Selberg's central limit theorem for log (1/2 + it)

Autor(en): **Radziwi, Maksym / Soundararajan, Kannan**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **63 (2017)**

Heft 1-2

PDF erstellt am: **09.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-760283

L'Enseignement Mathématique (2) 63 (2017), 1–19 DOI 10.4171/LEM/63-1/2-1

Selberg's central limit theorem for $\log|\zeta(1/2+it)|$

Maksym RADZIWIŁŁ and Kannan SOUNDARARAJAN

Abstract. We present a new and simple proof of Selberg's central limit theorem, according to which the logarithm of the Riemann zeta-function at height t is approximately normally distributed with mean 0 and variance $\frac{1}{2}\log\log t$.

Mathematics Subject Classification (2010). Primary: 11M06.

Keywords. Riemann zeta-function, value distribution, central limit theorem.

1. Introduction

Motivated by the Riemann hypothesis, a classical theme in analytic number theory has been to understand the value distribution of the Riemann zeta-function $\zeta(s)$. For example, fixing $\sigma = \mathrm{Re}(s)$ one may ask for the distribution of $\zeta(\sigma+it)$ as t varies in $[T, 2T]$ for large T. In view of the functional equation connecting $\zeta(s)$ to $\zeta(1-s)$, we may suppose here that $\sigma \geq \frac{1}{2}$, with the case $\sigma = \frac{1}{2}$ – the value distribution on the "critical line" – being of greatest interest.

When $\sigma > \frac{1}{2}$ is fixed, from the classical work of Bohr and Jessen [BJ1, BJ2] we have a good qualitative understanding of the distribution of $\zeta(\sigma+it)$. Suppose $X = X(T) \leq \log T$ is a parameter tending slowly to infinity with T (to fix ideas one can think of $X(T) = \sqrt{\log T}$). Then for typical $t \in [T, 2T]$ (by which we mean t lying outside a set of measure $o(T)$) one has

$$\zeta(\sigma+it) \sim \prod_{p\leq X}\Big(1-\frac{1}{p^{\sigma+it}}\Big)^{-1}.$$

In other words, $\zeta(\sigma+it)$ has an almost periodic structure and its value can be usually extracted from knowledge of p^{it} for small primes p. Further if X is suitably small, then Kronecker's theorem can be used to show that as t varies, the values p^{it} for $p \leq X$ are equidistributed on the unit circle, with each prime behaving "independently" of the others.

Here and throughout the paper, we use the standard Landau and Vinogradov notations of analytic number theory. Thus $f \sim g$ means that the ratio f/g tends to 1 for a suitable parameter (e.g., T above) tending to infinity; $f = o(g)$ means that $|f/g|$ tends to 0; and $f = O(g)$ and the Vinogradov notation $f \ll g$ both mean that $|f| \le C|g|$ for some absolute constant C and large values of the implicit parameter.

The value distribution on the critical line $\sigma = \frac{1}{2}$ is very different. Here $\zeta(\frac{1}{2} + it)$ no longer behaves like an almost periodic function, and these values are not typically determined by knowledge of p^{it} at small primes. Selberg [Sel2, Sel1] established the fundamental theorem that as t varies in $[T, 2T]$, the quantity $\log \zeta(\frac{1}{2}+it)/\sqrt{\frac{1}{2}\log\log t}$ behaves like a standard complex normal random variable – that is, its real and imaginary parts are distributed like independent normal random variables with mean 0 and variance 1 (see Theorem 1 for a precise statement for the real part). Further, Selberg's result holds not just for the Riemann zeta-function, but for a large class of L-functions arising from automorphic forms (provided one has some understanding of the distribution of zeros of such L-functions).

Selberg's work illuminates our understanding of zeta and L-functions on the critical line. Qualitatively, it shows that typical values of $|\zeta(\frac{1}{2} + it)|$ are either very small (say $\le 1/A$, for any A with $\log A = o(\sqrt{\log\log T})$) or very large ($> A$ with A as before), and that intermediate values appear only on a set of measure $o(T)$. This is in stark contrast to $|\zeta(\sigma+it)|$ for $\sigma > \frac{1}{2}$, which is typically of constant size. Here we may highlight the interesting open problem of whether the values $\zeta(\frac{1}{2} + it)$ as t varies are dense in the complex plane. An analogous result for $\zeta(\sigma + it)$ with $\frac{1}{2} < \sigma \le 1$ is known, but Selberg's result indicates why the problem for $\sigma = \frac{1}{2}$ has an entirely different flavor. Selberg's result is also a key to understanding questions such as the rate of growth of moments of zeta and L-functions (see [KS], [Sou]).

In this paper we give a new and simple proof of Selberg's influential theorem [Sel2, Sel1] for the real part of $\log \zeta(\frac{1}{2}+it)$. Thus, we establish that $\log|\zeta(\frac{1}{2}+it)|$ has an approximately normal distribution with mean zero and variance $\frac{1}{2}\log\log|t|$. Apart from some basic facts about the Riemann zeta function, we have tried to make our proof self-contained.

Theorem 1. *Let V be a fixed positive real number. Then as $T \to \infty$, uniformly for all $v \in [-V, V]$,*

$$\frac{1}{T} meas\Big\{T \le t \le 2T : \quad \log\big|\zeta(\tfrac{1}{2} + it)\big| \ge v\sqrt{\tfrac{1}{2}\log\log T}\Big\} \sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-u^2/2} du.$$

We now outline the proof, which is broadly based on four steps. The first step is to show that $\log|\zeta(\frac{1}{2}+it)|$ is usually close to $\log|\zeta(\sigma+it)|$ for suitable σ near $\frac{1}{2}$.

Proposition 1. *Let T be large, and suppose $T \le t \le 2T$. Then for any $\sigma > 1/2$ we have*

$$\int_{t-1}^{t+1} \Big| \log\big|\zeta(\tfrac{1}{2}+iy)\big| - \log\big|\zeta(\sigma+iy)\big| \Big| dy \ll (\sigma - \tfrac{1}{2}) \log T.$$

The proof of Proposition 1 is the only place where we will briefly need to mention the zeros of $\zeta(s)$. From now on, we set $\sigma_0 = \frac{1}{2} + \frac{W}{\log T}$, for a suitable parameter $W \ge 3$ to be chosen later. From Proposition 1 it follows that $\log|\zeta(\frac{1}{2}+it)|$ and $\log|\zeta(\sigma_0+it)|$ differ by at most AW except on a set of measure $O(T/A)$. If AW is small compared to $\sqrt{\log\log T}$, then this difference is negligible, and both quantities have the same distribution. Therefore we may focus on understanding the distribution of $\log|\zeta(\sigma_0+it)|$, which we may hope is an easier problem since we have moved away from the critical line.

There is considerable latitude in choosing parameters such as W, but to fix ideas we select

$$W = (\log\log\log T)^4, \quad X = T^{1/(\log\log\log T)^2}, \quad \text{and} \quad Y = T^{1/(\log\log T)^2}. \tag{1}$$

Here X and Y are two parameters that will appear shortly. Put

$$\mathcal{P}(s) = \mathcal{P}(s; X) = \sum_{2 \le n \le X} \frac{\Lambda(n)}{n^s \log n}, \tag{2}$$

where $\Lambda(n)$ denotes the von Mangoldt function, which is given by $\Lambda(n) = \log p$ if $n = p^k$ is a power of the prime p, and $\Lambda(n) = 0$ if n is not a prime power. By computing moments, in Section 3 we shall determine the distribution of Re $\mathcal{P}(\sigma_0+it)$.

Proposition 2. *As t varies in $T \le t \le 2T$, the distribution of $Re(\mathcal{P}(\sigma_0+it))$ is approximately normal with mean 0 and variance $\sim \frac{1}{2}\log\log T$. Precisely, if V is a fixed positive real number then as $T \to \infty$, uniformly for all $v \in [-V, V]$,*

$$\frac{1}{T} meas\Big\{T \le t \le 2T : Re\mathcal{P}(\sigma_0+it) \ge v\sqrt{\tfrac{1}{2}\log\log T}\Big\} \sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-u^2/2} du.$$

Our goal is now to connect $\text{Re}(\mathcal{P}(\sigma_0+it))$ with $\log|\zeta(\sigma_0+it)|$ for most values of t. This is done in two stages. First we introduce a Dirichlet polynomial $M(s)$ which we shall show is close to $\exp(-\mathcal{P}(s))$ except for $t \in [T, 2T]$ lying in a subset of measure $o(T)$. Define $a(n) = 1$ if n is composed of only primes below

X, and it has at most $100 \log\log T$ primes below Y, and at most $100 \log\log\log T$ primes between Y and X; set $a(n) = 0$ in all other cases. Put

$$M(s) = \sum_n \frac{\mu(n)a(n)}{n^s}. \tag{3}$$

Note that $a(n) = 0$ unless $n \le Y^{100 \log\log T} X^{100 \log\log\log T} < T^{\epsilon}$, and so $M(s)$ is a short Dirichlet polynomial. The motivation behind our definition of $M(s)$ will emerge in Section 4 where we establish the following proposition.

Proposition 3. *With notations as above, we have for* $T \le t \le 2T$

$$M(\sigma_0 + it) = \big(1 + o(1)\big) \exp\big(-\mathcal{P}(\sigma_0 + it)\big),$$

except perhaps on a subset of measure $o(T)$.

The final step of the proof shows that $\zeta(\sigma_0 + it)M(\sigma_0 + it)$ is typically close to 1.

Proposition 4. *With notations as above,*

$$\frac{1}{T}\int_T^{2T} \left|1 - \zeta(\sigma_0 + it)M(\sigma_0 + it)\right|^2 dt = o(1),$$

so that for $T \le t \le 2T$ *we have*

$$\zeta(\sigma_0 + it)M(\sigma_0 + it) = 1 + o(1),$$

except perhaps on a set of measure $o(T)$.

Proof of Theorem 1. We now show how to assemble the four propositions above to deduce Theorem 1. Proposition 4 shows that typically (that is for all $t \in [T, 2T]$ outside a set of measure $o(T)$) one has

$$\zeta(\sigma_0 + it) = \big(1 + o(1)\big)M(\sigma_0 + it)^{-1}.$$

Combining this with Proposition 3, outside a set of measure $o(T)$ we have

$$|\zeta(\sigma_0 + it)| = \big(1 + o(1)\big)\exp(\mathrm{Re}\ \mathcal{P}\big(\sigma_0 + it)\big).$$

Appealing now to Proposition 2, we conclude that $\log|\zeta(\sigma_0 + it)|$ is normally distributed with mean 0 and variance $\sim \sqrt{\frac{1}{2}\log\log T}$.

Now by Proposition 1 it follows that

$$\int_T^{2T} \Big|\log\big|\zeta(\tfrac{1}{2} + it)\big| - \log\big|\zeta(\sigma_0 + it)\big|\Big| dt \ll T(\sigma_0 - \tfrac{1}{2})\log T = WT,$$

so that outside a subset of $[T, 2T]$ of measure $O(T/W) = o(T)$ one has

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| = \log\left|\zeta(\sigma_0+it)\right| + O(W^2).$$

Since $W^2 = o(\sqrt{\log\log T})$, it follows that like $\log|\zeta(\sigma_0+it)|$, the quantity $\log|\zeta(\frac{1}{2}+it)|$ also has a normal distribution with mean 0 and variance $\sim \sqrt{\frac{1}{2}\log\log T}$. This completes the proof of Theorem 1. □

After developing the proofs of the propositions, in Section 7 we compare and contrast our approach with previous proofs, and also discuss possible extensions of this technique.

2. Proof of Proposition 1

Put $G(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)$ and let $\xi(s) = G(s)\zeta(s)$ denote the completed ζ-function. If t is large and $t-1 \le y \le t+1$, then by Stirling's formula

$$\left|\log\frac{G(\sigma+iy)}{G(1/2+iy)}\right| \ll (\sigma-1/2)\log t,$$

and so it is enough to prove that

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)}\right|\right|dy \ll (\sigma-\tfrac{1}{2})\log T.$$

Recall Hadamard's factorization formula

$$\xi(s) = e^{A+Bs}\prod_\rho\Big(1-\frac{s}{\rho}\Big)e^{s/\rho},$$

where A and B are constants with $B = -\sum_\rho \mathrm{Re}\,(1/\rho)$. Here the product is over all non-trivial zeros of $\zeta(s)$, which all lie in the region $0 \le \mathrm{Re}(\rho) \le 1$. Thus (assuming that y is not the ordinate of a zero of $\zeta(s)$)

$$\log\left|\frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)}\right| = \sum_\rho \log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|.$$

Integrating the above over $y \in (t-1, t+1)$ we get

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)}\right|\right|dy \le \sum_\rho\int_{t-1}^{t+1}\left|\log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|\right|dy. \tag{4}$$

Suppose $\rho = \beta+i\gamma$ is a zero of $\zeta(s)$. If $|t-\gamma| \ge 2$ then, for any $t-1 \le y \le t+1$

$$\left|\log\left|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\right|\right| = \left|\mathrm{Re}\log\Big(1-\frac{\sigma-\frac12}{\sigma+iy-\rho}\Big)\right|$$
$$= \left|\mathrm{Re}\frac{\sigma-\frac12}{\sigma+iy-\rho}\right| + O\Big(\frac{(\sigma-\frac12)^2}{(y-\gamma)^2}\Big) = O\Big(\frac{\sigma-\frac12}{(y-\gamma)^2}\Big),$$

so that

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\right|\right|dy \ll \frac{(\sigma-\frac12)}{(t-\gamma)^2}.$$

In the range $|t-\gamma|\le 2$ we use

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\right|\right|dy \le \frac12\int_{-\infty}^{\infty}\left|\log\frac{(\beta-\frac12)^2+x^2}{(\beta-\sigma)^2+x^2}\right|dx = \pi(\sigma-\tfrac12).$$

Thus in either case

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\frac12+iy-\rho}{\sigma+iy-\rho}\right|\right|dy \ll \frac{(\sigma-\frac12)}{1+(t-\gamma)^2}.$$

Inserting this in (4), and noting that there are $\ll \log(t+k)$ zeros with $k\le|t-\gamma|<k+1$, the proposition follows.

3. Proof of Proposition 2

Before proceeding to the proof of Proposition 2, we record a simple estimate which will be useful throughout the paper: for any two natural numbers m and n

$$\int_T^{2T}\Big(\frac{m}{n}\Big)^{it}dt = \begin{cases} T & \text{if } m=n,\\ O\Big(\min\big(T,\frac{1}{|\log(m/n)|}\big)\Big) & \text{if } m\ne n.\end{cases} \tag{5}$$

This follows upon evaluating the integral. In the case $m\ne n$, the following elementary estimates are also useful:

$$\frac{1}{|\log(m/n)|} \ll \begin{cases} 1 & \text{if } m\ge 2n, \text{ or if } m\le n/2;\\ m/|m-n| & \text{if } n/2<m\le 2n;\\ \sqrt{mn} & \text{always.}\end{cases} \tag{6}$$

We begin work on Proposition 2 by showing that we may restrict the sum in $\mathcal{P}(s)$ just to primes. The contribution of cubes and higher powers of primes is clearly $O(1)$, and we need only discard the contribution of squares of primes. By expanding out, and using (5) and (6),

$$\int_T^{2T} \Big| \sum_{p\le\sqrt{X}} \frac{1}{2p^{2(\sigma_0+it)}} \Big|^2 dt \ll T \sum_{\substack{p_1,p_2\le\sqrt{X}\\ p_1=p_2}} \frac{1}{(p_1p_2)^{2\sigma_0}} + \sum_{\substack{p_1,p_2\le\sqrt{X}\\ p_1\ne p_2}} \frac{1}{(p_1p_2)^{2\sigma_0}} \frac{1}{|\log(p_1/p_2)|}$$

$$\ll T + \sum_{\substack{p_1,p_2\le\sqrt{X}\\ p_1\ne p_2}} \frac{1}{(p_1p_2)^{2\sigma_0}} \sqrt{p_1p_2} \ll T.$$

Therefore, the measure of the set $t \in [T, 2T]$ with the contribution of prime squares being larger than L (say) is at most $\ll T/L^2$. Write

$$\mathcal{P}_0(\sigma_0+it) := \sum_{p\le X} \frac{1}{p^{\sigma_0+it}}.$$

In view of our estimate for the contribution of prime squares, to establish Proposition 2, it is enough to show that $\mathrm{Re}(\mathcal{P}_0(\sigma_0+it))$ has an approximately Gaussian distribution with mean 0 and variance $\sim \frac{1}{2}\log\log T$. We establish this by computing moments, keeping in mind that the Gaussian distribution is uniquely determined by its moments.

Lemma 1. *Suppose that k and ℓ are non-negative integers with $X^{k+\ell} \le T$. Then, if $k \ne \ell$,*

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \mathcal{P}_0(\sigma_0-it)^\ell dt \ll T,$$

while if $k = \ell$ we have

$$\int_T^{2T} \big|\mathcal{P}_0(\sigma_0+it)\big|^{2k} dt = k!T(\log\log T)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Write $\mathcal{P}_0(s)^k = \sum_n a_k(n)n^{-s}$, where $a_k(n) = 0$ unless n has the prime factorization $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ where $p_1, \ldots, p_r$ are distinct primes below X, and $\alpha_1+\ldots+\alpha_r = k$, in which case $a_k(n) = k!/(\alpha_1!\cdots\alpha_r!)$. Therefore, expanding out the integral, we obtain

$$\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \mathcal{P}_0(\sigma_0-it)^\ell dt$$
$$= T\sum_n \frac{a_k(n)a_\ell(n)}{n^{2\sigma_0}} + O\Big(\sum_{m\ne n} \frac{a_k(m)a_\ell(n)}{(mn)^{\sigma_0}} \frac{1}{|\log(m/n)|}\Big).$$

If $m \ne n$, then using the third estimate of (6), we see that the off-diagonal terms above contribute

$$\ll \sum_{m \neq n} a_k(m) a_\ell(n) \ll X^{k+\ell} \ll T.$$

Note that if $k \neq \ell$ then $a_k(n)a_\ell(n)$ is always zero, and the first statement of the lemma follows.

It remains, in the case $k = \ell$, to discuss the diagonal term, which is $T \sum_n a_k(n)^2/n^{2\sigma_0}$. Given positive integers $\alpha_1, \ldots, \alpha_r$ with $\alpha_1 + \ldots + \alpha_r = k$, the contribution of terms n of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is

$$\ll T \prod_{j=1}^{r} \Big(\sum_{p \le X} \frac{1}{p^{2\sigma_0 \alpha_j}} \Big) \ll T (\log\log T)^r.$$

Therefore, the contribution from terms n that are not square-free (in which case $r \le k-1$) is $O(T(\log\log T)^{k-1})$. Finally the square-free terms n (so that n is the product of k distinct primes) give

$$k! \sum_{\substack{p_1, \ldots, p_k \le X \\ p_j \text{ distinct}}} \frac{1}{(p_1 \cdots p_k)^{2\sigma_0}} = k! \Big(\sum_{p \le X} \frac{1}{p^{2\sigma_0}} \Big)^k + O_k\big((\log\log T)^{k-1}\big),$$

and the lemma follows upon recalling the definition (1). □

From Lemma 1 we see that if $X^k \le T$ then for odd k

$$\begin{aligned} \int_T^{2T} (\text{Re } \mathcal{P}_0(\sigma_0 + it))^k dt &= \frac{1}{2^k} \int_T^{2T} (\mathcal{P}_0(\sigma_0 + it) + \mathcal{P}_0(\sigma_0 - it))^k dt \\ &= \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} \int_T^{2T} \mathcal{P}_0(\sigma_0 + it)^\ell \mathcal{P}_0(\sigma_0 - it)^{k-\ell} dt \ll T, \end{aligned}$$

since $\ell \neq k - \ell$ for odd k. If k is even then, extracting the contribution of $\ell = k - \ell = k/2$ above, we obtain

$$\begin{aligned} &\frac{1}{T} \int_T^{2T} (\text{Re } \mathcal{P}_0(\sigma_0 + it)^k dt \\ &\qquad = 2^{-k} \binom{k}{k/2} (k/2)! (\log\log T)^{k/2} + O_k((\log\log T)^{k-1+\epsilon}). \end{aligned}$$

These moments match the moments of a Gaussian random variable with mean zero and variance $\sim \frac{1}{2} \log\log T$, and since the Gaussian is determined by its moments, Proposition 2 follows.

4. Proof of Proposition 3

Let us decompose $\mathcal{P}(s)$ as $\mathcal{P}_1(s)+\mathcal{P}_2(s)$, where

$$\mathcal{P}_1(s)=\sum_{2\le n\le Y}\frac{\Lambda(n)}{n^s\log n}\qquad\text{and}\qquad \mathcal{P}_2(s)=\sum_{Y<n\le X}\frac{\Lambda(n)}{n^s\log n}.$$

Put

$$\mathcal{M}_1(s)=\sum_{0\le k\le 100\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_1(s)^k\quad\text{and}\quad \mathcal{M}_2(s)=\sum_{0\le k\le 100\log\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_2(s)^k.$$

Recalling the definitions of X and Y (see (1)) we see that $\mathcal{M}_1$ and $\mathcal{M}_2$ are both short Dirichlet polynomials of length $\ll T^{\epsilon}$.

Lemma 2. *For $T\le t\le 2T$ we have*

$$|\mathcal{P}_1(\sigma_0+it)|\le\log\log T,\ \text{and}\ |\mathcal{P}_2(\sigma_0+it)|\le\log\log\log T, \tag{7}$$

except perhaps for a set of measure $\ll T/\log\log\log T$. When the bounds (7) hold, we have

$$\mathcal{M}_1(\sigma_0+it)=\exp\big(-\mathcal{P}_1(\sigma_0+it)\big)\Big(1+O\big((\log T)^{-99}\big)\Big), \tag{8}$$

and

$$\mathcal{M}_2(\sigma_0+it)=\exp\big(-\mathcal{P}_2(\sigma_0+it)\big)\Big(1+O\big((\log\log T)^{-99}\big)\Big). \tag{9}$$

Proof. Using (5) and the third estimate of (6), we find

$$\begin{aligned}\int_T^{2T}\big|\mathcal{P}_1(\sigma_0+it)\big|^2dt &\ll T\sum_{\substack{2\le n_1,n_2\le Y\\ n_1=n_2}}\frac{\Lambda(n_1)\Lambda(n_2)}{(n_1n_2)^{\sigma_0}\log n_1\log n_2}\\ &\quad+\sum_{\substack{2\le n_1,n_2\le Y\\ n_1\ne n_2}}\frac{\Lambda(n_1)\Lambda(n_2)}{(n_1n_2)^{\sigma_0}\log n_1\log n_2}\sqrt{n_1n_2}\\ &\ll T\log\log T.\end{aligned}$$

Similarly

$$\int_T^{2T}\big|\mathcal{P}_2(\sigma_0+it)\big|^2dt\ll T\log\log\log T.$$

The first assertion (7) follows.

Suppose $K\ge 1$ is a real number. If $|z|\le K$ then, using that $k!\ge(k/e)^k$,

$$\sum_{0\le k\le 100K}\frac{z^k}{k!}=e^z+O\Big(\sum_{k>100K}\frac{K^k}{k!}\Big)=e^z+O\Big(\sum_{k>100K}\Big(\frac{eK}{k}\Big)^k\Big)$$
$$=e^z+O(e^{-100K}).$$

Since $|z|\le K$, we may also write the right side above as $e^z(1+O(e^{-99K}))$. The estimate (8) holds upon taking $z=-\mathcal{P}_1(\sigma_0+it)$ and $K=\log\log T$, and similarly (9) follows. □

Put $a_1(n)=1$ if n is composed of at most $100\log\log T$ primes all below Y, and zero otherwise. Put $a_2(n)=1$ if n is composed of at most $100\log\log\log T$ primes all between Y and X, and zero otherwise. Note that $a_1(1)=a_2(1)=1$. Define

$$M_1(s)=\sum_n\frac{\mu(n)a_1(n)}{n^s}\qquad\text{and}\qquad M_2(s)=\sum_n\frac{\mu(n)a_2(n)}{n^s},$$

so that the Dirichlet polynomial $M(s)$ (introduced in (3)) may be factored as $M(s)=M_1(s)M_2(s)$. If we expand out $\mathcal{M}_j(s)$ as a Dirichlet polynomial, the result is similar to $M_j(s)$ but the two quantities are not identical. The next lemma shows that in mean square $\mathcal{M}_j(s)$ is indeed close to $M_j(s)$.

Lemma 3. *With notations as above, we have*

$$\int_T^{2T}\big|\mathcal{M}_1(\sigma_0+it)-M_1(\sigma_0+it)\big|^2dt\ll T(\log T)^{-60},$$

and

$$\int_T^{2T}\big|\mathcal{M}_2(\sigma_0+it)-M_2(\sigma_0+it)\big|^2dt\ll T(\log\log T)^{-60}.$$

Proof. We establish the first estimate, and the second follows similarly. Expand $\mathcal{M}_1(s)$ into a Dirichlet series $\sum_n b(n)n^{-s}$. Then we may see that $b(n)$ satisfies the following properties:

(i) $|b(n)|\le 1$ for all n,

(ii) $b(n)=0$ unless $n\le Y^{100\log\log T}$ is composed only of primes below Y, and

(iii) $b(n)=\mu(n)a_1(n)$ unless $\Omega(n)>100\log\log T$, or if there is a prime $p\le Y$ such that $p^k|n$ with $p^k>Y$.

Putting $c(n)=b(n)-\mu(n)a_1(n)$ temporarily, using (5) and (6) we obtain

$$\int_T^{2T}\big|\mathcal{M}_1(\sigma_0+it)-M_1(\sigma_0+it)\big|^2dt$$
$$\ll T\sum_{n_1=n_2}\frac{|c(n_1)c(n_2)|}{(n_1n_2)^{\sigma_0}}+\sum_{n_1\ne n_2}\frac{|c(n_1)c(n_2)|}{(n_1n_2)^{\sigma_0}}\sqrt{n_1n_2}.$$

The off-diagonal terms with $n_1 \neq n_2$ contribute

$$\ll \sum_{n_1 \neq n_2 \leq Y^{100 \log\log T}} 1 \ll T^{\epsilon}.$$

The diagonal terms $n_1 = n_2$ contribute, (recalling property (iii) above)

$$\ll T \sum_{\substack{p|n \implies p \leq Y \\ \Omega(n) > 100 \log\log T}} \frac{1}{n} + T\Big(\sum_{\substack{p \leq Y \\ p^k > Y}} \frac{1}{p^k} \Big)\Big(\sum_{p|n \implies p \leq Y} \frac{1}{n} \Big).$$

A small calculation shows that the second term above is $\ll T(\log Y)/\sqrt{Y} \ll T(\log T)^{-60}$. To handle the first term above, note that for any $1 < r < 2$ the quantity $r^{\Omega(n) - 100 \log\log T}$ is always non-negative, and is ≥ 1 on those n with $\Omega(n) > 100 \log\log T$. Therefore

$$\begin{aligned} T \sum_{\substack{p|n \implies p \leq Y \\ \Omega(n) > 100 \log\log T}} \frac{1}{n} &\ll T r^{-100 \log\log T} \prod_{p \leq Y} \Big(1 + \frac{r}{p} + \frac{r^2}{p^2} + \dots\Big) \\ &\ll T(\log T)^{-100 \log r} (\log T)^r. \end{aligned}$$

Choosing $r = e^{2/3} = 1.94\dots$, say, the above is $\ll T(\log T)^{-60}$, completing our proof. □

Proof of Proposition 3. From Lemma 3 it follows that except on a set of measure $o(T)$, one has

$$M_1(\sigma_0 + it) = \mathcal{M}_1(\sigma_0 + it) + O\big((\log T)^{-25}\big).$$

Moreover, from (8) (except on a set of measure $o(T)$) we note that

$$\mathcal{M}_1(\sigma_0 + it) = \exp\big(-\mathcal{P}_1(\sigma_0 + it)\big)(1 + O\big((\log T)^{-99}\big)).$$

Now, by (7) we see that $(\log T)^{-1} \ll |\mathcal{M}_1(\sigma_0 + it)| \ll \log T$ except on a set of measure $o(T)$, and therefore

$$M_1(\sigma_0+it) = \mathcal{M}_1(\sigma_0+it)+O\big((\log T)^{-25}\big) = \exp\big(-\mathcal{P}_1(\sigma_0+it)\big)(1+O((\log T)^{-20})).$$

Similarly, except on a set of measure $o(T)$, we have

$$\begin{aligned} M_2(\sigma_0 + it) &= \mathcal{M}_2(\sigma_0 + it) + O\big((\log\log T)^{-25}\big) \\ &= \exp(-\mathcal{P}_2(\sigma_0 + it))\Big(1 + O\big((\log\log T)^{-20}\big)\Big). \end{aligned}$$

Multiplying these estimates we obtain

$$M(\sigma_0 + it) = \exp\big(-\mathcal{P}(\sigma_0 + it)\big)\Big(1 + O\big((\log\log T)^{-20}\big)\Big),$$

completing our proof. □

5. Proof of Proposition 4

For $T \le t \le 2T$, one has $\zeta(\sigma_0 + it) = \sum_{n\le T} n^{-\sigma_0 - it} + O(T^{-\frac12})$ (see Theorem 4.11 of [Tit]), and so

$$\int_T^{2T} \zeta(\sigma_0+it)M(\sigma_0+it)dt = \sum_{n\le T}\sum_m \frac{a(m)\mu(m)}{(mn)^\sigma}\int_T^{2T}(mn)^{-it}dt + O(T^{\frac12+\epsilon})$$
$$= T + O(T^{\frac12+\epsilon}).$$

Therefore, expanding the square, we see that

$$\int_T^{2T} |1-\zeta(\sigma_0+it)M(\sigma_0+it)|^2 dt = \int_T^{2T} |\zeta(\sigma_0+it)M(\sigma_0+it)|^2 dt - T + O(T^{\frac12+\epsilon}). \tag{10}$$

It remains to evaluate the integral above, and to do this we shall use the following familiar lemma (see for example Lemma 6 of Selberg [Sel1]). For completeness we include a quick proof of the lemma in the next section; we give only a version sufficient for our purposes and not the sharpest known result.

Lemma 4. *Let h and k be non-negative integers, with $h,k \le T$. Then, for any $1 \ge \sigma > \frac12$,*

$$\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}|\zeta(\sigma+it)|^2dt$$
$$= \int_T^{2T}\Big(\zeta(2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{\sigma} + \Big(\frac{t}{2\pi}\Big)^{1-2\sigma}\zeta(2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{1-\sigma}\Big)dt + O\big(T^{1-\sigma+\epsilon}\min\{h,k\}\big).$$

Assuming Lemma 4, we now complete the proof of Proposition 4. In view of (10) we must show that

$$\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}}\int_T^{2T}\Big(\frac{h}{k}\Big)^{it}|\zeta(\sigma_0+it)|^2 \sim T, \tag{11}$$

and to do this we appeal to Lemma 4. Recall that $|a(n)| \le 1$ always, and that $a(n) = 0$ unless $n \le T^\epsilon$. Therefore, the error terms arising from Lemma 4 contribute an amount

$$\ll \sum_{h,k\le T^\epsilon}\frac{1}{(hk)^{\sigma_0}}T^{1-\sigma_0+\epsilon}\min\{h,k\} \ll T^{1-\sigma_0+\epsilon} = o(T).$$

We now focus on the main terms arising from Lemma 4, beginning with the first main term. This contributes

$$T\zeta(2\sigma_0)\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{2\sigma_0}}(h,k)^{2\sigma_0}. \tag{12}$$

Write $h = h_1h_2$ where h_1 is composed only of primes below Y, and h_2 of primes between Y and X, and then $a(h) = a_1(h_1)a_2(h_2)$ in the notation of section 3. Writing similarly $a(k) = a_1(k_1)a_2(k_2)$, we see that the quantity in (12) factors as

$$T\zeta(2\sigma_0)\Big(\sum_{h_1,k_1}\frac{\mu(h_1)\mu(k_1)a_1(h_1)a_1(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}\Big)\Big(\sum_{h_2,k_2}\frac{\mu(h_2)\mu(k_2)a_2(h_2)a_2(k_2)}{(h_2k_2)^{2\sigma_0}}(h_2,k_2)^{2\sigma_0}\Big). \tag{13}$$

Consider the first factor in (13). If we ignore the condition that h_1 and k_1 must have at most $100\log\log T$ prime factors, then the resulting sum is simply

$$\sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\le Y}}\frac{\mu(h_1)\mu(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0} = \prod_{p\le Y}\Big(1-\frac{1}{p^{2\sigma_0}}\Big).$$

In approximating the first factor by the product above, we incur an error term which is at most

$$\ll \sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\le Y\\ \Omega(h_1)>100\log\log T}}\frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0},$$

where we used symmetry to assume that h_1 has many prime factors. Since $e^{\Omega(h_1)-100\log\log T}$ is always non-negative, and is ≥ 1 for those h_1 with $\Omega(h_1)\ge 100\log\log T$, the above may be bounded by

$$\ll e^{-100\log\log T}\sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\le Y}}\frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}e^{\Omega(h_1)}$$

$$\ll (\log T)^{-100}\prod_{p\le Y}\Big(1+\frac{1+2e}{p}\Big) \ll (\log T)^{-90}.$$

Thus the first factor in (13) is

$$\prod_{p\le Y}\Big(1-\frac{1}{p^{2\sigma_0}}\Big) + O((\log T)^{-90}) \sim \prod_{p\le Y}\Big(1-\frac{1}{p^{2\sigma_0}}\Big).$$

Similarly one obtains that the second factor in (13) is

$$\prod_{Y<p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)+O\big((\log\log T)^{-90}\big)\sim\prod_{Y<p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big).$$

Using these in (13), we obtain that the first main term is

$$\sim T\zeta(2\sigma_0)\prod_{p\le X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)=T\prod_{p>X}\Big(1-\frac{1}{p^{2\sigma_0}}\Big)^{-1}\sim T,$$

since, recalling from (1) the definitions of σ_0, W, and X and using the prime number theorem and partial summation,

$$\sum_{p>X}\frac{1}{p^{2\sigma_0}}\ll\int_X^\infty\frac{1}{t^{2\sigma_0}}\frac{dt}{\log t}\ll\frac{X^{1-2\sigma_0}}{(2\sigma_0-1)\log X}=o(1).$$

In the same way we see that the second main term arising from Lemma 4 is

$$\begin{aligned}\zeta(2-2\sigma_0)&\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{1-2\sigma_0}dt\Big)\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{hk}(h,k)^{2-2\sigma_0}\\&\sim\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{1-2\sigma_0}dt\Big)\zeta(2-2\sigma_0)\prod_{p\le X}\Big(1-\frac{2}{p}+\frac{1}{p^{2\sigma_0}}\Big)\\&\ll T^{2-2\sigma_0}\frac{1}{(2\sigma_0-1)}\prod_{p\le X}\Big(1-\frac{1}{p}\Big)=o(T).\end{aligned}$$

This completes our proof of (11), and hence of Proposition 4.

6. Proof of Lemma 4

Put $G(s)=\pi^{-s/2}s(s-1)\Gamma(s/2)$, so that $\xi(s)=G(s)\zeta(s)=\xi(1-s)$ is the completed zeta function. Define for any given $s\in\mathbb{C}$

$$I(s)=I(\overline{s})=\frac{1}{2\pi i}\int_{(c)}\xi(z+s)\xi(z+\overline{s})e^{z^2}\frac{dz}{z},$$

where the integral is over the line from $c-i\infty$ to $c+i\infty$ for any $c>0$. By moving the line of integration to the left, and using the functional equation $\xi(z+s)\xi(z+\overline{s})=\xi(-z+(1-s))\xi(-z+(1-\overline{s}))$ we obtain that

$$|\zeta(s)|^2=\frac{1}{|G(s)|^2}\big(I(s)+I(1-s)\big). \tag{14}$$

From now on suppose that $s=\sigma+it$ with $T\le t\le 2T$, and $1\ge\sigma\ge\frac{1}{2}$. If z is a complex number with real part $c=1-\sigma+1/\log T$, then an application of Stirling's formula gives

$$\frac{G(z+s)G(z+\overline{s})}{|G(s)|^2} = \Big(\frac{t}{2\pi}\Big)^z \Big(1 + O\Big(\frac{|z|}{T}\Big)\Big).$$

Therefore, we see that

$$\frac{I(s)}{|G(s)|^2} = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)} \Big(\frac{t}{2\pi}\Big)^z \zeta(z+s)\zeta(z+\overline{s})e^{z^2}\frac{dz}{z} + O(T^{-\sigma+\epsilon}).$$

Since we are in the region of absolute convergence of $\zeta(z+s)$ and $\zeta(z+\overline{s})$, we obtain

(15) $$\int_T^{2T} \Big(\frac{h}{k}\Big)^{it} \frac{I(s)}{|G(s)|^2}dt = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)} \frac{e^{z^2}}{z}\sum_{m,n=1}^{\infty}\frac{1}{(mn)^{z+\sigma}}\Big(\int_T^{2T}\Big(\frac{hm}{kn}\Big)^{it}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz + O(T^{1-\sigma+\epsilon}).$$

In the integral in (15), we distinguish the diagonal terms $hm = kn$ from the off-diagonal terms $hm \neq kn$. The diagonal terms $hm = kn$ may be parametrized as $m = Nk/(h,k)$ and $n = Nh/(h,k)$, and therefore these terms contribute

(16) $$\frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)} \frac{e^{z^2}}{z}\zeta(2z+2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz.$$

As for the off-diagonal terms, the inner integral over t may be bounded by $\ll T^{1-\sigma}\min(T, 1/|\log(hm/kn)|)$, and therefore these contribute

(17) $$\ll T^{1-\sigma}\sum_{\substack{m,n=1\\ hm\neq kn}}^{\infty}\frac{1}{(mn)^{1+1/\log T}}\min\Big(T, \frac{1}{|\log(hm/kn)|}\Big) \ll \min\{h,k\}T^{1-\sigma+\epsilon}.$$

The final estimate above follows by first discarding terms with $hm/(kn) > 2$ or $< 1/2$, and for the remaining terms (assume that $k \leq h$) noting that for a given m the sum over values n may be bounded by kT^{ϵ} (here it may be useful to distinguish the cases $hm > T$ and $hm < T$).

From (15), (16) and (17), we conclude that

(18) $$\int_T^{2T} \Big(\frac{h}{k}\Big)^{it} \frac{I(s)}{|G(s)|^2}dt = \frac{1}{2\pi i}\int_{(1-\sigma+1/\log T)} \frac{e^{z^2}}{z}\zeta(2z+2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^z dt\Big)dz + O(\min\{h,k\}T^{1-\sigma+\epsilon}).$$

A similar argument gives

$$\int_T^{2T} \Big(\frac{h}{k}\Big)^{it} \frac{I(1-s)}{|G(s)|^2} dt = O\big(T^{1-\sigma+\epsilon} \min\{h,k\}\big) \tag{19}$$

$$+\frac{1}{2\pi i}\int_{(\sigma+1/\log T)} \frac{e^{z^2}}{z}\zeta(2z+2-2\sigma)\Big(\frac{(h,k)^2}{hk}\Big)^{z+1-\sigma}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{z+1-2\sigma} dt\Big)dz.$$

With a suitable change of variables, we can combine the main terms in (18) and (19) as

$$\frac{1}{2\pi i}\int_{(1+1/\log T)} \zeta(2z)\Big(\frac{(h,k)^2}{hk}\Big)^{z}\Big(\int_T^{2T}\Big(\frac{t}{2\pi}\Big)^{z-\sigma} dt\Big)\Big(\frac{e^{(z-\sigma)^2}}{z-\sigma}+\frac{e^{(z-1+\sigma)^2}}{z-1+\sigma}\Big)dz,$$

and moving the line of integration to the left we obtain the main term of the lemma as the residues of the poles at $z=\sigma$ and $z=1-\sigma$ (note that the potential pole at $z=1/2$ from $\zeta(2z)$ is canceled by a zero of $e^{(z-\sigma)^2}/(z-\sigma)+e^{(z-1+\sigma)^2}/(z-1+\sigma)$ there). This completes our proof of Lemma 4.

7. Discussion

In common with Selberg's proof of Theorem 1 our proof relies on the Gaussian distribution of short sums over primes, as in Proposition 2. In contrast with Selberg's proof, we do not need to invoke delicate zero density estimates for $\zeta(s)$ (see (1.14$'$) of [Sel1]), the easier mean-value theorem in Proposition 4 provides for us a sufficient substitute; nevertheless, our ideas are closely related to the mollifier technique that underlies zero-density results (going back to work of Bohr and Landau). Selberg's original proof also used an intricate argument expressing $\log\zeta(s)$ in terms of primes and zeros; an elegant alternative approach was given by Bombieri and Hejhal [BH], although they too require a strong zero density result near the critical line. We should also point out that by just focussing on the central limit theorem, we have not obtained asymptotic formulae for the moments of $\log|\zeta(\frac12+it)|$ which Selberg established.

In Selberg's approach, it was easier to handle Im $\log\zeta(\frac12+it)$, and the case of $\log|\zeta(\frac12+it)|$ entailed additional technicalities. In contrast, our method works well for $\log|\zeta(\frac12+it)|$ but requires substantial modifications to handle Im $\log\zeta(\frac12+it)$. The reason is that Proposition 4 guarantees that typically $|\zeta(\sigma_0+it)| \approx |M(\sigma_0+it)|^{-1}$, but it could be that Im $\log\zeta(\frac12+it)$ and Im $\log M(\sigma_0+it)^{-1}$ are not typically close but differ by a substantial integer multiple of 2π. In this respect our argument shares some similarities with Laurinčikas's proof of Selberg's central limit theorem [Lau], which relies on bounding small moments of $|\zeta(\frac12+it)|$ using Heath-Brown's work on fractional moments [HB]. In particular, Laurinčikas's argument also breaks down for the imaginary part of $\log\zeta(\frac12+it)$.

We can quantify the argument given here, providing a rate of convergence to the limiting distribution, but we have not pursued the matter as it did not seem to yield anything stronger than what is known. We also remark that the argument also gives the joint distribution of $\log|\zeta(\frac{1}{2}+it)|$ and $\log|\zeta(\frac{1}{2}+it+i\alpha)|$ (for any fixed non-zero $\alpha \in \mathbb{R}$) and shows that these are distributed like independent Gaussians. One can allow for more than one shift, and also keep track of the uniformity in α.

Our proof of Proposition 3 (in Section 4) involved splitting the mollifier $M(s)$ into two factors, or equivalently of the prime sum $\mathcal{P}(s)$ into two pieces. We would have liked to get away with just one prime sum, but this barely fails. In order to use Proposition 1 successfully, we are forced to take $W = o(\sqrt{\log\log T})$. To mollify successfully on the $\frac{1}{2}+\frac{W}{\log T}$ line (see Proposition 4) we need to work with primes going up to roughly $T^{\frac{1}{W}}$. If $W = o(\sqrt{\log\log T})$ then this length is $T^{A/\sqrt{\log\log T}}$ for a large parameter A, and if we try to expand $\exp(\mathcal{P}(s))$ into a series (as in Section 4) we will be forced to take more than $\sqrt{\log\log T}$ terms in the exponential series. This leads to Dirichlet polynomials that are just a little too long. We resolve this (see Section 4) by splitting $\mathcal{P}$ into two terms, exploiting the fact that the longer sum $\mathcal{P}_2$ has a significantly smaller variance.

Propositions 1, 2, and 3 in our argument are quite general and analogues may be established for higher degree automorphic L-functions in the t-aspect. An analogue of Proposition 4 however can at present only be established for L-functions of degree 2 (relying here upon information on the shifted convolution problem), and unknown for degrees 3 or higher. However, some hybrid results are possible. For example, by adapting the techniques in [CIS2, CIS1] we can establish an analogue of Proposition 4 for twists of a fixed $GL(3)$ L-function by primitive Dirichlet characters with conductor below Q. In this way one can show that as χ ranges over all primitive Dirichlet characters with conductor below Q, and t ranges between -1 and 1, the distribution of $\log|L(\frac{1}{2}+it, f\times\chi)|$ is approximately normal with mean 0 and variance $\sim \frac{1}{2}\log\log Q$; here f is a fixed eigenform on $GL(3)$.

Keating and Snaith [KS] have conjectured that central values of L-functions in families have a log normal distribution with an appropriate mean and variance depending on the family. For example, we may consider the family of quadratic Dirichlet L-functions $L(\frac{1}{2}, \chi_d)$ where d ranges over fundamental discriminants of size X. In this setting, we may carry out the arguments of Propositions 2, 3 and 4 and conclude that $\log L(\sigma_0, \chi_d)$ has a normal distribution with mean $\sim \frac{1}{2}\log\log X$ and variance $\sim \log\log X$, provided that $\sigma_0 = \frac{1}{2}+\frac{W}{\log X}$ where W is any function with $W\to\infty$ as $X\to\infty$ and with $\log W = o(\log\log X)$. However in this situation we do not have an analogue of Proposition 1 allowing us to pass from this to the central value; indeed, our knowledge at present does not exclude the possibility that $L(\frac{1}{2}, \chi_d) = 0$ for a positive proportion of discriminants d.

Finally we remark that the proof presented here was suggested by earlier work of the authors [RS], where general one sided central limit theorems towards the Keating-Snaith conjectures are established.

Acknowledgments. The first author was partially supported by NSF grant DMS-1128155, NSERC Discovery Grant, CRC program, and an Alfred P. Sloan Foundation fellowship. The second author is partially supported by NSF grant DMS-1500237, and a Simons Investigator award from the Simons Foundation.

We thank the referee and editors for helpful comments, which improved our exposition.

References

[BJ1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **54** (1930), 1–35. JFM 56.0287.01 MR 1555301

[BJ2] —— Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.* **58** (1932), 1–55. Zbl 0003.38901 MR 1555343

[BH] E. Bombieri and D. A. Hejhal, On the distribution of zeros of linear combinations of Euler products. *Duke Math. J.* **80** (1995), 821–862. Zbl 0853.11074 MR 1370117

[CIS1] J. B. Conrey, H. Iwaniec and K. Soundararajan, The sixth power moment of Dirichlet L-functions. *Geom. Funct. Anal.* **22** (2012), 1257–1288. Zbl 1270.11082 MR 2989433

[CIS2] —— Critical zeros of Dirichlet L-functions. *J. Reine Angew. Math.* **681** (2013), 175–198. Zbl 1357.11070 MR 3181494

[HB] D. R. Heath-Brown, Fractional moments of the Riemann zeta function. *J. London Math. Soc.* (2) **24** (1981), 65–78. Zbl 0431.10024 MR 0623671

[KS] J. P. Keating and N. C. Snaith, Random matrix theory and L-functions at $s = 1/2$. *Comm. Math. Phys.* **214** (2000), 91–110. Zbl 1051.11047 MR 1794267

[Lau] A. Laurinchikas, A limit theorem for the Riemann zeta-function on the critical line. II. *Litovsk. Mat. Sb.* **27** (1987), 489–500. MR 0925354

[RS] M. Radziwiłł and K. Soundararajan, Moments and distribution of central L-values of quadratic twists of elliptic curves. *Invent. Math.* **202** (2015), 1029–1068. Zbl 06518187 MR 3425386

[Sel1] A. Selberg, Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.* **48** (1946), 89–155. Zbl 0061.08402 MR 0020594

[Sel2] —— Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, pages 367–385. Univ. Salerno, Salerno, 1992. Zbl 0787.11037 MR 1220477

[Sou] K. Soundararajan, Moments of the Riemann zeta function. *Ann. of Math.* (2) **170** (2009), 981–993. Zbl 1251.11058 MR 2552116

[Tit] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*. The Clarendon Press, Oxford University Press, New York, second edition, (1986). Edited and with a preface by D. R. Heath-Brown. Zbl 0601.10026 MR 0882550

(*Reçu le 29 septembre 2015*)

Maksym Radziwiłł, Department of Mathematics, McGill University,
805 Sherbrooke Street West, Montreal, H3A 0B9, Canada

e-mail: maksym.radziwill@gmail.com

Kannan Soundararajan, Department of Mathematics, Stanford University,
450 Serra Mall, Bldg. 380, Stanford, CA 94305-2125, USA

e-mail: ksound@math.stanford.edu