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# SUBTLETIES OF THE MINIMAX SELECTOR

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ABSTRACT. In this note, we show that the minimax and maximin critical values of a function quadratic nondegenerate at infinity are equal when defined in homology or cohomology with coefficients in a field. However, by an example of F. Laudenbach, this is not always true for coefficients in a ring and, even in the case of a field, the minimax-maximin depends on the field.

## 1. INTRODUCTION

Given a Lagrangian submanifold $L$ in the cotangent bundle of a closed manifold $M$, obtained by Hamiltonian deformation of the zero section, the minimax selector introduced by J.-C. Sikorav [14] provides an almost everywhere defined section $M \to L$ of the projection $T^*M \to M$ restricted to $L$. As noticed by M. Chaperon [5, 6], this defines weak solutions of smooth Cauchy problems for Hamilton-Jacobi equations; in the classical case of a convex Hamiltonian, the minimax is a minimum and the minimax solution coincides with the viscosity solution, which is not always the case for nonconvex Hamiltonians. For a recent use of the minimax selector in weak KAM theory, see [1].

The minimax has been defined using homology or cohomology with various coefficient rings, for example $\mathbf{Z}$ in [5, 15], $\mathbf{Q}$ in [3] and $\mathbf{Z}_2$ in [13]. Also, in [15], the maximin was mentioned as a natural analogue to the minimax. But there is no evidence showing that all these critical values coincide. G. Capitanio has given a proof [3] that the maximin and minimax for homology with coefficients in $\mathbf{Q}$ are equal, but the criterion he uses (Proposition 2 in [3]) is not correct — see Remark 3.11 hereafter.

In this note, we investigate the maximin and minimax for a general function quadratic at infinity, not necessarily related to Hamilton-Jacobi equations. We give both algebraic and geometric proofs that the minimax and maximin with coefficients in a field coincide; the geometric proof, based on Barannikov's Jordan normal form for the boundary operator of the Morse complex, improves our understanding of the problem. The Barannikov normal form also plays a crucial role in the proof of Arnold's 4 cusps conjecture [7].

A counterexample for coefficients in $\mathbf{Z}$, due to F. Laudenbach [11], is constructed using Morse homology; in this example, moreover, the minimax-maximin for coefficients in $\mathbf{Z}_2$ is not the same as for coefficients in $\mathbf{Q}$. However, if the minimax and maximin for coefficients in $\mathbf{Z}$ coincide, then all three minimax-maximin critical values are equal.

## 2. Maximin and minimax

Hypotheses and notation. We denote by $X$ the vector space $\mathbf{R}^n$ and by $f$ a real function on $X$, *quadratic at infinity* in the sense that it is continuous and there exists a nondegenerate quadratic form $Q\colon X \to \mathbf{R}$ such that $f$ coincides with $Q$ outside a compact subset.

Let $f^c := \{x \mid f(x) \leq c\}$ denote the sub-level sets of $f$. Note that for $c$ large enough, the homotopy types of $f^c$, $f^{-c}$ do not depend on $c$, we may denote them as $f^{\infty}$ and $f^{-\infty}$. Suppose the quadratic form $Q$ has Morse index $\lambda$, then the homology groups with coefficient ring $R$ are

$$H_*(f^{\infty},f^{-\infty};R) \simeq \begin{cases} R & \text{in dimension } \lambda \\ 0 & \text{otherwise}. \end{cases}$$

Consider the homomorphism of homology groups

$$i_{c*}\colon H_*(f^c,f^{-\infty};R) \to H_*(f^{\infty},f^{-\infty};R)$$

induced by the inclusion $i_c\colon (f^c,f^{-\infty}) \hookrightarrow (f^{\infty},f^{-\infty})$.

Definition 2.1. If $\Xi$ is a generator of $H_\lambda(f^{\infty},f^{-\infty};R)$, we let

$$\underline{\gamma}(f,R) := \inf\{c : \Xi \in \mathrm{Im}(i_{c*})\},$$

i.e. $\underline{\gamma}(f,R) = \inf\{c : i_{c*}H_\lambda(f^c,f^{-\infty};R) = H_\lambda(f^{\infty},f^{-\infty};R)\}$.

Similarly, we can consider the homology group

$$H_*(X \setminus f^{-\infty}, X \setminus f^{\infty}; R) \simeq \begin{cases} R & \text{in dimension } n-\lambda \\ 0 & \text{otherwise}, \end{cases}$$

and the homomorphism

$$j_{c*}\colon H_*(X \setminus f^c, X \setminus f^{\infty}; R) \to H_*(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)$$

induced by $j_c\colon (X \setminus f^c, X \setminus f^{\infty}) \hookrightarrow (X \setminus f^{-\infty}, X \setminus f^{\infty})$.

DEFINITION 2.2. If $\Delta$ is a generator of $H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)$, we let

$$\begin{aligned}\overline{\gamma}(f,R) &:= \sup\{c : \Delta \in \operatorname{Im}(j_{c*})\} \\ &= \sup\{c : j_{c*}H_{n-\lambda}(X \setminus f^c, X \setminus f^{\infty}; R) = H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^{\infty}; R)\}.\end{aligned}$$

LEMMA 2.3. *One has that*

$$\underline{\gamma}(f,R) = \inf\max f := \inf_{[\sigma]=\Xi}\max_{x\in|\sigma|} f(x)$$

$$\overline{\gamma}(f,R) = \sup\min f := \sup_{[\sigma]=\Delta}\min_{x\in|\sigma|} f(x),$$

*where $\sigma$ is a relative cycle and $|\sigma|$ denotes its support. We call $\sigma$ a descending (resp. ascending) simplex if $[\sigma] = \Xi$ (resp. $[\sigma] = \Delta$).*

*Proof.* A descending simplex $\sigma$ defines an element of $H_\lambda(f^c, f^{-\infty}; R)$ if and only if $|\sigma| \subset f^c$, in which case one has $\max_{x\in|\sigma|} f(x) \le c$, hence $\underline{\gamma}(f,R) \ge \inf\max f$; choosing $c = \max_{x\in|\sigma|} f(x)$, we get equality. The case of $\overline{\gamma}$ is identical.

DEFINITION 2.4. $\underline{\gamma}(f,R)$ is called a *minimax* of $f$ and $\overline{\gamma}(f,R)$, a *maximin*.

REMARK. As we shall see later, in view of Morse homology, these names are proper for excellent Morse functions.

One can also consider cohomology instead of homology and define

$$\underline{\alpha}(f,R) := \inf\{c\colon i_c^* \neq 0\}, \quad i_c^*\colon H^\lambda(f^{\infty}, f^{-\infty}; R) \to H^\lambda(f^c, f^{-\infty}; R)$$

$$\overline{\alpha}(f,R) := \sup\{c\colon j_c^* \neq 0\}, \quad j_c^*\colon H^{n-\lambda}(X\setminus f^{-\infty}, X\setminus f^{\infty}; R) \to H^{n-\lambda}(X\setminus f^c, X\setminus f^{\infty}; R).$$

PROPOSITION 2.5 ([15], Proposition 2.4). *When $X$ is $R$-oriented,*

$$\overline{\alpha}(f,R) = \underline{\gamma}(f,R) \quad \textit{and} \quad \underline{\alpha}(f,R) = \overline{\gamma}(f,R).$$

*Proof.* We establish for example the first identity: one has the commutative diagram

$$\begin{array}{ccc} H_\lambda(f^c, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^c; R) \\ \downarrow i_{c*} & & \downarrow \\ H_\lambda(f^\infty, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R) \\ \downarrow & & \downarrow j_c^* \\ H_\lambda(f^\infty, f^c; R) & \simeq & H^{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R) \end{array}$$

where the horizontal isomorphisms are given by Alexander duality ([9], section 3.3) and the columns are exact. It does follow that $i_{c*}$ is onto if and only if $j_c^*$ is zero.

DEFINITION 2.6 ([8]). As long as $X$ is finite dimensional, the *Clarke generalized derivative* of a locally Lipschitzian function $f\colon X \to \mathbf{R}$ can be defined as follows: by Rademacher's theorem, the set $\mathrm{dom}(df)$ of differentiability points of $f$ is dense in $X$; we let $\partial f(x)$ be the convex hull of the set of limits of convergent sequences $df(x_n)$ with $\lim x_n = x$. A point $x \in X$ is called a *critical point* of $f$ if $0 \in \partial f(x)$.

PROPOSITION 2.7. *If $f$ is $C^2$ then $\underline{\gamma}(f,R)$ and $\overline{\gamma}(f,R)$ are critical values of $f$; they are critical values of $f$ in the sense of Clarke when $f$ is locally Lipschitzian.*

*Proof.* Take $\underline{\gamma}$ for example: if $c = \underline{\gamma}(f,R)$ is not a critical value then, for small $\epsilon > 0$, $f^{c-\epsilon}$ is a deformation retract of $f^{c+\epsilon}$ via the flow of $-\frac{\nabla f}{\|\nabla f\|^2}$, hence $\underline{\gamma}(f,R) \leq c - \epsilon$, a contradiction. The same argument applies when $f$ is only locally Lipschitzian, replacing $\nabla f$ by a pseudo-gradient [4].

LEMMA 2.8. *If $f$ is locally Lipschitzian, then*

$$\overline{\gamma}(f,R) = -\underline{\gamma}(-f,R).$$

*Proof.* Using a (pseudo-)gradient of $f$ as previously, one can see that $X \setminus f^c$ and $(-f)^{-c}$ have the same homotopy type when $c$ is not a critical value of $f$. Otherwise, choose a sequence of non-critical values $c_n \nearrow c = \overline{\gamma}(f,R)$, then $-c_n \geq \underline{\gamma}(-f,R)$, taking the limit, we have $\overline{\gamma}(f,R) \leq -\underline{\gamma}(-f,R)$. Similarly, taking $c_n' \searrow \underline{\gamma}(-f,R)$, then $-c_n' \leq \overline{\gamma}(f,R)$, from which the limit gives us the reverse inequality $-\underline{\gamma}(-f,R) \leq \overline{\gamma}(f,R)$.

REMARK. The extension of the minimax selector to Lipschitzian functions is natural in the framework of Hamilton-Jacobi equations: even for smooth initial data, the minimax solution at time $t$ is not smooth in general, but it is Lipschitzian; now, it can be interesting to take it as a new Cauchy datum.

The following two questions arise naturally:
(1) Do we have $\underline{\gamma}(f,R) = \overline{\gamma}(f,R)$ ?
(2) Do $\underline{\gamma}(f,R)$ and $\overline{\gamma}(f,R)$ depend on the coefficient ring $R$ ?
Here are two obvious elements for an answer:

PROPOSITION 2.9. *One has* $\underline{\gamma}(f,\mathbf{Z}) \geq \overline{\gamma}(f,\mathbf{Z})$.

*Proof.* As the intersection number of $\Xi$ and $\Delta$ is $\pm 1$, the support of any descending simplex $\sigma$ must intersect the support of any ascending simplex $\tau$ at some point $\bar{x}$, hence $\max\limits_{x\in|\sigma|} f(x) \geq f(\bar{x}) \geq \min\limits_{x\in|\tau|} f(x)$.

PROPOSITION 2.10. *One has* $\underline{\gamma}(f,\mathbf{Z}) \geq \underline{\gamma}(f,R)$ *and* $\overline{\gamma}(f,\mathbf{Z}) \leq \overline{\gamma}(f,R)$ *for every ring* $R$.

*Proof.* A simplex $\sigma$ whose homology class generates $H_\lambda(f^\infty, f^{-\infty};\mathbf{Z})$ induces a simplex whose homology class generates $H_\lambda(f^\infty, f^{-\infty};R)$, whence the first inequality and, mutatis mutandis, the second one.

THEOREM 2.11. *If* $\mathbf{F}$ *is a field, then* $\underline{\gamma}(f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{F})$.

*Proof.* By Proposition 2.5, it is enough to prove that

$$\underline{\gamma}(f,\mathbf{F}) = \underline{\alpha}(f,\mathbf{F}).$$

Recall that $\underline{\gamma}(f,\mathbf{F})$ (resp. $\underline{\alpha}(f,\mathbf{F})$) is the infimum of the real numbers $c$ such that $i_{c*}\colon H_\lambda(f^c, f^{-\infty};\mathbf{F}) \to H_\lambda(f^\infty, f^{-\infty};\mathbf{F})$ is onto (resp. such that $i_c^*\colon H^\lambda(f^\infty, f^{-\infty};\mathbf{F}) \to H^\lambda(f^c, f^{-\infty};\mathbf{F})$ is nonzero). Now, as $H_\lambda(f^\infty, f^{-\infty};\mathbf{F})$ is a one-dimensional vector space over $\mathbf{F}$, the linear map $i_{c*}$ is onto if and only if it is nonzero, i.e. if and only if the transposed map $i_c^*$ is nonzero.

REMARK. This proof is invalid for coefficients in $\mathbf{Z}$ since a $\mathbf{Z}$-linear map to $\mathbf{Z}$, for example $\mathbf{Z} \ni m \to km$, $k \in \mathbf{Z}$, $k > 1$, can be nonzero without being onto; we shall see in Section 4 that Theorem 2.11 itself is not true in that case.

COROLLARY 2.12. *If* $\underline{\gamma}(f,\mathbf{Z}) = \overline{\gamma}(f,\mathbf{Z}) = \gamma$ *then* $\underline{\gamma}(f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{F}) = \gamma$ *for every field* $\mathbf{F}$.

*Proof.* This follows at once from Theorem 2.11 and Proposition 2.10.

COROLLARY 2.13. *Let* $\gamma \in \mathbf{R}$ *have the following property: there exist both a descending simplex over* $\mathbf{Z}$ *along which* $\gamma$ *is the maximum of* $f$ *and an ascending simplex over* $\mathbf{Z}$ *along which* $\gamma$ *is the minimum of* $f$. *Then,* $\underline{\gamma}(f,\mathbf{Z}) = \overline{\gamma}(f,\mathbf{Z}) = \underline{\gamma}(f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{F}) = \gamma$ *for every field* $\mathbf{F}$.

*Proof.* We have $\underline{\gamma}(f;\mathbf{Z}) \leq \gamma \leq \overline{\gamma}(f;\mathbf{Z})$ by Lemma 2.3 and $\overline{\gamma}(f;\mathbf{Z}) \leq \underline{\gamma}(f;\mathbf{Z})$ by Proposition 2.9, hence our result by Corollary 2.12.

## 3. MORSE COMPLEXES AND THE BARANNIKOV NORMAL FORM

The previous proof of Theorem 2.11, though simple, is quite algebraic. We now give a more geometric proof, which we find more concrete and illuminating, based on Barannikov's canonical form of Morse complexes. It will provide a good setting for the counterexample in Section 4.

First, there is a continuity result for the minimax and maximin:

PROPOSITION 3.1 ([14, 16]). *If* $f$ *and* $g$ *are two continuous functions quadratic at infinity with the same reference quadratic form, then*

$$|\underline{\gamma}(f,R) - \underline{\gamma}(g,R)| \leq |f-g|_{C^0}$$
$$|\overline{\gamma}(f,R) - \overline{\gamma}(g,R)| \leq |f-g|_{C^0}.$$

*Proof.* For $f \leq g$, from Lemma 2.3, it is easy to see that $\underline{\gamma}(f) \leq \underline{\gamma}(g)$. In the general case, this implies $\underline{\gamma}(g) \leq \underline{\gamma}(f + |g-f|) \leq \underline{\gamma}(f) + |g-f|_{C^0}$; exchanging $f$ and $g$, we get $\underline{\gamma}(f) \leq \underline{\gamma}(g) + |f-g|_{C^0}$.

COROLLARY 3.2. *To prove Theorem* 2.11, *it suffices to establish it for* excellent Morse *functions* $f\colon X \to \mathbf{R}$, *i.e. smooth functions having only nondegenerate critical points, each of which corresponds to a different value of* $f$.

*Proof.* By a standard argument, given a non-degenerate quadratic form $Q$ on $X$, the set of all continuous functions on $X$ equal to $Q$ off a compact subset contains a $C^0$-dense subset consisting of excellent Morse functions; our result follows by Proposition 3.1.

To prove Theorem 2.11 for excellent Morse functions, we will use Morse homology.

HYPOTHESES. We consider an excellent Morse function $f$ on $X$, quadratic at infinity [1]; for each pair of regular values $b < c$ of $f$, we denote by $f_{b,c}$ the restriction of $f$ to $f^c \cap (-f)^{-b} = \{b \leq f \leq c\}$.

MORSE COMPLEXES. Let

$$C_k(f_{b,c}) := \{\xi^k_\ell : 1 \leq \ell \leq m_k\}$$

denote the set of critical points of index $k$ of $f_{b,c}$, ordered so that $f(\xi^k_\ell) < f(\xi^k_m)$ for $\ell < m$. Given a generic gradient-like vector field $V$ for $f$ such that $(f, V)$ is Morse-Smale [2], the *Morse complex* of $(f_{b,c}, V)$ over $R$ consists of the free $R$-modules

$$M_k(f_{b,c}, R) := \{\sum_\ell a_\ell \xi^k_\ell, \quad a_\ell \in R\}$$

together with the boundary operator $\partial\colon M_k(f_{b,c}, R) \to M_{k-1}(f_{b,c}, R)$ given by

$$\partial \xi^k_\ell := \sum_m \nu_{f,V}(\xi^k_\ell, \xi^{k-1}_m)\xi^{k-1}_m$$

where, with given orientations for the stable manifolds (hence co-orientations for unstable manifolds), $\nu_{f,V}$ is the intersection number of the stable manifold $W^s(\xi^k_\ell)$ of $\xi^k_\ell$ and the unstable manifold $W^u(\xi^{k-1}_m)$ of $\xi^{k-1}_m$, i.e. the algebraic number of trajectories of $V$ connecting $\xi^k_\ell$ and $\xi^{k-1}_m$; note that

- $\nu_{f,V}(\xi^k_\ell, \xi^{k-1}_m)$ is the same for all $b, c$ with $f(\xi^k_\ell), f(\xi^{k-1}_m)$ in $[b, c]$;
- $\nu_{f,V}(\xi^k_\ell, \xi^{k-1}_m) \neq 0$ implies $f(\xi^k_\ell) > f(\xi^{k-1}_m)$: otherwise, the stable manifold of $\xi^{k-1}_m$ and the unstable manifold of $\xi^k_\ell$ for $V$, which cannot be transversal because of their dimensions, would intersect, contradicting the genericity of $V$.
- $\nu_{f,V}(\xi^k_\ell, \xi^k_m) = 0$ for two distinct critical points of the same index.

This does define a complex, i.e. $\partial \circ \partial = 0$: see for example [10, 12]. The homology $HM_*(f_{b,c}, R) := H_*(M_*(f_{b,c}, R))$ is called the *Morse homology* [3] of $f_{b,c}$.

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[1]) The theory applies as well to functions on a closed manifold, for example.

[2]) Being *Morse-Smale* means that the stable and unstable manifolds of all the critical points are transversal.

[3]) Morse homology is defined in general for any Morse function.

LEMMA 3.3 (Barannikov [2]). *If $R$ is a field $\mathbf{F}$, then this boundary operator $\partial$ has a special kind of Jordan normal form as follows: each $M_k(f_{b,c},\mathbf{F})$ has a basis*

$$\Xi^k_\ell := \sum_{i\leq\ell} \alpha_{\ell,i}\xi^k_i\,, \quad \alpha_{\ell,\ell}\neq 0 \tag{1}$$

*such that either $\partial\Xi^k_\ell = 0$ or $\partial\Xi^k_\ell = \Xi^{k-1}_m$ for some $m$, in which case no $\ell'\neq\ell$ satisfies $\partial\Xi^k_{\ell'} = \Xi^{k-1}_m$. If $(\Theta^k_\ell)$ is another such basis, then $\partial\Xi^k_\ell = \Xi^{k-1}_m$ (resp. 0) is equivalent to $\partial\Theta^k_\ell = \Theta^{k-1}_m$ (resp. 0); in other words, the matrix of $\partial$ in all such bases is the same.*

*Proof.* We prove existence by induction. Given nonnegative integers $k,i$ with $i<m_k$, suppose that vectors $\Xi^p_q$ of the form (1) have been obtained for all $(p,q)$ with either $p<k$, or $p=k$ and $q\leq i$, possessing the required property that either $\partial\Xi^p_q = \Xi^{p-1}_{j_p(q)}$ (with $j_p(q)\neq j_p(q')$ for $q\neq q'$) or $\partial\Xi^p_q = 0$. If $\partial\xi^k_{i+1}=0$ (e.g., when $k=0$), we take $\xi^k_{i+1} := \Xi^k_{i+1}$ and continue the induction. Otherwise, $\partial\xi^k_{i+1} = \sum\alpha_j\Xi^{k-1}_j$, $\alpha_j\in\mathbf{F}$. Moving all the terms $\Xi^{k-1}_{j_k(q)} = \partial\Xi^k_q, q\leq i$ from the right-hand side to the left, we get

$$\partial\big(\xi^k_{i+1} - \sum_{q\leq i}\alpha_{j_k(q)}\Xi^k_q\big) = \sum_j \beta_j\Xi^{k-1}_j\,.$$

Let

$$\Xi^k_{i+1} := \xi^k_{i+1} - \sum_{q\leq i}\alpha_{j_k(q)}\Xi^k_q\,.$$

If $\beta_j=0$ for all $j$, then $\partial\Xi^k_{i+1}=0$ and the induction can go on. Otherwise,

$$\partial\Xi^k_{i+1} = \sum_{j\leq j_0}\beta_j\Xi^{k-1}_j =: \tilde{\Xi}^{k-1}_{j_0} \text{ with } \beta_{j_0}\neq 0\,;$$

as $\partial\tilde{\Xi}^{k-1}_{j_0} = \partial\partial\Xi^k_{i+1} = 0$, we can replace $\Xi^{k-1}_{j_0}$ by $\tilde{\Xi}^{k-1}_{j_0}$ and continue the induction[4]).

DEFINITION 3.4. Under the hypotheses and with the notation of the Barannikov lemma, two critical points $\xi^k_\ell$ and $\xi^{k-1}_m$ of $f_{b,c}$ are *coupled* if $\partial\Xi^k_\ell = \Xi^{k-1}_m$. A critical point is *free* (over $\mathbf{F}$) when it is not coupled with any other critical point.

In other words, $\xi^k_\ell$ is free if and only if $\Xi^k_\ell$ is a cycle of $M_k(f_{b,c},\mathbf{F})$ but not a boundary, hence the following result:

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[4]) Note that if $\mathbf{F}$ were not a field, this would not provide a basis for noninvertible $\beta_{j_0}$.

COROLLARY 3.5. *For each integer $k$, the Betti number* $\dim_{\mathbf{F}} HM_k(f_{b,c},\mathbf{F})$ *is the number of free critical points of index $k$ of $f_{b,c}$ over* $\mathbf{F}$. □

THEOREM 3.6.

(1) *The Barannikov normal form of the Morse complex of $f_{b,c}$ over* $\mathbf{F}$ *is independent of the gradient-like vector field $V$.*

(2) *So is the Morse homology $HM_*(f_{b,c},R)$; it is isomorphic to $H_*(f^c,f^b;R)$.*

(3) *For $b' \le b < c \le c'$, the inclusion $i\colon f^c \hookrightarrow f^{c'}$, restricted to the critical set $C_*(f_{b,c})$, induces a linear map $i_*\colon M_*(f_{b,c},R) \to M_*(f_{b',c'},R)$ such that $\partial \circ i_* = i_* \circ \partial$ and therefore a linear map $i_*\colon HM_*(f_{b,c},R) \to HM_*(f_{b',c'},R)$, which is the usual $i_*\colon H_*(f^c,f^b;R) \to H_*(f^{c'},f^{b'};R)$ modulo the previous isomorphism.*

*Idea of the proof* [10]. (1) Connecting two generic gradient-like vector fields $V_0$, $V_1$ for $f$ by a generic family, one can prove that each of the Morse complexes defined by $V_0$ and $V_1$ is obtained from the other by a change of variables whose matrix is upper-triangular with all diagonal entries equal to 1.

(2) When there is no critical point of $f$ in $\{b \le f \le c\}$, both $HM_*(f_{b,c},R)$ and $H_*(f^c,f^b;R)$ are trivial (the flow of $V$ defines a retraction of $f^c$ onto $f^b$).

When there is only one critical point $\xi$ of $f$ in $\{b \le f \le c\}$, of index $\lambda$,

$$HM_k(f_{b,c},R) \simeq H_k(f^c,f^b;R) \simeq \begin{cases} R, & \text{if } k=\lambda, \\ 0 & \text{otherwise}: \end{cases}$$

the class of $\xi$ obviously generates $HM_\lambda(f_{b,c},R)$, whereas a generator of $H_\lambda(f^c,f^b;R)$ is the class of a cell of dimension $\lambda$, namely the stable manifold of $\xi$ for $V|_{\{b\le f\le c\}}$; the isomorphism associates the second class to the first.

In the general case, one can consider a subdivision $b = b_0 < \cdots < b_N = c$ consisting of regular values of $f$ such that each $f_{b_j,b_{j+1}}$ has precisely one critical point. One can show that the boundary operator $\partial$ of the relative singular homology $\partial\colon H_{k+1}(f^{b_{i+1}},f^{b_i}) \to H_k(f^{b_i},f^{b_{i-1}})$ can be interpreted as the intersection number of the stable manifold of the critical point in $\{b_i \le f \le b_{i+1}\}$ and the unstable manifold of that in $\{b_{i-1} \le f \le b_i\}$, i.e., their algebraic number of connecting trajectories.

(3) The first claims are easy. The last one follows from what has just been sketched. □

COROLLARY 3.7. *If $f$ is an excellent Morse function quadratic at infinity, then it has precisely one free critical point $\xi$ over $\mathbf{F}$; its index $\lambda$ is that of the reference quadratic form $Q$ and*

$$\underline{\gamma}(f,\mathbf{F}) = f(\xi)\,.$$

*Proof.* Clearly, the dimension of

$$HM_k(f,\mathbf{F}) = HM_k(f_{-\infty,\infty},\mathbf{F}) \simeq H_k(f^{\infty},f^{-\infty};\mathbf{F}) = H_k(Q^{\infty},Q^{-\infty};\mathbf{F})$$

is 1 if $k=\lambda$ and 0 otherwise. The first two assertions follow by Corollary 3.5. To prove $\underline{\gamma}(f,\mathbf{F}) = f(\xi)$, note that $\underline{\gamma}(f)$ is the infimum of the regular values $c$ of $f$ such that the class of $\xi$ in $HM_\lambda(f_{-\infty,\infty},\mathbf{F})$ lies in the image of $i_{c*}\colon HM_\lambda(f_{-\infty,c},\mathbf{F}) \to HM_\lambda(f_{-\infty,\infty},\mathbf{F})$ by Theorem 3.6(3), which means $c \geq f(\xi)$.

PROPOSITION 3.8. *The excellent Morse function $-f_{b,c} = (-f)_{-c,-b}$ has the same free critical points over the field $\mathbf{F}$ as $f_{b,c}$.*

*Proof.* Assuming $V$ fixed, this is essentially easy linear algebra:

- One has $C_k(-f) = C_{n-k}(f)$ and the ordering of the corresponding critical values is reversed. Thus, the lexicographically ordered basis of $M_*(-f)$ corresponding to $(\xi^k_\ell)_{1\leq\ell\leq m_k,0\leq k\leq n}$ is $(\xi^{n-k}_{m_{n-k}-\ell+1})_{1\leq\ell\leq m_{n-k},0\leq k\leq n}$.
- The vector field $-V$ has the same relations with $-f$ as $V$ has with $f$, hence $\nu_{-f,-V}(\xi^{n-k}_{m_{n-k}-\ell+1},\xi^{n-(k-1)}_{m_{n-(k-1)}-m+1}) = \nu_{f,V}(\xi^{n-(k-1)}_{m_{n-(k-1)}-m+1},\xi^{n-k}_{m_{n-k}-\ell+1})$.

That is, the matrix of the boundary operator of $M_*(-f_{b,c})$ in the basis $(\xi^{n-k}_{m_{n-k}-\ell+1})$ is the matrix $\widetilde{M}$ obtained from the matrix $A$ of the boundary operator of $M_*(f_{b,c})$ in the basis $(\xi^k_\ell)$ by symmetry with respect to the second diagonal (i.e. by reversing the order of both the lines and columns of the transpose of $A$).

Lemma 3.3 can be rephrased as follows: there exists a block-diagonal matrix

$$P = \operatorname{diag}(P_0,\ldots,P_n)$$

where each $P_k \in \mathrm{GL}(m_k,\mathbf{F})$ is upper triangular, such that

$$P^{-1}AP = B \tag{2}$$

is a Barannikov normal form, meaning the following: the entries of the column of indices ${}^k_\ell$ are 0 except possibly one, equal to 1, which must lie on the line of indices ${}^{k-1}_m$ for some $m$ and be the only nonzero entry on this line. The normal form $B$ is the same for every choice of $P$ and $V$. Clearly, $\xi^k_\ell$ is a free critical point of $f_{b,c}$ if and only if both the line and column of indices ${}^{k}_{\ell}$ of $B$ are zero.

Equation (2) reads

$$\widetilde{P}\widetilde{A}\widetilde{P}^{-1} = \widetilde{B}. \tag{3}$$

Now, $\widetilde{P}^{-1}$ and $\widetilde{P} = (\widetilde{P}^{-1})^{-1}$ are block diagonal upper triangular matrices whose $k^{th}$ diagonal block lies in $\mathrm{GL}(m_{n-k}, \mathbf{F})$; therefore, by (3), as $\widetilde{B}$ is a Barannikov normal form for the ordering associated to $-f$, it is *the* Barannikov normal form of the boundary operator of $M_*(-f_{b,c})$, from which our result follows at once.

COROLLARY 3.9. *For any excellent Morse function $f$ quadratic at infinity, the sole free critical point of $-f$ over* $\mathbf{F}$ *is the free critical point $\xi$ of $f$; hence $\underline{\gamma}(f,\mathbf{F}) = f(\xi) = -(-f)(\xi) = -\underline{\gamma}(-f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{F})$ by Corollary* 3.7 *and Lemma* 2.8*, which proves Theorem* 2.11. □

Before we give an example where $\underline{\gamma}(f,\mathbf{Z}) > \overline{\gamma}(f,\mathbf{Z})$, here is a situation where this cannot occur:

PROPOSITION 3.10. *Assume that $M_*(f,\mathbf{Z})$ can be put into Barannikov normal form by a basis change* (1) *of the free* $\mathbf{Z}$*-module $M_*(f,\mathbf{Z})$:*

$$\Xi^k_\ell := \sum_{i\le \ell} \alpha^k_{\ell,i}\xi^k_i\,, \quad \alpha^k_{\ell,i} \in \mathbf{Z}\,, \quad \alpha^k_{\ell,\ell} = \pm 1\,. \tag{4}$$

*Then, $\underline{\gamma}(f,\mathbf{Z}) = \overline{\gamma}(f,\mathbf{Z}) = f(\xi)$, where $\xi$ is the sole free critical point of $f$ over* $\mathbf{Z}$.

*Proof.* We are in the situation of the proof of Proposition 3.8 with $P_k \in \mathrm{GL}(m_k,\mathbf{Z})$, which implies that the Barannikov normal form $B$ of the boundary operator is the same for $\mathbf{Z}$ as for $\mathbf{Q}$; it does follow that there is a unique free critical point $\xi$ of $f$ over $\mathbf{Z}$ (the same as over $\mathbf{Q}$) and that it is the unique free critical point of $-f$ over $\mathbf{Z}$; moreover, the proof of Corollary 3.7 shows that $\underline{\gamma}(f,\mathbf{Z}) = \overline{\gamma}(f,\mathbf{Z}) = f(\xi)$. We conclude as in Corollary 3.9.

Now that the coefficients are in $\mathbf{Z}$, the classical method called *handle sliding* [10, 12] states that, under an additional condition imposed on the index of the change of basis in (4), namely $2 \le k \le n-2$, the Barannikov normal form can be realized by a gradient-like vector field for $f$.

More precisely, let $P\colon M_*(f) \to M_*(f)$ be a transformation matrix where $P = \mathrm{diag}(P_0,\dots,P_n)$ with each $P_k \in \mathrm{GL}(m_k,\mathbf{Z})$ such that $P_k = id$ for $k = 0,1$

or $n-1, n$, and $P_k$ is upper triangular with $\pm 1$ in the diagonal entries for $2 \leq k \leq n-2$. Then one can construct a gradient-like vector field $V'$ such that, if the matrix of the boundary operator for a given gradient-like vector field $V$ is $A$, then the matrix for $V'$ is given by $B = P^{-1}AP$.

Roughly speaking, one modifies $V$, each time for one $i \leq \ell$, by sliding the stable sphere[5]) $S_L(\xi^k_\ell)$ of $\xi^k_\ell$ for $V$ so that it sweeps across the unstable sphere $S_R(\xi^k_i)$ of $\xi^k_i$ with indicated intersection number. In other words, $S'_L(\xi^k_\ell)$ for the resulted $V'$ is the connected sum of $S_L(\xi^k_\ell)$ and the boundary of a meridian disk of $S_R(\xi^k_i)$ described in section 4.4 of [10]. One may refer to the Basis Theorem (Theorem 7.6 in [12]) for a detailed construction of $V'$.

REMARK 3.11 (on the "proof" of Corollary 3.9 in [3]). Capitanio uses the following:

CRITERION. *A critical point $\xi$ of $f$ is free (over* $\mathbf{Q}$*) if and only if, for any critical point $\eta$ incident to $\xi$, there is a critical point $\xi'$, incident to $\eta$, such that*

$$|f(\xi') - f(\eta)| < |f(\xi) - f(\eta)|,$$

*where, given a generic gradient-like vector field $V$ for $f$, two critical points are called* incident *if their algebraic number of connecting trajectories is nonzero.*

Unfortunately, this is not true: one can construct a function $f\colon \mathbf{R}^{2n} \to \mathbf{R}$, $n \geq 2$, quadratic at infinity with Morse index $n$, having five critical points, two of index $n-1$ and three of index $n$, whose gradient vector field $V$ defines the Morse complex

$$\partial\xi^n_1 = \xi^{n-1}_2, \qquad \partial\xi^n_2 = \xi^{n-1}_1, \qquad \partial\xi^n_3 = 0.$$

This complex can be reformulated into

$$\begin{aligned} \partial\xi^n_1 &= (\xi^{n-1}_2 - \xi^{n-1}_1) + \xi^{n-1}_1 \\ \partial(\xi^n_2 + \xi^n_1) &= (\xi^{n-1}_2 - \xi^{n-1}_1) + 2\xi^{n-1}_1 \\ \partial(\xi^n_3 + \xi^n_2) &= \xi^{n-1}_1. \end{aligned}$$

Hence, for a change of basis

$$\xi^{n-1}_2 \mapsto \xi^{n-1}_2 - \xi^{n-1}_1, \quad \xi^n_2 \mapsto \xi^n_2 + \xi^n_1, \quad \xi^n_3 \mapsto \xi^n_3 + \xi^n_2$$

[5]) The *stable* and *unstable spheres* are $S_L(\xi^k_\ell) = W^s(\xi^k_\ell) \cap L$ and $S_R(\xi^k_i) = W^u(\xi^k_i) \cap L$ where $L = f^{-1}(c)$ for some $c \in \big(f(\xi^k_i), f(\xi^k_\ell)\big)$.

one can construct a gradient-like vector field $V'$ for $f$ by sliding handles, such that

$$\partial\xi_1^n = \xi_2^{n-1} + \xi_1^{n-1}, \quad \partial\xi_2^n = \xi_2^{n-1} + 2\xi_1^{n-1}, \quad \partial\xi_3^n = \xi_1^{n-1}.$$

Obviously, $\xi_3^n$ is the only free critical point, but $\xi_2^n$ satisfies the criterion (with incidences under $V'$). □

## 4. An example of Laudenbach

PROPOSITION 4.1. *There exists an excellent Morse function $f\colon \mathbf{R}^{2n} \to \mathbf{R}$ as follows:*

1. *it is quadratic at infinity and the reference quadratic form has index and coindex $n > 1$;*
2. *it has exactly five critical points: three of index $n$, one of index $n-1$ and one of index $n+1$;*
3. *its Morse complex over $\mathbf{Z}$ is given by*

$$(5)\qquad \begin{aligned} &\partial\xi_1^{n-1} = 0 \\ &\partial\xi_1^n = \xi_1^{n-1}, \quad \partial\xi_2^n = -2\xi_1^{n-1}, \quad \partial\xi_3^n = -\xi_1^{n-1} \\ &\partial\xi_1^{n+1} = \xi_2^n - 2\xi_3^n, \end{aligned}$$

*hence, for any field $\mathbf{F}_2$ of characteristic 2 and any field $\mathbf{F}$ of characteristic $\neq 2$,*

$$(6)\quad \begin{aligned} \underline{\gamma}(f,\mathbf{Z}) = \underline{\gamma}(f,\mathbf{F}_2) = \overline{\gamma}(f,\mathbf{F}_2) = f(\xi_3^n) \\ > f(\xi_2^n) = \underline{\gamma}(f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{F}) = \overline{\gamma}(f,\mathbf{Z}). \end{aligned}$$

*Proof that* (5) *implies* (6). The Morse complex of $f$ over $\mathbf{F}_2$ is written

$$\begin{aligned} &\partial\xi_1^{n-1} = 0 \\ &\partial\xi_1^n = \xi_1^{n-1}, \quad \partial\xi_2^n = 0, \quad \partial(\xi_3^n + \xi_1^n) = 0 \\ &\partial\xi_1^{n+1} = \xi_2^n, \end{aligned}$$

implying that $\xi_3^n$ is the only free critical point, hence, by Corollary 3.7,

$$\underline{\gamma}(f,\mathbf{F}_2) = \overline{\gamma}(f,\mathbf{F}_2) = f(\xi_3^n);$$

as $\underline{\gamma}(f,\mathbf{Z}) \geq \underline{\gamma}(f,\mathbf{F}_2)$ by Proposition 2.10 and $\underline{\gamma}(f,\mathbf{Z}) \leq f(\xi_3^n)$, we do have

$$\underline{\gamma}(f,\mathbf{Z}) = f(\xi_3^n).$$

Similarly (keeping the numbering of the critical points defined by $f$) the Morse complex of $-f$ over $\mathbf{F}$ has the Barannikov normal form

$$\begin{aligned}
&\partial(-2\xi_1^{n+1}) = 0\\
&\partial\xi_3^n = -2\xi_1^{n+1}, \quad \partial(\xi_2^n + \frac{1}{2}\xi_3^n) = 0, \quad \partial(-\xi_3^n - 2\xi_2^n + \xi_1^n) = 0\\
&\partial\xi_1^{n-1} = -\xi_3^n - 2\xi_2^n + \xi_1^n,
\end{aligned}$$

showing that the free critical point is $\xi_2^n$; hence, by Corollary 3.7 and Proposition 3.8,

$$\overline{\gamma}(f,\mathbf{F}) = \underline{\gamma}(f,\mathbf{F}) = f(\xi_2^n);$$

finally, as we have $\overline{\gamma}(f,\mathbf{Z}) \leq \overline{\gamma}(f,\mathbf{F})$ by Proposition 2.10, and $\overline{\gamma}(f,\mathbf{Z}) \geq f(\xi_1^n)$, we should prove $\overline{\gamma}(f,\mathbf{Z}) > f(\xi_1^n)$, which is obvious since $\xi_1^n$ and $\xi_1^{n+1}$ are boundaries in $M_*(-f,\mathbf{Z})$.

*How to construct such a function* $f$. It is easy to construct a function $f_0\colon \mathbf{R}^{2n} \to \mathbf{R}$ with properties (1) and (2) required in the proposition and whose gradient vector field $V_0$ provides a Morse complex given by

$$\begin{aligned}
&\partial\xi_1^{n-1} = 0\\
&\partial\xi_1^n = \xi_1^{n-1}, \quad \partial\xi_2^n = 0, \quad \partial\xi_3^n = 0\\
&\partial\xi_1^{n+1} = \xi_3^n.
\end{aligned}$$

For a change of basis

$$\xi_2^n \mapsto \xi_2^n - \xi_1^n, \qquad \xi_3^n \mapsto \xi_3^n - 2(\xi_2^n - \xi_1^n)$$

one can construct a gradient-like vector field $V'$ for $f_0$ by sliding handles, such that

$$\begin{aligned}
&\partial\xi_1^{n-1} = 0\\
&\partial\xi_1^n = \xi_1^{n-1}, \quad \partial\xi_2^n = -\xi_1^{n-1}, \quad \xi_3^n = -2\xi_1^{n-1}\\
&\partial\xi_1^{n+1} = -2\xi_2^n + \xi_3^n.
\end{aligned}$$

Since $(f_0, V')$ is Morse-Smale, the invariant manifolds of those critical points of the same index are disjoint, hence one can modify $f_0$ to $f$ such that

- $f$ has the same critical points as $f_0$;
- the ordering of critical points for $f$ is $f(\xi_2^n) > f(\xi_3^n) > f(\xi_1^n)$;
- $V'$ is a gradient-like vector field for $f$.

This can be realized by the preliminary rearrangement theorem (Theorem 4.1 in [12]).

In other words, we have made a change of critical points $\xi_2^n \leftrightarrow \xi_3^n$, hence obtain the required Morse complex in the proposition.

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