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THE TAMBARA-YAMAGAMI CATEGORIES AND 3-MANIFOLD INVARIANTS

by Vladimir TURAEV and Leonid VAINERMAN

ABSTRACT. We prove that if two Tambara-Yamagami categories $\mathcal{TY}(A,\chi,\nu)$ and $\mathcal{TY}(A',\chi',\nu')$ give rise to the same state sum invariants of 3-manifolds and the order of one of the groups A, A' is odd, then $\nu = \nu'$ and there is a group isomorphism $A \approx A'$ carrying χ to χ'. The proof is based on an explicit computation of the state sum invariants for the lens spaces of type $(k,1)$.

INTRODUCTION

One of the fundamental achievements of quantum topology was a discovery of a non-trivial connection between monoidal categories and state-sum 3-manifold invariants. This connection was first observed by O. Viro and V. Turaev and later generalized in the papers of J. Barrett, B. Westbury, A. Ocneanu, S. Gelfand, D. Kazhdan and others. Their results may be summarized by saying that every spherical fusion category $\mathcal{C}$ over $\mathbf{C}$ with $\dim(\mathcal{C}) \neq 0$ gives rise to a numerical topological invariant $|M|_{\mathcal{C}} \in \mathbf{C}$ of any closed oriented 3-dimensional manifold M. A prototypical example of a spherical fusion category is the category REP(G) of finite-dimensional complex representations of a finite group G. This category allows nice operations on objects and morphisms: direct sums, tensor products, left and right dualization. Moreover, REP(G) contains a finite family of "simple" objects (= irreducible representations) such that all objects split as direct sums of the objects of this family. Certainly, the sets of morphisms in REP(G) are finite-dimensional complex vector spaces. Axiomatizing these properties, one obtains a notion of a *fusion category*, see [4]. The condition of *sphericity* on a fusion category $\mathcal{C}$ is more technical and basically says that all objects of $\mathcal{C}$ have a well-defined numerical dimension invariant under isomorphisms in $\mathcal{C}$, see [2]. A spherical fusion category $\mathcal{C}$ has a numerical dimension defined as the sum of the squares of the dimensions of the isomorphism classes of simple objects (for example, $\dim(\mathrm{REP}(G)) = |G|$). The class of spherical fusion categories includes the categories of type $\mathrm{REP}(G)$ and many other categories some of which will be mentioned below. The class of spherical fusion categories is believed to be "big but not too big" so that one may hope for some kind of classification.

The invariant of a 3-manifold M associated with $\mathrm{REP}(G)$ is nothing but the number of homomorphisms from the fundamental group of M to G. In general, the invariant $|M|_{\mathcal{C}}$ associated with a spherical fusion category $\mathcal{C}$ can not be defined in terms of the fundamental group. The definition of $|M|_{\mathcal{C}}$ proceeds in terms of state sums on a triangulation of M. The key algebraic ingredients of these state sums are the so-called *6j-symbols* associated with $\mathcal{C}$.

The formula $(M, \mathcal{C}) \mapsto |M|_{\mathcal{C}}$ defines a pairing between homeomorphism classes of closed oriented 3-manifolds and spherical fusion categories of non-zero dimension. A study of this pairing leads to natural questions both in algebra and topology. One usually studies the topological aspects. Is the pairing $(M, \mathcal{C}) \mapsto |M|_{\mathcal{C}}$ sufficiently strong to distinguish the 3-sphere from other 3-manifolds? (The answer is "yes".) Is it sufficiently strong to distinguish arbitrary 3-manifolds up to homeomorphism? (The answer is "no", see [5].)

We shall focus on algebraic questions and specifically on the following *reconstruction problem*: To what extent can a spherical fusion category be reconstructed from the associated 3-manifold invariants? The rational for this problem is that the number $|M|_{\mathcal{C}}$ may be viewed as a generalized dimension of $\mathcal{C}$ determined by M. The reconstruction problem is intriguing already for the categories of type $\mathrm{REP}(G)$. Is it true that for any non-isomorphic finite groups G_1, G_2 there is a closed oriented 3-manifold M such that the numbers of homomorphisms from $\pi_1(M)$ to G_1 and G_2 are different? We do not know the answer.

In this paper, we study the reconstruction problem for a class of spherical fusion categories introduced by Tambara and Yamagami [13]. The origin of their work is as follows. On the one hand, studying bimodule categories in the theory of operator algebras, Yamagami [17] constructed examples of non-isomorphic semisimple Hopf algebras with equivalent categories of representations. On the other hand, Tambara and Yamagami attempted to distinguish three existing 8-dimensional non-commutative semisimple Hopf algebras by their categories of representations. These Hopf algebras are the Kac-Paljutkin algebra [8] and the group algebras of the dihedral group D_8 and the quaternion group Q_8. It was known that the representation categories of these Hopf algebras had the same Grothendieck ring, but it was unknown whether or not these tensor categories themselves were equivalent. Tambara and Yamagami completely classified semisimple tensor categories with Grothendieck ring of the above mentioned type, and deduced that the categories of representations of the three Hopf algebras in question are not equivalent to each other.

A *Tambara-Yamagami category* $\mathcal{TY}(A,\chi,\nu)$ is determined by a bicharacter χ on a finite abelian group A and a sign $\nu = \pm 1$. By a *bicharacter* on A we mean a non-degenerate symmetric bilinear pairing $\chi\colon A\times A \to S^1$; the *non-degeneracy* of χ means that the adjoint homomorphism $A \to \mathrm{Hom}(A, S^1)$ is bijective. The pair (A,χ) is called a *bicharacter pair*. It is known that the category $\mathcal{TY}(A,\chi,\nu)$ has a canonical structure of a spherical fusion category and its dimension is non-zero.

Two bicharacter pairs (A,χ) and (A',χ') are said to be *isomorphic* if there is an isomorphism $A \cong A'$ transforming χ into χ'. It is known that two Tambara-Yamagami categories, $\mathcal{TY}(A,\chi,\nu)$ and $\mathcal{TY}(A',\chi',\nu')$, are monoidally equivalent if and only if the pairs (A,χ) and (A',χ') are isomorphic and $\nu = \nu'$. Moreover, the monoidal equivalence, if it exists, may always be chosen to preserve the structure of a spherical category.

Each bicharacter pair (A,χ) splits uniquely as an orthogonal sum

$$(A,\chi) = \bigoplus_p (A^{(p)}, \chi^{(p)}),$$

where p runs over all prime natural numbers, $A^{(p)} \subset A$ is the abelian p-group consisting of the elements of A annihilated by a sufficiently big power of p, and $\chi^{(p)}\colon A^{(p)} \times A^{(p)} \longrightarrow S^1$ is the restriction of χ to $A^{(p)}$. In the sequel, the order of a group A is denoted $|A|$.

THEOREM 0.1. *Let $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ and $\mathcal{C}' = \mathcal{TY}(A',\chi',\nu')$ be two Tambara-Yamagami categories such that $|M|_{\mathcal{C}} = |M|_{\mathcal{C}'}$ for all closed oriented 3-manifolds M.*

(a) *We have $|A| = |A'|$ and if $|A|$ is not a positive power of 4, then $\nu = \nu'$.*
(b) *For every odd prime p, the pairs $(A^{(p)}, \chi^{(p)})$ and $(A'^{(p)}, \chi'^{(p)})$ are isomorphic.*

Combining the claims (a) and (b) we obtain the following corollary.

COROLLARY 0.2. *Let $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ and $\mathcal{C}' = \mathcal{TY}(A',\chi',\nu')$ be two Tambara-Yamagami categories such that $|M|_{\mathcal{C}} = |M|_{\mathcal{C}'}$ for all closed oriented 3-manifolds M. If $|A|$ is odd, then the bicharacter pairs (A,χ) and (A',χ') are isomorphic and $\nu = \nu'$.*

We conjecture a similar claim in the case where $|A|$ is even.

The proof of Theorem 0.1 is based on an explicit computation of $|M|_{\mathcal{C}}$ for the *lens spaces* $L_k = L_{k,1}$ with $k = 0, 1, 2, \dots$ Recall that L_k is the closed oriented 3-manifold obtained from the 3-sphere S^3 by surgery along a trivial knot in S^3 with framing k. In particular, $L_0 = S^1 \times S^2$, $L_1 = S^3$, and $L_2 = \mathbf{R}P^3$. The manifolds $\{L_k\}_k$ are pairwise non-homeomorphic; they are distinguished by the fundamental group $\pi_1(L_k) = \mathbf{Z}/k\mathbf{Z}$.

To formulate our computation of $|L_k|_{\mathcal{C}}$, we recall the notion of a Gauss sum. Let A be a finite abelian group and $\chi\colon A \times A \longrightarrow S^1$ be a symmetric bilinear form (possibly degenerate). A *quadratic map* associated with χ is a map $\mu\colon A \to S^1$ such that for all $a, b \in A$,

$$\mu(a+b) = \chi(a,b)\,\mu(a)\,\mu(b).$$

In other words, the coboundary of μ is equal to χ. Such a μ always exists (see, for example, [9]) and determines the normalized *Gauss sum*

$$\gamma(\mu) = |A|^{-1/2}|A^{\perp}_{\chi}|^{-1/2}\sum_{a\in A}\mu(a) \in \mathbf{C},$$

where

$$A^{\perp}_{\chi} = \{a \in A \mid \chi(a,b) = 1 \text{ for all } b \in A\}$$

is the annihilator of χ. (If χ is a bicharacter, then $A^{\perp}_{\chi} = \{0\}$.) The normalization is chosen so that either $\gamma(\mu) = 0$ or $|\gamma(\mu)| = 1$ (see Lemma 2.1 below).

Denote by Q_χ the set of quadratic maps associated with χ. This set has precisely $|A|$ elements; this follows from the fact that any two quadratic maps associated with χ differ by a homomorphism $A \to S^1$. Every integer $k \geq 0$ determines a subgroup $A_k = \{a \in A \mid ka = 0\}$ of A and a number

$$\zeta_k(\chi) = |A|^{-1/2}|A_k|^{-1/2}\sum_{\mu\in Q_\chi}\gamma(\mu)^k \in \mathbf{C}.$$

For example, $A_0 = A$ and $\zeta_0(\chi) = 1$.

THEOREM 0.3. *Let $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ be a Tambara–Yamagami category. For any odd integer $k \geq 1$, we have*

$$|L_k|_{\mathcal{C}} = \frac{|A_k|}{2|A|}. \tag{0.1}$$

For any even integer $k \geq 0$, we have

$$|L_k|_{\mathcal{C}} = \frac{|A_k| + \nu^{k/2}|A|^{1/2}|A_{k/2}|^{1/2}\zeta_{k/2}(\chi)}{2|A|}. \tag{0.2}$$

For $k = 0$, Formula (0.2) gives $|S^1 \times S^2|_{\mathcal{C}} = 1$ which is known to be true for all spherical fusion categories $\mathcal{C}$.

Our proof of Theorem 0.3 is based on two results. The first is the equality $|M|_{\mathcal{C}} = \tau_{\mathcal{Z}(\mathcal{C})}(M)$ recently established in [16]. Here $\mathcal{C}$ is an arbitrary spherical fusion category of non-zero dimension, $\mathcal{Z}(\mathcal{C})$ is the *Drinfeld-Joyal-Street center* of $\mathcal{C}$, and $\tau_{\mathcal{Z}(\mathcal{C})}(M)$ is the *Reshetikhin-Turaev invariant* of M. The second result is the computation of the center of $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ in [6].

The paper is organized as follows. In Section 1 we recall the Tambara-Yamagami category and its center and prove Theorem 0.3. In Sections 2 and 3 we prove respectively claims (a) and (b) of Theorem 0.1.

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1. The Tambara-Yamagami categories and their centers

In this section, (A, χ) is a bicharacter pair, $\nu = \pm 1$, and $n = |A|$.

1.1 The category $\mathcal{TY}(A, \chi, \nu)$

The simple objects of the Tambara-Yamagami category $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ are all elements a of A and an additional object m. The unit object of $\mathcal{C}$ is the zero element $0 \in A$. All other objects of $\mathcal{C}$ are finite direct sums of the simple objects. The tensor product in $\mathcal{C}$ is determined by the following *fusion rules*:

$$a \otimes b = a + b \quad \text{and} \quad a \otimes m = m \otimes a = m \quad \text{for all} \quad a, b \in A,$$
$$\text{and} \quad m \otimes m = \bigoplus_{a \in A} a.$$

The category $\mathcal{C}$ is associative but generally speaking not strictly associative. For any simple objects U, V, W of $\mathcal{C}$, the associativity isomorphism $\phi_{U,V,W} \colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by the following formulas (where a, b, c run over A):

$$\phi_{a,b,c} = id_{a+b+c}\,, \qquad \phi_{a,b,m} = id_m\,, \qquad \phi_{m,a,b} = id_m\,,$$

$$\phi_{a,m,b} = \chi(a,b) id_m\,, \qquad \phi_{a,m,m} = \bigoplus_{b\in A} id_b\,, \quad \phi_{m,m,a} = \bigoplus_{b\in A} id_b\,,$$

$$\phi_{m,a,m} = \bigoplus_{b\in A} \chi(a,b) id_b\,, \quad \phi_{m,m,m} = \quad (\nu n^{-1/2}\chi(a,b)^{-1} id_m)_{a,b}\,.$$

The unit isomorphisms are trivial. The duality in $\mathcal{C}$ is defined by $a^* = -a$ for all $a \in A$ and $m^* = m$. The left duality morphisms in $\mathcal{C}$ are the identity maps $0 \to a \otimes a^*$, $a^* \otimes a \to 0$ for $a \in A$, the inclusion $0 \hookrightarrow m \otimes m$ and $\nu n^{1/2}$ times the obvious projection $m \otimes m \to 0$. The right duality morphisms in $\mathcal{C}$ are the identity maps $0 \to a^* \otimes a$, $a \otimes a^* \to 0$ for $a \in A$, ν times the inclusion $0 \hookrightarrow m \otimes m$ and $n^{1/2}$ times the obvious projection $m \otimes m \to 0$.

The functor $X \mapsto X^{**}$, where X is an object of $\mathcal{TY}(A, \chi, \nu)$, equals to the identity functor. There is a *pivotal structure* j on $\mathcal{TY}(A, \chi, \nu)$ defined by $j(a) = id_a$ for all $a \in A$ and $j(m) = sign(\nu) id_m$, where *sign* means the sign of a real number. This structure is canonical in the sense that the corresponding pivotal dimensions are equal to the Perron-Frobenius dimensions of objects: $\dim(a) = 1$ for all $a \in A$ and $\dim(m) = \sqrt{|A|}$.

We define a *fusion category* as a $\mathbf{C}$-linear monoidal category with compatible left and right dualities such that all objects are direct sums of simple objects, the number of isomorphism classes of simple objects is finite, and the unit object is simple. (An object V is *simple* if $\mathrm{End}(V) = \mathbf{C}\, id_V$.) The condition of *sphericity* says that the left and right dimensions of all objects are equal. A spherical fusion category has a *numerical dimension* defined as the sum of the squares of the dimensions of the (isomorphism classes of) simple objects. A basic reference on the theory of fusion categories is [4].

It is easy to see that the above mentioned pivotal structure in $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ is spherical. It turns $\mathcal{C}$ into a spherical fusion category of dimension $2n$.

1.2 The center

The center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ was computed in [6], Prop. 4.1. The category $\mathcal{Z}(\mathcal{C})$ has three types of simple objects whose description together with the corresponding quantum dimensions and twists is as follows:

(1) $2n$ invertible objects $X_{(a,\varepsilon)}$, where a runs over A and ε runs over complex square roots of $\chi(a,a)^{-1}$. Here $\dim(X_{(a,\varepsilon)}) = 1$ and $\theta_{(a,\varepsilon)} = \chi(a,a)^{-1}$;

(2) $\frac{n(n-1)}{2}$ objects $Y_{(a,b)}$ parameterized by unordered pairs (a, b), where $a, b \in A$, $a \neq b$. Here $\dim(Y_{(a,b)}) = 2$ and $\theta_{(a,b)} = \chi(a,b)^{-1}$;

(3) $2n$ objects $Z_{(\mu,\Delta)}$, where μ runs over Q_χ and Δ runs over the square roots of $\nu\gamma(\mu)$. Here $\dim(Z_{(\mu,\Delta)}) = n^{1/2}$ and $\theta_{(\mu,\Delta)} = \Delta$.

Denote by I the set of the (isomorphism classes of) simple objects of $\mathcal{Z}(\mathcal{C})$. The dimension of $\mathcal{Z}(\mathcal{C})$ is computed by

$$\dim \mathcal{Z}(\mathcal{C}) = \sum_{i\in I} (\dim(i))^2 = 2n \times 1 + \frac{n(n-1)}{2} \times 4 + 2n \times n = 4n^2.$$

We will need the following more general computation.

LEMMA 1.1. *For an integer* $k \geq 0$, *set* $\tau_k = \sum_{i\in I} \theta_i^k (\dim(i))^2$, *where* θ_i *and* $\dim(i)$ *are the twist and the dimension of* $i \in I$. *If* k *is odd, then* $\tau_k = 2n|A_k|$. *If* k *is even, then* $\tau_k = 2n(|A_k| + \nu^{k/2}|A|^{1/2}|A_{k/2}|^{1/2}\zeta_{k/2}(\chi))$.

Proof. A direct computation shows that $\tau_k = 2u_k + nv_k$, where

$$u_k = \sum_{a\in A} \chi(a,a)^{-k} + \sum_{(a,b)\in A^2, a\neq b} \chi(a,b)^{-k}$$

and $v_k = \sum_{(\mu,\Delta)} \Delta^k$. Since χ is non-degenerate,

$$u_k = \sum_{a,b\in A} \chi(a,b)^{-k} = \sum_{a,b\in A} \chi(a, b^{-k}) = n\,|A_k|.$$

If k is odd, then the contributions of the pairs (μ,Δ) and $(\mu,-\Delta)$ to v_k cancel so that $v_k = 0$ and $\tau_k = 2n|A_k|$. For even k,

$$v_k = \sum_{\mu} 2(\nu\gamma(\mu))^{k/2} = 2\nu^{k/2}|A|^{1/2}|A_{k/2}|^{1/2}\zeta_{k/2}(\chi).$$

1.3 PROOF OF THEOREM 0.3

Since $\mathcal{C} = \mathcal{TY}(A,\chi,\nu)$ is a spherical fusion category of non-zero dimension, it determines for any closed oriented 3-manifold M a *state sum invariant* $|M|_{\mathcal{C}} \in \mathbf{C}$, see [15], [1]. By a theorem of Müger [10], the category $\mathcal{Z}(\mathcal{C})$ is *modular* in the sense of [14]. A modular category endowed with a square root $\mathcal{D}$ of its dimension gives rise to the Reshetikhin-Turaev invariant of any M as above. The RT-invariant of M determined by $\mathcal{Z}(\mathcal{C})$ and the square root $\mathcal{D} = 2n = \tau_1$ of $\dim \mathcal{Z}(\mathcal{C})$ will be denoted by $\tau_{\mathcal{Z}(\mathcal{C})}(M)$. A theorem of Virelizier and Turaev [16] implies that $|M|_{\mathcal{C}} = \tau_{\mathcal{Z}(\mathcal{C})}(M)$ for all M. By [14], Chapter II, 2.2, for all $k \geq 0$,

$$\tau_{\mathcal{Z}(\mathcal{C})}(L_k) = \mathcal{D}^{-2} \sum_{i\in I} \theta_i^k (\dim(i))^2 = 4n^{-2}\tau_k.$$

Substituting the expression for τ_k provided by Lemma 1.1, we obtain the claim of the theorem.

2. Proof of Theorem 0.1(a)

We start with a well-known lemma. In this lemma we call a quadratic map $\mu\colon A \to S^1$ *homogeneous* if $\mu(na) = (\mu(a))^{n^2}$ for all $n \in \mathbf{Z}$ and $a \in A$.

LEMMA 2.1. *Let A be a finite abelian group and $\mu\colon A \to S^1$ be a quadratic map associated with a symmetric bilinear form $\chi\colon A \times A \to S^1$. Set $A^\perp = A^\perp_\chi \subset A$.*
- *If $\mu(A^\perp) \neq 1$, then $\gamma(\mu) = 0$.*
- *If $\mu(A^\perp) = 1$, then $|\gamma(\mu)| = 1$.*
- *If $\mu(A^\perp) = 1$ and μ is homogeneous, then $\gamma(\mu)$ is an 8-th complex root of unity.*

Proof. We have

$$|A|\,|A^\perp|\,|\gamma(\mu)|^2 = |\sum_{a\in A}\mu(a)|^2 = \sum_{a,b\in A}\mu(a)\overline{\mu(b)} = \sum_{a,b\in A}\mu(a)\mu(b)^{-1}$$
$$= \sum_{a,b\in A}\mu(a+b)\mu(b)^{-1} = \sum_{a,b\in A}\chi(a,b)\mu(a)\,.$$

When b runs over A, the complex number $\chi(a,b)$ runs over a finite subgroup of S^1. We have $\sum_{b\in A}\chi(a,b) = 0$ unless this subgroup is trivial. The latter holds if and only if $a \in A^\perp$ and in this case $\sum_{b\in A}\chi(a,b) = |A|$. Therefore,

$$|A|\,|A^\perp|\,|\gamma(\mu)|^2 = |A|\sum_{a\in A^\perp}\mu(a)\,.$$

The restriction of μ to $A^\perp$ is a group homomorphism $A^\perp \to S^1$. If $\mu(A^\perp) \neq 1$, then $\sum_{a\in A^\perp}\mu(a) = 0$ and therefore $\gamma(\mu) = 0$. Suppose now that $\mu(A^\perp) = 1$. Then $\sum_{a\in A^\perp}\mu(a) = |A^\perp|$ and therefore $|\gamma(\mu)| = 1$. The equality $\mu(A^\perp) = 1$ also ensures that μ is the composition of the projection $A \to A' = A/A^\perp$ with a quadratic map $\mu'\colon A' \to S^1$ associated with the non-degenerate symmetric bilinear form $A' \times A' \to S^1$ induced by χ. It follows from the definitions that $\gamma(\mu) = \gamma(\mu')$. If μ is homogeneous, then so is μ'. It is known (see, for instance, [11], Chapter 5, Section 2) that for any homogeneous quadratic map on a finite abelian group associated with a non-degenerate symmetric bilinear form, the corresponding invariant γ is an 8-th root of unity. This implies the last claim of the lemma.

LEMMA 2.2. *Let (A,χ) be a bicharacter pair. For any integer $k \geq 1$, either $\zeta_k(\chi) = 0$ or $\zeta_k(\chi)$ is an 8-th root of unity. If $k = 1$ or k is divisible by $8|A|$, then $\zeta_k(\chi) = 1$.*

Proof. Pick a quadratic map $\mu_0\colon A \to S^1$ associated with χ. Observe that for every integer k, the function $\mu_0^k\colon A \to S^1$ carrying any $c \in A$ to $(\mu_0(c))^k$ is a quadratic map associated with the symmetric bilinear form $\chi^k\colon A\times A \to S^1$ defined by $\chi^k(a,b) = (\chi(a,b))^k$. We claim that for all $k \in \mathbf{Z}$,

$$\zeta_k(\chi) = \gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k. \tag{2.1}$$

Indeed, since χ is non-degenerate, any quadratic map $\mu\colon A \to S^1$ associated with χ can be expanded in the form $\mu(a) = \chi(a,c)\,\mu_0(a)$ for a unique $c = c(\mu) \in A$. Since $\chi(a,c)\,\mu_0(a) = \mu_0(a+c)\,\mu_0(c)^{-1}$ for all $a, c \in A$, we have

$$\begin{aligned}
\zeta_k(\chi) &= |A|^{-1/2}|A_k|^{-1/2} \sum_{\mu\in Q_\chi} (|A|^{-1/2} \sum_{a\in A} \mu(a))^k \\
&= |A|^{-1/2}|A_k|^{-1/2} \sum_{c\in A} (|A|^{-1/2} \sum_{a\in A} \chi(a,c)\,\mu_0(a))^k \\
&= \{|A|^{-1/2}|A_k|^{-1/2} \sum_{c\in A} \mu_0(c)^{-k}\}\{|A|^{-1/2} \sum_{b\in A} \mu_0(b)\}^k \\
&= \gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k.
\end{aligned}$$

In the last equality we use the obvious fact that $A^{\perp}_{\chi^{-k}} = A_k$.

We can always choose $\mu_0\colon A \to S^1$ to be homogeneous. Then μ_0^{-k} also is homogeneous. Since χ is non-degenerate, the previous lemma implies that $\gamma(\mu_0)$ is an 8-th root of unity and $\gamma(\mu_0^{-k})$ is either zero or an 8-th root of unity. This implies the first claim of the lemma.

For $k = 1$, Formula (2.1) gives

$$\zeta_1(\chi) = \gamma(\mu_0^{-1})\,\gamma(\mu_0) = \gamma(\overline{\mu_0})\,\gamma(\mu_0) = \overline{\gamma(\mu_0)}\,\gamma(\mu_0) = 1,$$

where the overbar is the complex conjugation.

Observe that $\mu_0^{2n} = 1$ for $n = |A|$. Indeed, for any $a \in A$,

$$\begin{aligned}
1 = \mu_0(0) = \mu_0(2na) &= (\mu_0(a))^{2n}\chi(a,a)^{n(n-1)} \\
&= (\mu_0(a))^{2n}\chi(na,(n-1)a) = (\mu_0(a))^{2n}.
\end{aligned}$$

Therefore for all $k \in 2n\mathbf{Z}$, we have $\gamma(\mu_0^{-k}) = 1$. If $k \in 8\mathbf{Z}$, then $(\gamma(\mu_0))^k = 1$. Hence, if $k \in 8n\mathbf{Z}$, then $\zeta_k(\chi) = \gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k = 1$.

2.1 Proof of Theorem 0.1(a)

For $k = 1$, Formula (0.1) gives $|L_1|_{\mathcal{C}} = (2|A|)^{-1}$. Thus,

$$|A| = |L_1|_{\mathcal{C}}^{-1}/2 = |L_1|_{\mathcal{C}'}^{-1}/2 = |A'|.$$

This and Formula (0.1) imply that $|A_k| = |A'_k|$ for all odd $k \geq 1$.

Set $n = |A| = |A'|$. Suppose that $\nu \neq \nu'$. Assume for concreteness that $\nu = -1$ and $\nu' = +1$. Formula (0.2) with $k = 2$ and Lemma 2.2 show that

$$|A_2| - n^{1/2} = 2n|L_2|_{\mathcal{C}} = 2n|L_2|_{\mathcal{C}'} = |A'_2| + n^{1/2}.$$

Thus, $|A_2| - |A'_2| = 2n^{1/2}$. Therefore, $n = m^2$ for an integer $m \geq 1$. Since n is not a positive power of 4, either $m = 1$ or m is not a power of 2. If $m = 1$, then $A = A' = \{0\}$ and so $A_2 = A'_2 = \{0\}$ which contradicts the equality $|A_2| - |A'_2| = 2m$.

Suppose that $m = n^{1/2}$ is not a power of 2. Pick an odd divisor $\ell \geq 3$ of m. Applying Formula (0.2) to $k = 2\ell$, we obtain

$$|A_k| - m|A_\ell|^{1/2}\zeta_\ell(\chi) = |A'_k| + m|A'_\ell|^{1/2}\zeta_\ell(\chi').$$

Note that $|A_k| = |A_2|\,|A_\ell|$ and similarly for A'. Since ℓ is odd, we have $|A_\ell| = |A'_\ell|$. Therefore

$$|A_2| - |A'_2| = m|A_\ell|^{-1/2}(\zeta_\ell(\chi') + \zeta_\ell(\chi)).$$

The right-hand side of this equality must be a real number that cannot exceed $2m|A_\ell|^{-1/2}$ by Lemma 2.2. Thus, $|A_2| - |A'_2| \leq 2m|A_\ell|^{-1/2}$. Since ℓ divides n, we have $A_\ell \neq 1$ so that $|A_\ell| \geq 2$. This gives $|A_2| - |A'_2| \leq 2m/\sqrt{2}$ which contradicts the equality $|A_2| - |A'_2| = 2m$. This contradiction shows that $\nu = \nu'$.

2.2 REMARKS

(i) It is easy to extend the above argument to show that the conclusion of Theorem 0.1(a) also holds for $|A| = 4$.

(ii) Let in the proof above $|A| = |A'| = n$ be a positive power of 2 and $\nu = -1, \nu' = 1$. Formula (0.2) with $k = 2\ell$, where $\ell \geq 3$ is odd, shows that

$$|A_{2\ell}| - n^{1/2}|A_\ell|^{1/2}\zeta_\ell(\chi) = 2n|L_{2\ell}|_{\mathcal{C}} = 2n|L_{2\ell}|_{\mathcal{C}'} = |A'_{2\ell}| + n^{1/2}|A'_\ell|^{1/2}\zeta_\ell(\chi').$$

But now $A_\ell = \{0\}$, so $|A_\ell| = 1$, $|A_{2\ell}| = |A_2|$ and similarly for A'. This gives $|A_2| - |A'_2| = n^{1/2}(\zeta_\ell(\chi') + \zeta_\ell(\chi))$. Comparing with the equality $|A_2| - |A'_2| = 2n^{1/2}$ obtained above, we conclude that $\zeta_\ell(\chi') + \zeta_\ell(\chi) = 2$. By Lemma 2.2, this is possible if and only if $\zeta_\ell(\chi) = \zeta_\ell(\chi') = 1$ for all odd $\ell \geq 3$.

(iii) The number $\zeta_k(\chi)$ is closely related to the *Frobenius-Schur indicator* $\nu_{2k}(m)$ of the object m of the category $\mathcal{C} = \mathcal{TY}(A, \chi, \nu)$ computed by Shimizu [12]. Indeed, substituting $n = 2k, V = m$ in formula (3) of [12] and taking into account that $\dim(\mathcal{C}) = 2|A|$, $\theta_m = \Delta$, $\dim(m) = |A|^{1/2}$, we obtain

$$\nu_{2k}(m) = \frac{1}{2|A|^{1/2}} \sum_{\mu,\Delta} \Delta^{2k} = |A|^{-1/2} \sum_{\mu \in Q_\chi} [\nu\gamma(\mu)]^k = \nu^k |A_k|^{1/2} \zeta_k(\chi)$$

(our sign ν is equal to Shimizu's $sgn(\tau)$). This and Lemma 2.2 give another proof of the following results of Shimizu (see [12], Theorem 3.5): the number $|A_k|^{-1/2}\nu_{2k}(m)$ is either 0 or an 8-th complex root of unity for all k; this number is 0 if and only if for some (and then for any) $\mu \in Q_\chi$, there is $a = a_\chi \in A_k$ such that $\mu(a)^k = 1$. The latter claim follows from Lemma 2.1, Formula (2.1), and the equality $A^{\perp}_{\chi^{-k}} = A_k$.

3. Proof of Theorem 0.1(b)

3.1 Preliminaries on bicharacters

Any finite abelian group A splits uniquely as a direct sum $A = \oplus_p A^{(p)}$, where $p \geq 2$ runs over all prime integers and $A^{(p)}$ consists of all elements of A annihilated by a sufficiently big power of p. The group $A^{(p)}$ is a p-group, i.e., an abelian group annihilated by a sufficiently big power of p. Given a bicharacter χ of A, we have $\chi(A^{(p)}, A^{(p')}) = 1$ for any distinct p, p'. Therefore the restriction, $\chi^{(p)}$, of χ to $A^{(p)}$ is a bicharacter and we have an orthogonal splitting $(A, \chi) = \oplus_p (A^{(p)}, \chi^{(p)})$.

Fix a prime integer $p \geq 2$ and recall the properties of bicharacters on p-groups, see, for example, [3] for a survey. Given a bicharacter χ on a finite abelian p-group A, there is an orthogonal splitting $(A, \chi) = \oplus_{s \geq 1}(A_s, \chi_s)$, where A_s is a direct sum of several copies of $\mathbf{Z}/p^s\mathbf{Z}$ and $\chi_s : A_s \times A_s \to S^1$ is a bicharacter. The rank of A_s as a $\mathbf{Z}/p^s\mathbf{Z}$-module depends only on A and is denoted $r_{p,s}(A)$.

Assume from now on that $p \neq 2$. Then the splitting $(A, \chi) = \oplus_{s \geq 1}(A_s, \chi_s)$ is unique up to isomorphism and each χ_s is an orthogonal sum of bicharacters on $r_s(A)$ copies of the cyclic abelian group $\mathbf{Z}/p^s\mathbf{Z}$. Using the canonical injection $\mathbf{Z}/p^s\mathbf{Z} \hookrightarrow S^1, z \mapsto e^{2\pi i z/p^s}$, we can view χ_s as a pairing with values in the ring $\mathbf{Z}/p^s\mathbf{Z}$. This allows us to consider the determinant $\det \chi_s \in \mathbf{Z}/p^s\mathbf{Z}$ of χ_s. Since χ_s is non-degenerate, $\det \chi_s$ is coprime with p. Let

$$\sigma_{p,s}(\chi) = \left(\frac{\det \chi_s}{p}\right) \in \{\pm 1\}$$

be the corresponding Legendre symbol. Recall that for an integer d coprime with p, the *Legendre symbol* $\left(\frac{d}{p}\right)$ is equal to 1 if $d \pmod p$ is a quadratic residue and to -1 otherwise, see, for example, [7]. If $r_{p,s}(A) = 0$, then by definition $\sigma_{p,s} = 1$. It follows from the definitions that the integers $\{r_{p,s}\}_s$ are additive and the signs $\{\sigma_{p,s}\}_s$ are multiplicative with respect to orthogonal summation of bicharacter pairs. A theorem due to H. Minkowski, E. Seifert, and C.T.C. Wall says that these invariants form a complete system: two bicharacters, χ_1 and χ_2, on p-groups A_1 and A_2, respectively, are isomorphic if and only if $r_{p,s}(A_1) = r_{p,s}(A_2)$ and $\sigma_{p,s}(\chi_1) = \sigma_{p,s}(\chi_2)$ for all $s \geq 1$.

For brevity, when p is specified, we denote $r_{p,s}(A)$ and $\sigma_{p,s}(\chi)$ by $r_s(A)$ and $\sigma_s(\chi)$, respectively.

3.2 Computation of ζ_k

Consider the $\mathbf{C}$-valued invariants $\{\zeta_k\}_{k\geq 1}$ of bicharacters defined in the introduction. It is easy to deduce from the definitions that $\zeta_k(\chi\oplus\chi') = \zeta_k(\chi)\,\zeta_k(\chi')$ for any bicharacters χ, χ' and any k. Thus, the formula $\chi \mapsto \zeta_k(\chi)$ defines a multiplicative function from the semigroup of bicharacter pairs (with the orthogonal sum $\oplus$ as operation) to $\mathbf{C}$.

Fix an odd prime $p \geq 3$. We now compute ζ_k on the bicharacters on p-groups. For any odd integer a, set $\varepsilon_a = i = \sqrt{-1}$ if $a \equiv 3 \pmod 4$ and $\varepsilon_a = 1$ otherwise. For any integers $k, s \geq 1$, we have $\gcd(k, p^s) = p^t$ with $0 \leq t \leq s$. Set

$$\alpha_{k,s} = ks + s - t \quad \text{and} \quad \beta_{k,s} = \frac{\varepsilon_{p^s}^k}{\varepsilon_{p^{s-t}}} \left(\frac{h}{p}\right)^{ks+s-t} \left(\frac{k'}{p}\right)^{s-t} \in \{\pm 1, \pm i\},$$

where $h = (p^s + 1)/2 \in \mathbf{Z}$ and $k' = k/p^t \in \mathbf{Z}$. Note that $\gcd(h, p) = 1$ so that the Legendre symbol $\left(\frac{h}{p}\right)$ is defined. If $t < s$, then $\gcd(k', p) = 1$ so that the Legendre symbol $\left(\frac{k'}{p}\right)$ is defined; if $t = s$, then by definition, $\left(\frac{k'}{p}\right)^{s-t} = 1$.

LEMMA 3.1. *For any $k \geq 1$ and any bicharacter χ on a p-group A,*

$$\zeta_k(\chi) = \prod_{s\geq 1} \beta_{k,s}^{r_s(A)}\, [\sigma_s(\chi)]^{\alpha_{k,s}}. \tag{3.1}$$

Proof. The proof is based on the following classical Gauss formula: for any integer d coprime with p,

$$\sum_{j=0}^{p^s-1} \exp\left(\frac{2\pi i}{p^s} dj^2\right) = p^{\frac{s}{2}} \varepsilon_{p^s} \left(\frac{d}{p}\right)^s. \tag{3.2}$$

A more general formula holds for any integer d: if $\gcd(d,p^s) = p^t$ with $0 \leq t \leq s$ and $d' = d/p^t$, then

$$\sum_{j=0}^{p^s-1} \exp\left(\frac{2\pi i}{p^s} dj^2\right) = p^t \sum_{j=0}^{p^{s-t}-1} \exp\left(\frac{2\pi i}{p^{s-t}} d' j^2\right) = p^{\frac{s+t}{2}} \varepsilon_{p^{s-t}} \left(\frac{d'}{p}\right)^{s-t}, \tag{3.3}$$

where, by definition, for $t = s$, the expression $\left(\frac{d'}{p}\right)^{s-t}$ is equal to 1.

We now prove (3.1). It is clear that both sides of (3.1) are multiplicative with respect to orthogonal summation of bicharacters. The results stated in Section 3.1 allow us to reduce the proof of (3.1) to the case where $A = \mathbf{Z}/p^s\mathbf{Z}$ for some $s \geq 1$. We must prove that for any bicharacter $\chi\colon A \times A \to S^1$,

$$\zeta_k(\chi) = \beta_{k,s}\, [\sigma_s(\chi)]^{\alpha_{k,s}}. \tag{3.4}$$

Set as above $h = (p^s+1)/2$ and $k' = k/p^t$, where $\gcd(k,p^s) = p^t$ with $0 \leq t \leq s$. The bicharacter χ is given by $\chi(a,b) = \exp(\frac{2\pi i}{p^s}\Delta ab)$ for all $a,b \in A$, where Δ is an integer coprime with p. Observe that the map $\mu_0\colon A \to S^1$ carrying any $a \in A$ to $\exp(\frac{2\pi i}{p^s} h\Delta a^2)$ is a quadratic map associated with χ. Formula (3.2) and the multiplicativity of the Legendre symbol imply that

$$\gamma(\mu_0) = p^{-s/2} \sum_{j=0}^{p^s-1} \exp\left(\frac{2\pi i}{p^s} h\Delta j^2\right) = \varepsilon_{p^s} \left(\frac{h}{p}\right)^s \left(\frac{\Delta}{p}\right)^s = \varepsilon_{p^s} \left(\frac{h}{p}\right)^s [\sigma_s(\chi)]^s.$$

Similarly, Formula (3.3) implies that

$$\begin{aligned}\sum_{c\in A} \mu_0(c)^{-k} &= \sum_{j=0}^{p^s-1} \overline{\exp\left(\frac{2\pi i}{p^s} kh\Delta j^2\right)} \\ &= p^{(s+t)/2} \varepsilon_{p^{s-t}}^{-1} \left(\frac{h}{p}\right)^{s-t} \left(\frac{k'}{p}\right)^{s-t} [\sigma_s(\chi)]^{s-t}.\end{aligned}$$

Since $|A| = p^s$ and $|A_k| = \gcd(k,p^s) = p^t$, we have

$$\gamma(\mu_0^{-k}) = |A|^{-1/2} |A_k|^{-1/2} \sum_{c\in A} \mu_0(c)^{-k} = \varepsilon_{p^{s-t}}^{-1} \left(\frac{h}{p}\right)^{s-t} \left(\frac{k'}{p}\right)^{s-t} [\sigma_s(\chi)]^{s-t}.$$

These computations and Formula (2.1) imply that

$$\zeta_k(\chi) = \gamma(\mu_0^{-k})\,(\gamma(\mu_0))^k = \frac{\varepsilon_{p^s}^k}{\varepsilon_{p^{s-t}}} (\frac{h}{p})^{ks+s-t} (\frac{k'}{p})^{s-t} [\sigma_s(\chi)]^{ks+s-t}.$$

This is equivalent to Formula (3.4).

Note one special case of Lemma 3.1: if k is divisible by $2|A|$, then $\zeta_k(\chi) = \prod_{s\geq 1} \beta_{k,s}^{r_s(A)}$. Indeed, in this case for all s such that $\mathbf{Z}/p^s\mathbf{Z}$ is a direct summand of A, we have $\gcd(k, p^s) = p^s$ and $\alpha_{k,s} = ks \in 2\mathbf{Z}$. For all other s, we have $\sigma_s(\chi) = 1$. Therefore $[\sigma_s(\chi)]^{\alpha_{k,s}} = 1$ for all s.

3.3 Proof of Theorem 0.1(b)

We begin with a few remarks concerning the subgroups $(A_k)_k$ of A defined in the introduction. Using the splitting $A = \oplus_p A^{(p)}$, one easily checks that $A_{kl} = A_k \oplus A_l$ for any relatively prime integers k, l. For any prime p, the integers $(|A_{p^m}|)_{m\geq 1}$ depend only on the group $A^{(p)}$ and determine the isomorphism class of $A^{(p)}$. Indeed, $A^{(p)} = \oplus_{s\geq 1}(\mathbf{Z}/p^s\mathbf{Z})^{r_{p,s}}$ for $r_{p,s} = r_{p,s}(A) \geq 0$. Given $m \geq 1$,

$$A_{p^m} = (A^{(p)})_{p^m} = \bigoplus_{s=1}^{m} (\mathbf{Z}/p^s\mathbf{Z})^{r_{p,s}} \oplus \bigoplus_{s>m} (\mathbf{Z}/p^m\mathbf{Z})^{r_{p,s}}.$$

Hence,

$$\log_p(|A_{p^{m+1}}|/|A_{p^m}|) = r_{p,m+1} + r_{p,m+2} + \cdots .$$

Therefore, the sequence $(|A_{p^m}|)_{m\geq 1}$ determines the sequence $\{r_{p,s}(A)\}_{s\geq 1}$ and so determines the isomorphism type of $A^{(p)}$.

Formula (0.1) and the assumptions of the theorem imply that, for all odd $k \geq 1$,

$$|A_k| = 2n|L_k|_{\mathcal{C}} = 2n|L_k|_{\mathcal{C}'} = |A'_k|,$$

where $n = |A| = |A'|$. By the previous paragraph, $A^{(p)} \cong A'^{(p)}$ for all prime $p \neq 2$, and for all $s \geq 1$,

$$r_{p,s}(A^{(p)}) = r_{p,s}(A) = r_{p,s}(A') = r_{p,s}(A'^{(p)}). \tag{3.5}$$

Since $n = \prod_{p\geq 2} |A^{(p)}|$, we also have $|A^{(2)}| = |A'^{(2)}|$.

Let $N \geq 2$ be a positive power of 2 annihilating both $A^{(2)}$ and $A'^{(2)}$. Then $A_N = A^{(2)}$ and $A'_N = A'^{(2)}$. For any odd integer $\ell \geq 1$,

$$|A_{N\ell}| = |A_N|\,|A_\ell| = |A^{(2)}|\,|A_\ell| = |A'^{(2)}|\,|A'_\ell| = |A'_N|\,|A'_\ell| = |A'_{N\ell}|.$$

Similarly, $|A_{2N\ell}| = |A'_{2N\ell}|$. Applying (0.2) to $k = 2N\ell$, we obtain $\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi')$.

Fix from now on an odd prime p. The identity (3.5) shows that to prove that the bicharacter pairs $(A^{(p)}, \chi^{(p)})$ and $(A'^{(p)}, \chi'^{(p)})$ are isomorphic, it is enough to verify that $\sigma_s(\chi^{(p)}) = \sigma_s(\chi'^{(p)})$ for all $s \geq 1$. Set

$$\ell = \frac{|A|}{|A^{(2)}||A^{(p)}|} = \prod_{q\geq 3, q\neq p} |A^{(q)}| = \prod_{q\geq 3, q\neq p} |A'^{(q)}| = \frac{|A'|}{|A'^{(2)}||A'^{(p)}|},$$

where q runs over all odd primes distinct from p. Clearly, ℓ is an odd integer. For any N as above, $\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi')$. Observe that

$$\zeta_{N\ell}(\chi) = \zeta_{N\ell}(\chi^{(2)}) \prod_{q \geq 3} \zeta_{N\ell}(\chi^{(q)}),$$

where q runs over all odd primes. Since $N\ell$ is divisible by $2|A^{(q)}|$ for $q \neq p$, the remark at the end of Section 3.2 implies that $\zeta_{N\ell}(\chi^{(q)}) = \zeta_{N\ell}(\chi'^{(q)}) \neq 0$ for all $q \neq p$. Replacing if necessary N by a bigger power of 2, we can assume that N is divisible by $8|A^{(2)}| = 8|A'^{(2)}|$. The last claim of Lemma 2.2 yields $\zeta_{N\ell}(\chi^{(2)}) = \zeta_{N\ell}(\chi'^{(2)}) = 1$. Combining these equalities, we obtain $\zeta_{N\ell}(\chi^{(p)}) = \zeta_{N\ell}(\chi'^{(p)})$. Expanding both sides as in Formula (3.1) and using Formula (3.5) and the inclusions $\sigma_s(\chi^{(p)}), \sigma_s(\chi'^{(p)}) \in \{\pm 1\}$, we obtain

$$\prod_{\text{odd}\, s \geq 1} \sigma_s(\chi^{(p)}) = \prod_{\text{odd}\, s \geq 1} \sigma_s(\chi'^{(p)}).$$

Replacing in this argument ℓ by $\ell p, \ell p^2, \ell p^3, \ldots$, we similarly obtain that for all odd $u \geq 1$ and even $v \geq 2$,

$$\prod_{\text{odd}\, s \geq u} \sigma_s(\chi^{(p)}) = \prod_{\text{odd}\, s \geq u} \sigma_s(\chi'^{(p)}), \qquad \prod_{\text{even}\, s \geq v} \sigma_s(\chi^{(p)}) = \prod_{\text{even}\, s \geq v} \sigma_s(\chi'^{(p)}).$$

These equalities easily imply that $\sigma_s(\chi^{(p)}) = \sigma_s(\chi'^{(p)})$ for all s.

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Vladimir Turaev

Department of Mathematics
Indiana University
Bloomington, IN47405
U.S.A.
e-mail: vtouraev@indiana.edu

Leonid Vainerman

Department of Mathematics
University of Caen
14000 Caen
France
e-mail: leonid.vainerman@math.unicaen.fr