In this case we call $\omega_{X/Y}$ the *relative dualising sheaf.*

THEOREM 5.18 ([Kle80], Thm. 21, [Kle80], Prop. 9). *Let $\varphi\colon X \to Y$ be a flat projective morphism of schemes. Then relative duality holds if and only if all the fibres are Cohen-Macaulay. In this case the relative dualising sheaf $\omega_{X/Y}$ is flat over Y. Let $\psi\colon Y' \to Y$ be any (not necessarily flat) base change, then relative duality holds for the flat morphism $\varphi'\colon X' \to Y'$ and*

$$\omega_{X'/Y'} \simeq (\psi')^* \omega_{X/Y} . \tag{5.19}$$

THEOREM 5.20 ([Kle80], p. 58; see also Theorem 5.15). *Let $\varphi\colon X \to Y$ be a flat projective morphism. If X is Cohen-Macaulay and Y is Gorenstein, relative duality holds and*

$$\omega_{X/Y} \simeq \omega_X \otimes \varphi^* \omega_Y^{-1} . \tag{5.21}$$

This allows us to extend slightly the definition of the relative dualising sheaf to the non-flat case.

DEFINITION 5.22. Let $\varphi\colon X \to Y$ be a fibration such that X is quasi-projective Cohen-Macaulay and Y is Gorenstein. Then we set $\omega_{X/Y} := \omega_X \otimes \varphi^* \omega_Y^{-1}$.

REMARK 5.23. We will call $\omega_{X/Y}$ the *relative dualising sheaf*, even in the non-flat setting. Note that this is a heavy abuse of language, since $\omega_{X/Y}$ is definitely not a relative dualising sheaf in the sense of our definition. In particular $\omega_{X/Y}$ does not have good properties with respect to base change, fibre products etc.

COROLLARY 5.24. *Let $\varphi\colon X \to Y$ and $\psi\colon Y' \to Y$ be flat Cohen-Macaulay fibrations. Then η is a flat Cohen-Macaulay fibration. Furthermore we have*

$$\omega_{X'/Y} = (\varphi')^* \omega_{Y'/Y} \otimes (\psi')^* \omega_{X/Y}. \tag{5.25}$$

If X and Y' are Gorenstein, X' is Gorenstein. If X and Y' are integral, X' is integral.

Proof. STEP 1: *the Cohen-Macaulay case.* Since X (resp. Y') is Cohen-Macaulay, the fibres of the morphism φ (resp. ψ) are Cohen-Macaulay by Theorem 5.15, so relative duality holds for ψ' by Theorem 5.18. So all the fibres of the induced morphism $\psi'\colon X' \to X$ are Cohen-Macaulay. Since X is Cohen-Macaulay, this implies by Theorem 5.15 that X' is Cohen-Macaulay. Since the general fibres of φ and ψ are irreducible, this holds for the general fibre of η. By [Cam04, Lemma 1.10] this shows that there exists an open subset $Y^* \subset Y$ such that $\eta^{-1}(Y^*)$ is irreducible. Since η is flat, it is an open mapping, so $\eta^{-1}(Y^*)$ is dense in X'. This shows the irreducibility of X'. By [Kle80, p. 58] we have

$$\omega_{X'/Y} = \omega_{X'/X} \otimes (\psi')^* \omega_{X/Y},$$

so (5.19) implies (5.25).

STEP 2: *the Gorenstein case.* Since Y' is Gorenstein, we know by (5.21) that $\omega_{X'/X} = (\varphi')^* \omega_{Y'/Y}$ is locally free. Using the same formula we see that in this case the fibres of ψ' are even Gorenstein. Since X is Gorenstein, this implies by Theorem 5.20 that X' is Gorenstein.

STEP 3: *integrality.* Since X and Y' are integral, they admit non-empty open subsets $X_0 \subset X$ and $Y_0 \subset Y'$ that are smooth. By generic smoothness applied to the induced morphisms $X_0 \to Y$ and $Y_0 \to Y$ we can suppose up to restricting a bit further that they are smooth over a smooth base. Therefore $X_0 \times_Y Y_0$ is smooth and dense in X', in particular X' is generically reduced. Since X' is flat over the integral scheme Y, it follows from [Laz04a, p. 246] that X' is reduced.

We conclude with an absolutely trivial, and nevertheless crucial remark.

COROLLARY 5.26. *Let* $\varphi\colon X \to Y$ *be a flat Cohen-Macaulay fibration, and let* $\mathcal{S}$ *be a locally free sheaf on* X. *Then*

$$\varphi_*(\mathcal{S} \otimes \omega_{X/Y})$$

is reflexive.

Proof. Since $\mathcal{S}$ is locally free, we have

$$\mathcal{S} \otimes \omega_{X/Y} \simeq \mathcal{H}om_X(\mathcal{S}^*, \omega_{X/Y})$$

so relative duality implies that

$$\varphi_*(\mathcal{S} \otimes \omega_{X/Y}) \simeq \varphi_*\mathcal{H}om_X(\mathcal{S}^*, \omega_{X/Y}) \simeq \mathcal{H}om_Y(R^{\dim X - \dim Y}\varphi_*\mathcal{S}^*, \mathcal{O}_Y)\,.$$

The dual sheaf of a coherent sheaf is reflexive.

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