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Branched coverings and equivariant smoothings of 4-manifolds

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Abstract. This paper describes some new families of finite group actions on 4-manifolds, including infinite families of smoothly inequivalent actions which are topologically equivalent and locally linear actions which are not smoothable. The constructions involve a variety of results on 4-manifolds and branched coverings. One common feature is that these families are counterexamples to the Lashof–Rothenberg homotopy classification results for equivariant smoothings which hold for actions with no 4-dimensional strata. Related examples with 4-dimensional fixed point sets are also described.

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Let M be an unbounded topological n-manifold, where $n \neq 4$. By [KiS], suitably defined equivalence classes of smooth structures on M are classified by bundle data; more precisely, one can define a topological tangent bundle $\tau_M \downarrow M$ whose fibers are homeomorphic to $\mathbb{R}^n$, and classes of smooth structures are in 1-1 correspondence with vector bundle structures on τ_M. In [LR] R. Lashof and M. Rothenberg proved a similar result for manifolds with suitably restricted actions of a finite group G. Specifically, they consider actions which are locally linear (or locally smooth [Bre]); for such actions one has an equivariant bundle structure $\tau_{M,G}$ on the topological tangent bundle; and classes of equivariant smoothings of a locally linear G-manifold M correspond to G-vector bundle structures on the equivariant bundle $\tau_{M,G}$ if one avoids 4-dimensional problems. More precisely, for each subgroup $H \subset G$ one must assume that no component of the fixed point set M^H is 4-dimensional.

Advances in 4-manifold theory since the appearance of [LR] have yielded many examples which show that the conclusions of [KiS] do not extend to 4-

manifolds (e.g., see [FQ], [Gom1], [Don1], [Don2]), and these results indicate that counterexamples should also exist for the equivariant smoothing theory of [LR] if some fixed set M^H has a 4-dimensional component.

The main purpose of this paper is to describe relatively simple 4-dimensional examples by combining several advances in 4-manifold theory with the theory of cyclic branched coverings (cf. Fox [Fox], Reddy [Red] and [Sch]). We shall also discuss a few other examples and closely related results. One approach to this problem would be to study free G-actions by considering candidates for orbit spaces. We shall mention a few examples of this type, but our main interest will concern actions which are not free. One reason for doing so is that questions about smoothings free actions quickly reduce to similar questions about their orbit spaces, and numerous examples have been constructed (e.g., see the results of I. Hambleton and M. Kreck in [HK] or [Kre] and of M. Ue in [Ue1] or [Ue2]; see also [Tor]). An equally important reason is that the underlying manifolds are often very familiar objects, and a third reason is to construct examples with nontrivial fixed point data.

For the sake of conciseness, we have only described limited families of examples in this paper, with emphasis on methods developed during the nineteen eighties and nineties. Other advances in 4-manifold theory – particularly from the past two decades – clearly yield many other examples like those considered here.

Here is a brief outline of the paper. The first section discusses some preliminary results: One is a smoothability result for certain noncompact locally linear 4-dimensional G-manifolds, which parallels the smoothing theorem for noncompact 4-manifolds due to Freedman and Quinn [FQ] and (independently) Lashof and Taylor [LT]. Another is a variation of the results about smooth structures for certain orbit spaces in [Sch]. In the second section we shall use branched covering to construct locally linear 4-dimensional G-manifolds of the following two types:

(1) The equivariant tangent bundle reduces to a G-vector bundle, but the manifold is not equivariantly smoothable.

(2) There are nondiffeomorphic equivariant smoothings which correspond to the same G-vector bundle structure on the equivariant tangent bundle.

The third section constructs uncountable families of nondiffeomorphic smoothings for linear actions on $\mathbb{R}^4$; since linear actions on $\mathbb{R}^4$ are equivariantly contractible, there is a unique isotopy class of G-vector bundle structures on their equivariant tangent bundles, so these actions are noncompact counterexamples to a 4-dimensional analog of [LR]. The existence of equivariant smoothings on certain exotic 4-spaces has been known for some time (cf. [Gom1], [FT], [Gom4]), but most of the uncountable families in this paper have not previously appeared in the literature. Our examples include some families which can be realized as G-invariant open subsets of the corresponding linear action on $\mathbb{R}^4$, and others which are not (nonequivariantly) diffeomorphic to open subsets of $\mathbb{R}^4$ with the usual smooth structure. Finally, Section 4 considers some related questions involving higher dimensional G-manifolds, in particular we construct higher dimensional counterexamples to the conclusions of [LR] for locally linear G-manifolds whose fixed point sets have 4-dimensional components.

There is a limited amount of overlap between the results of this paper and the theorems and problems in A. Edmonds' excellent survey of group actions on 4-manifolds [Edm2]. This may reflect the present state of 4-manifold theory, in which we are still only beginning to understand the wide-ranging aspects of this subject. A more systematic understanding of smooth simply connected 4-manifolds might provide a framework for organizing the sorts of examples which are currently known to exist.

1. Preliminaries

This section considers two unrelated questions. One is the equivariant smoothability of certain noncompact 4-dimensional G-manifolds and the other involves some extensions of standard results about smooth structures on certain orbit spaces (see [Sch], Section 1).

1.1. Noncompact locally linear 4-manifolds. By [FQ] and [LT], every connected noncompact 4-manifold is smoothable, and we shall prove an analog for locally linear group actions on such manifolds. To simplify the discussion we shall restrict attention to certain group actions which are semifree (the only isotropy subgroups are G and $\{1\}$).

Theorem 1.1. *Let G be a finite group, and let M be a connected,* ***noncompact****, unbounded locally linear 4-dimensional G-manifold. Assume that G acts semifreely on M and each component of M^G is even-dimensional. Then M is equivariantly smoothable.*

Remark 1. The condition on M^G is always satisfied if G has odd order.

Remark 2. A more detailed analysis shows that the theorem and its proof extend to locally linear actions satisfying the following condition: For each subgroup $H \subset G$, every component of M^H is even dimensional.

Remark 3. Results of [KwL] and [KwS] imply that every locally linear G-manifold of dimension ≤ 3 has a unique equivalence class of equivariant smoothings, so it follows that every noncompact, connected, semifree locally linear G-manifold of dimension ≤ 4 is equivariantly smoothable if G has odd order. The analogous statement is false in all higher dimensions. Specifically, let M^4 be the closed simply connected manifold whose intersection form on $H_2(M;\mathbb{Z})$ is given by the E_8 matrix, and consider the locally linear $\mathbb{Z}_p$ actions on M^4 in [Edm1], where $p \geq 5$ is prime. Since the Kirby-Siebenmann smoothing obstruction for M^4 is non-trivial (cf. [FQ]), the product manifolds $M^4 \times \mathbb{R}^k (k \geq 1)$ are not even smoothable as manifolds in the non-equivariant sense.

Proof of Theorem 1.1. The hypotheses imply that G acts freely on $M - M^G$ and M^G is a union of pairwise disjoint components which are isolated points and surfaces.

Claim. There is a G-invariant closed neighborhood N of M^G such that N is an equivariantly smoothable G-manifold with boundary.

If the claim is true, then the theorem follows immediately from [FQ] and [LT]; by the claim and the Equivariant Collar Neighborhood Theorem, there is an equivariant smoothing of an open neighborhood for N in M, and the results of [FQ] and [LT] imply that the induced smoothing of $\partial N/G$ extends to a smoothing of $(M - \operatorname{Int} N)/G$ because the latter is noncompact. If we pull this smoothing back to the covering space $M - \operatorname{Int} N$ and attach it to the given equivariant smoothing on a neighborhood of N, we obtain an equivariant smoothing of M.

We shall now prove the claim. It suffices to prove this for each component of M^G separately. If a component of M^G consists of a single isolated point, then the conclusion follows immediately by local linearity, for we can take N to be a small orthogonal disk centered at the point. If a component C of M^G is a surface, then there is an invariant open neighborhood U of C such that U/G is a 4-manifold and $C \subset U/G$ is a locally flat surface (see [Bre], Lemma IV.4.2 for the assertion that U/G is a manifold; local flatness follows because the action is locally equivalent to $\mathbb{R}^2 \times V$, where G acts trivially on $\mathbb{R}$ and V is a 2-dimensional G-representation with $V^G = \{0\}$). We can now use [FQ] to conclude that C has a closed vector bundle neighborhood D^* in U/G, and it follows that the inverse image $D \subset U$ of $D^* \subset U/G$ is the unit disk bundle of a G-vector bundle; furthermore, since G acts semifreely on M, this G-vector bundle must come from a complex line bundle. Since surfaces are smoothable and G-vector bundles over smooth manifolds are smoothable, it follows that D is equivariantly smoothable, and this completes the proof. □

1.2. Remarks on branched coverings. Let G be a finite cyclic group, and let M be a locally linear semifree G-manifold such that M^G is a codimension 2 submanifold. As noted in the preceding discussion, the orbit space $M^* = M/G$ is a topological manifold such that M/G is a locally flat submanifold and the orbit space projection $M \to M^*$ is a regular branched covering whose branch set is M^G (compare [Fox], [Red]). Furthermore, if G acts smoothly on M, then there is a canonical smooth structure on M^* such that $M \to M^*$ is a smooth branched covering map (see [Sch], Section 1). In the remainder of this section we shall collect a few simple results that will be needed later.

Proposition 1.2. *Let G be a finite cyclic group, and let M^n and N^n be smooth semifree G-manifolds whose fixed point sets are $(n-2)$-dimensional, and take the smooth structure on M^n/G and N^n/G given in* [Sch]. *If $f : M^n \to N^n$ is a G-equivariant diffeomorphism, then the induced map of orbit spaces $f/G : M^n/G \to N^n/G$ is topologically isotopic to a diffeomorphism.*

Proof. This result is a straightforward elementary exercise if $M^G = N^G = \varnothing$ – in which case f/G itself is a diffeomorphism – but otherwise one must look more closely at the construction of the smooth structures on the orbit spaces in Section 1 of [Sch].

Suppose now that $f : M \to N$ is an equivariant diffeomorphism, and let $\mathcal{A}$ and $\mathcal{B}$ be smooth structures on M/G and N/G which satisfy the conditions of [Sch]. The smooth structures involve choices of good G-invariant Riemannian metrics $f^*\mathcal{B}$ on M, then one can check directly that the induced homeomorphism $f/G : M^n/G \to N^n/G$ defines a diffeomorphism from $(M/G, f^*\mathcal{B})$ to $(N/G, \mathcal{B})$. By our previous comments the identity map from $(M/G, \mathcal{A})$ to $(M/G, f^*\mathcal{B})$ is topologically isotopic to a homeomorphism, and therefore it follows that $f/G : (M/G, \mathcal{A}) \to (N/G, \mathcal{B})$ is topologically isotopic to a diffeomorphism. □

We shall also use the following result on connected sums.

Proposition 1.3. *Let G be a finite group, and let M^n be a locally linear semifree G-manifold such that every component of M^G is $(n-2)$-dimensional. Assume further that M is oriented (hence G acts orientation-preservingly). Let N^n be a compact oriented n-manifold, and let $\#^G N$ denote the connected sum of $|G|$ copies of N. Then the following hold:*

(i) *There is a locally linear action of G on $M^n \# (\#^G N)$ such that the fixed point set is M^G and the orbit space is homomorphic to $M/G \# N$.*

(ii) *If the group action on M is smooth, and N is smooth, then there is a smooth action of G on $M^n \# (\#^G N)$ such that the fixed point set is M^G and the orbit space is diffeomorphic to $M/G \# N$.*

Proof. Take an equivariant connected sum of M and $\#^G N$ by choosing an orientation preserving equivariant embedding of $G \times D^n$ in $M - M^/G$ and an orientation preserving embedding of D^n in N, removing the interiors of the embedded disks from $M - M^G$ and $G \times N$ respectively, and identifying the common boundaries by a suitable equivariant homeomorphism of $G \times S^{n-1}$. If everything is smooth, choose embeddings which are smooth and identify the copies of $G \times S^{n-1}$ by an equivariant diffeomorphism. This yields an action on $M^n \# (\#^G N)$ whose fixed point set and orbit space have the desired properties, and the construction yields a smooth G-manifold whose fixed point set is diffeomorphic to M^G if the G-manifold M and the manifold N are smooth. □

2. Compact examples

In contrast to all other dimensions, a compact 4-manifold might not be smoothable even if its topological tangent bundle comes from a vector bundle. For example, if M^4 is the closed simply connected manifold whose intersection form on $H_2(M;\mathbb{Z})$ is the E_8 matrix, then the tangent bundle of the connected sum $M\#M$ is isomorphic to a vector bundle by [KiS], [FQ] and [LT], but by [Don1] this manifold is not smoothable. Here are some corresponding examples of locally linear G-manifolds.

Proposition 2.1. *Let $k \geq 1$ be an integer, let V be a nontrivial irreducible 1-dimensional unitary representation of $\mathbb{Z}_k$, and consider the $\mathbb{Z}_k$-action on the $2k$-fold connected sum $\#^{2k} M$ which is an equivariant connected sum of the linear action on the 4-dimensional unit sphere $S(\mathbb{R}^3 \oplus V)$ and $\mathbb{Z}_k \times (M\#M)$ as in Proposition 1.3. Then the equivariant tangent bundle of this action reduces to a $\mathbb{Z}_k$-vector bundle, but the group action is not equivariantly smoothable.*

Proof. The result of [Don1] imply that $\#^{2k} M$ is not smoothable, so the group action certainly cannot be equivariantly smoothable. Therefore the proof reduces to showing that the $\mathbb{Z}_k$-tangent bundle of the group action comes from a $\mathbb{Z}_k$-vector bundle.

Let $D \subset S(\mathbb{R}^3 \oplus V)$ be a disk which is disjoint from the fixed point set and is so small that $g_1 \cdot D \cap g_2 \cdot D = \varnothing$ if $g_1 \neq g_2$ in $\mathbb{Z}_k$ (hence $\mathbb{Z}_k \times D$ is embedded in the complement of the fixed point set). Since $\mathbb{Z}_k$ acts smoothly on $S(\mathbb{R}^3 \oplus V)$, the restriction of the equivariant tangent bundle to $S(\mathbb{R}^3 \oplus V) - \mathbb{Z}_k \times \mathrm{Int} D$ comes from a $\mathbb{Z}_k$-vector bundle. The construction in Proposition 1.3 expresses the group action on $\#^{2k} M$ as the union of $S(\mathbb{R}^3 \oplus V) - \mathbb{Z}_k \times \mathrm{Int} D$ with $\mathbb{Z}_k \times M \# M - \mathrm{Int} E$, where $E \in M \# M$ is a closed disk and one identifies the boundaries $\mathbb{Z}_k \times \partial D \cong \mathbb{Z}_k \times \partial E$ by a homeomorphism $\partial E \cong \partial D$. This homeomorphism determines a vector bundle structure on the restriction of the tangent bundle $\tau_{M \# M}$ to ∂E, and it follows that the $\mathbb{Z}_k$-tangent bundle to the group action comes from a vector bundle if and only if the vector bundle structure over ∂E extends to a vector bundle structure over $M \# M - \mathrm{Int} E$.

In fact, there is a vector bundle structure on ∂E which extends to a vector bundle structure over $M \# M - \mathrm{Int}\, E$ because the tangent bundle $\tau_{M \# M}$ is isomorphic to a vector bundle. Furthermore, the two structure on ∂E are equivalent if the homeomorphism $\partial D \to \partial E$ is topologically isotopic to a diffeomorphism. Since every homeomorphism of 3-manifolds is isotopic to a diffeomorphism, this condition is fulfilled and hence the first vector bundle structure on ∂E extends to $M \# M - \mathrm{Int}\, E$. Therefore the $\mathbb{Z}_k$-topologial tangent bundle to the $\mathbb{Z}_k$-action on $\#^{2k} M$ comes from a $\mathbb{Z}_k$-vector bundle. □

There are also examples of nonsmoothable locally linear actions which satisfy the tangent bundle condition and act on smoothable closed simply connected 4-manifolds. Our discussion will use the following generalization of a formula due to F.Hirzebruch [Hir] (specifically, formula [15]) for smooth actions:

Proposition 2.2. *Let $d \geq 2$, and let M^4 be a closed oriented 4-manifold with a locally linear $\mathbb{Z}_d$-action. Assume all components of $F = M^{\mathbb{Z}_d}$ are 2-dimensional. Then $M^* = M/\mathbb{Z}_d$ is a closed oriented 4-manifold such that the image F^* of F is a locally flat submanifold and the signature of M and M^* satisfy the following identity:*

$$\mathrm{sgn}(M) = d\, \mathrm{sgn}(M^*) - \frac{d^2-1}{3d} \left(F^* \cdot F^*\right)$$

where $F^* \cdot F^*$ denotes the self-intersection number of F^* in M^*.

Proof. The generalization of Hirzebruch's result for smooth actions is essentially a special case of Theorem 14B.2 in Wall's book [Wal2] because the argument proving Theorem 1.1 in this paper shows that the components of F in M all have equivariant tubular neighborhoods given by G-vector bundles. Both sides of Hirzebruch's branched covering formula are oriented equivariant bordism invariants, so one can apply Lemma 14.3 in [Wal2] to conclude that the result remains true in the topological setting. It is worth noting that the special argument in the 4-dimensional case of Wall's proof can be avoided because subsequent results in [FQ] imply a topological transversality theorem for 4-manifolds. □

The preceding results yield some relatively simple examples of nonsmoothable actions on smoothable simply connected 4-manifolds.

Theorem 2.3. *Let* $d \geq 6$ *satisfy* $d \equiv 2 \bmod 4$. *Then there is a locally linear* $\mathbb{Z}_d$*-action on a simply connected* 4*-manifold* M_d^4 *with the following properties:*

(i) *The equivariant tangent bundle of the action comes from a* $\mathbb{Z}_d$*-vector bundle, but the action is not smoothable.*

(ii) *For all but at most finitely many choices of* d, *the manifold* M_d^4 *is nonequivariantly smoothable.*

Proof. The idea behind the construction is fairly direct and very similar to the approach of Fintushel, Stern and Sunukjian in [FSS] for constructing topologically equivalent smooth actions. Results of R. Lee and D. Wilczyński [LW] imply that d times the generator of $H_2(\mathbb{CP}^2) \cong \mathbb{Z}$ can be represented by a simply topologically embedded surface Σ_g of genus $g = \frac{1}{4}d^2 - 1$. If ω denotes the pullback of the canonical complex line bundle $E \downarrow S^2$ to Σ_g by a map of degree 1, then the normal bundle of the embedded surface is $E(\otimes^{d^2}\omega)$, where $\otimes^m \omega$ is the m-fold tensor product of a complex line bundle with itself. The construction in [LW] yields an embedding such that $\pi_1(\mathbb{CP}^2 - \Sigma_g)$ is isomorphic to $\mathbb{Z}_d$ (by [LW] the fundamental group is abelian, and direct computation shows that $H^3(\mathbb{CP}^2, \Sigma_g) \cong H_1(\mathbb{CP}^2 - \Sigma_g) \cong \mathbb{Z}_d$). Therefore we can construct a locally linear action on some closed 4-manifold M_d^4 such that the orbit space projection $M_d^4 \to M_d^4/\mathbb{Z}_d$ is a regular d-fold cyclic branched covering whose branch set is Σ_g.

We shall first prove that the $\mathbb{Z}_d$-tangent bundle of this action comes from a $\mathbb{Z}_d$-vector bundle. It will be useful to begin by considering the more general question of analyzing the equivariant tangent bundle associated to a cyclic branched covering $M \to M^*$ of an closed oriented 4-manifold along an connected oriented surface $F \subset M$; let F^* denote the image of F in M^*. Let $T \subset M$ be a closed tubular neighborhood of F in M, and let T^* be its image in M^* (hence T^* is a closed tubular neighborhood of F^*). Then the vector bundle structures on T^* and the unique smooth structure on F^* define an equivariant smoothing of a neighborhood of T^* and we can pull this back as in [Sch] to obtain an equivariant smoothing of an invariant neighborhood of T. Also, if M^* is a smooth manifold there is an induced smoothing of $M^* - F^*$, and since $M - F \to M^* - F^*$ is a regular covering there is an induced equivariant smoothing of $M - F$. On the overlap set $\operatorname{Int}(T) - F$ one has two smoothings from these constructions; if F^* is smoothly embedded these two smoothings coincide, and this yields the standard smoothing of the branched covering.

Suppose now that $F^* \subset M^*$ is not smoothable, and consider the problem of putting a linear structure on the equivariant tangent bundle τ_M. The preceding methods now yield two linear structures on the equivariant tangent bundle restricted to $\mathrm{Int}(T) - F$; these are pullbacks of two linear structures on $\mathrm{Int}(T^*) - F^*$ under the (unbranched) covering space map $\mathrm{Int}(T) - F \to \mathrm{Int}(T^*) - F^*$, and the equivariant tangent bundle to M will have a linear structure if and only if the two smoothings of $\mathrm{Int}(T^*) - F^*$ determine the same linearization of the latter's nonequivariant tangent bundle.

We can reformulate this problem homotopically as follows: Let $S^* \subset \mathrm{Int}(T^*)$ be a closed tubular subneighborhood given by a disk bundle of sufficiently small radius, and let W^* be the cylinder $T^* - \mathrm{Int}(S^*)$, so that $\partial W^* = \partial T^* \cup \partial S^*$. Then the obstructions to linearization lie in the cohomology groups

$$H^i\big(W^*, \partial W^*; \pi_i(\mathrm{Top}_4/\mathrm{O}_4)\big)$$

where $0 \leq i \leq 4$. Results of [LT] and [Qui] imply that $\pi_i(\mathrm{Top}_4/\mathrm{O}_4) \cong \pi_i(\mathrm{Top}/\mathrm{O})$ $(\cong \pi_i(K(\mathbb{Z}_2, 3)))$ in this range, and therefore the only possible nonzero obstruction lies in $H^4(W^*, \partial W^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since the composite

$$H^4(W^*, \partial W^*; \mathbb{Z}_2) \cong H^4(M^*, S^* \cup M^* - \mathrm{Int}(T^*); \mathbb{Z}_2) \to H^4(M^*, M^* - \mathrm{Int}(T^*); \mathbb{Z}_2)$$

is an isomorphism, it follows that this obstruction is equal to the obstruction for extending the linear structure on the tangent bundle to $M^* - \mathrm{Int}(T^*)$, which is given by the smoothability of M^*, to a linear structure on the tangent bundle to M^*. Since the original smooth structure on M^* extends the smooth structure on $M^* - \mathrm{Int}(S^*)$, this obstruction vanishes. This completes the proof that the equivariant tangent bundle of M comes from a $\mathbb{Z}_d$-vector bundle.

The nonsmoothability of the group action may be seen as follows: If there were an equivariant smoothing of M, then we could use [Sch] to find a smooth structure on $\mathbb{CP}^2$ (possibly not the usual one) such that Σ_g is smoothly embedded. However, by results of D. Kotschick and G. Matić ([KM], Theorem 3.1 and Corollary 1.2) no such smooth structure exists.

For the remainder of the proof, we return to the special case where $M^* = \mathbb{CP}^2, M = M_d$, and $F \cong F^*$ is a closed simply embedded surface of genus $\frac{1}{4}d^2 - 1$ representing d times the generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$. The only statement left to prove is the smoothability of M_d. The results of [FQ] imply that the closed simply connected manifold M_d is smoothable if the rank $b_2(M_d)$ of $H_2(M_d; \mathbb{Z})$ is at least $\frac{11}{8}$ times the absolute value $|\sigma(M_d)|$ of the signature: If the intersection form on $H_2(M_d; \mathbb{Z})$ is odd, then we have shown that the Kirby-Siebenmann invariant vanishes (since τ_{M_d} comes from a vector bundle), and $b_2(M_d) \geq 11\,|\sigma(M_d)|/8$ implies that the form is indefinite, so by [FQ] the manifold M_d is homeomorphic to a connected sum of $\pm\mathbb{CP}^2$'s. On the other hand, if the form is even, then we may proceed as follows: Since the surface $F^* \subset \mathbb{CP}^2$ has degree d, it follows that its self-intersection number is d^2. If we substitute this into Hirzebruch's formula for the signature of a branched covering, then the latter reduces to

$$\sigma(M_d) = d \; - \; 2\cdot\binom{d+1}{3}$$

and since $d \equiv 2 \bmod 4$ it follows that the right hand side is divisible by 16. Therefore $H_2(M_d;\mathbb{Z})$ is isomorphic to a direct sum of an even number of copies of E_8 and also some copies of the intersection form for $S^2 \times S^2$. If $b_2(M_d) \geq \frac{11}{8}|\sigma(M_d)|$ then by [FQ] the branched covering M_d is homeomorphic to a connected sum of copies of $S^2 \times S^2$ with copies of $K3$ surfaces and hence M_d is smoothable.

We shall prove that the $\frac{11}{8}$ inequality holds for all but at most finitely many choices of d. Predictably, this requires information about $b_2(M_d) = \chi(M_d) - 2$, and this follows from elementary considerations and the genus equation $g = \frac{1}{4}d^2 - 1$:

$$\begin{aligned}\chi(M_d) &= \chi(\Sigma_g) + \chi(M_d, \Sigma_g) = (2-2g) + d(2g+1)\\ &= (d-1)2g + d + 2 = (d-1)(2g+1) + 2 = (d-1)\left(\frac{d^2-4}{2} + 2\right) + 3\\ &= \frac{d^3}{2} + \text{lower terms.}\end{aligned}$$

It follows that

$$\lim_{d\to\infty} \frac{b_2(M_d)}{|\sigma(M_d)|} = \frac{3}{2} \quad > \quad \frac{11}{8}$$

and by the preceding discussion we have shown that M_d is nonequivariantly smoothable. □

2.1. Other examples. There are numerous results which yield other examples of nonsmoothable locally linear actions on closed 4-manifolds (e.g., see [Kiy], [Nak] and the references listed in [LN]). In particular, the results of [Kiy] produce infinite families of such actions on connected sums of two or more copies of $S^2 \times S^2$, where the group is a cyclic group of order $p \geq 19$ and the singular set consists of isolated points (in contrast, for most of our examples the fixed point sets are 2-dimensional).

Claim. The equivariant (topological) tangent bundles for the Kiyono examples in [Kiy] come from equivariant vector bundles.

Proof. Given a locally linear action of an odd order group G on a connected sum of two or more copies of $S^2 \times S^2$ with an isolated singular set S, let M be the connected sum. By local linearity the action is smooth near the singular set, and the obstruction to extending the linear structure from a neighborhood of the singular set to all of M lies in

$$H^4\big(M/G, S/G; \pi_3(\mathrm{Top}_4/\mathrm{O}_4)\big) \cong \mathbb{Z}_2.$$

Since G has odd order, the image of this obstruction in $H^4(M,s) \cong H^4(M)$ is zero if and only if the obstruction itself is zero. However, by construction the image of the obstruction in $H^4(M)$ is just the Kirby-Siebenmann obstruction to linearizing the nonequivariant tangent bundle of M. Since M is a connected sum of copies of $S^2 \times S^2$, we know that M is smoothable and hence the Kirby-Siebenmann obstruction is trivial. Therefore the obstruction to linearizing the equivariant tangent bundle is also trivial. □

2.2. Smoothly inequivalent group actions with equivalent (equivariant) tangent bundle linearizations. We have already noted that the proof of the preceding result is in some respect similar to that of [FSS], which yields infinite families of smoothly inequivalent, but topologically equivalent, actions of finite cyclic groups on simply connected 4-manifolds. Their methods also involve sophisticated applications of gauge theory methods and delicate surgery constructions. It is natural to ask whether branched covering methods yield further examples of smooth finite group actions which are topologically but not smoothly equivalent, and the remainder of this section is devoted to describing such families of group actions.

We shall describe examples constructed from simply connected Dolgachev surfaces (see [FM1], [FM2], [FM3] and [OV]); there is an infinite family of such 4-manifolds X_k, indexed by the nonnegative integers, such that they are all homeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ (the bar denotes the opposite orientation) but no two are diffeomorphic. Since these closed manifolds are simply connected, it follows that $H^3(X_k; \mathbb{Z}_2) = 0$ and hence all of these smoothings correspond to the standard linearization of the topological tangent bundle over the smooth manifold X_0.

As in [Ue1] and [Ue2], one method of constructing exotic actions is to form suitably chosen equivariant connected sums, and this will be the basis for all our examples. In many cases the key point is to recognize that such connected sums of 4-manifolds have alternate descriptions (unlike the situation for 3-manifolds). Our first examples use a result of R. Gompf [Gom3]; namely, if we generically let $\bar{M}$ denote M with the opposite orientation, then for each k the manifold $X_k \# \overline{X_k}$ is orientation presevingly diffeomorphic to $\#^{10}\left(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\right)$.

Proposition 2.4. *There are infinitely many smoothly inequivalent smooth involutions on* $\#^{10}(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ *which are topologically equivalent and define equivalent linearizations of the equivariant tangent bundles.*

Proof. Start with the orientation-reversing involution on S^4 given by hyperplane reflection, and take equivariant connected sums with $X_k \amalg \overline{X_k}$ for each nonnegative integer k. By Gompf's result, one obtains a family of smooth involutions Φ_k on $\#^{10}(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$. These actions are topologically equivalent because the manifolds X_k are homeomorphic to each other. If two actions were smoothly equivalent, then an equivariant diffeomorphism would preserve the components of the fixed point sets' complements, and likewise for their closures. Since the latter are the punctured manifolds $X_k - \operatorname{Int}(D^4)$, it would follow that the latter would be diffeomorphic for two choices $m \neq n$ of k. Therefore X_n would be obtainable from X_m by cutting out an open disk and attaching a closed disk along the boundary by some diffeomorphism of S^3. Since every diffeomorphism of S^3 extends to D^4 by a fundamental theorem of J. Cerf [Cer], it follows that X_m and X_n would be diffeomorphic. Therefore the smooth involutions Φ_m and Φ_n are smoothly inequivalent if $m \neq n$. Since $H^3(X_0; \mathbb{Z}_2) = 0$ all of the smoothings $X_k \to X_0$ determine the same linearization of the tangent bundle, and the reasoning in the proof of Proposition 2.1 now shows that the linearizations for the equivariant tangent bundles of the actions Φ_k are equivariant. □

Similar considerations easily yield other infinite families of smoothly inequivalent, but topologically equivalent, actions that are orientation-preserving, but in these cases we cannot determine whether any of these manifolds with group actions are nonequivariantly diffeomorphic.

Proposition 2.5. *Let* $n \geq 2$ *be an integer, let* V *be a* 1*-dimensional unitary representation of* $\mathbb{Z}_n$*, and for each* $k \geq 0$ *consider the smooth* $\mathbb{Z}_n$ *action* Φ_k *on the* n*-fold connected sum* $\#^n\ X_k$ *which is an equivariant connected sum of the linear action on* $S(\mathbb{R}^3 \oplus V)$ *with* $\mathbb{Z}_n \times X_k$ *as in Proposition* 1.3. *Then the actions* Φ_k *are topologically equivalent, but no two are smoothly equivalent. However, these smooth actions determine equivalent linearizations for the topological equivariant tangent bundles associated to* Φ_0.

As noted above, it would be enlightening to know if for some $n \geq 2$ there is an infinite family of actions as above on the same smooth manifold:

Question. For some $n \geq 2$ is there an infinite family of smooth manifolds $\{\#^n\ X_{k(m)}\}$ that are pairwise (orientation preservingly) diffeomorphic?

We shall conclude this section with another example in which the smoothly inequivalent actions all operate on the same smooth manifold. To simplify the notation, if p and q are positive integers we shall let $p\,\mathbb{CP}^2\,\#\,q\,\overline{\mathbb{CP}^2}$ denote the (oriented) connected sum of p copies of $\mathbb{CP}^2$ and q copies of $\overline{\mathbb{CP}^2}$.

Theorem 2.6. *There is an infinite family of inequivalent smooth involutions* Φ_k *on* $3\,\mathbb{CP}^2\,\#\,18\,\overline{\mathbb{CP}^2}$ *such that the involutions are topologically equivalent and determine equivalent linearizations of the equivariant topological tangent bundle.*

Proof. The construction starts with the conjugation involution on $\mathbb{CP}^2$, which takes a point with homogeneous coordinates (a,b,c) to the point with homogeneous coordinates $(\overline{a},\overline{b},\overline{c})$. The fixed point set of this action is $\mathbb{RP}^2$ (viewed as the subset of points representable by real homogeneous coordinates), and it is well known that the orbit space $\mathbb{CP}^2/\mathbb{Z}_2$ is diffeomorphic to S^4 (e.g., see Atiyah–Berndt [AB], Kuiper [Kui] or Massey [Mas]). Since the normal bundle of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ is not orientable, this example does not quite fit into the setting of Section 1 in [Sch], so comments are in order regarding the smooth structure on the orbit space discussed in [AB] and [Kui]. In the situations of interest, there are compatible orientations on the manifold with involution M and its orbit space $M/\mathbb{Z}_2$ such that the orbit space projection has positive degree (equal to 2). Therefore, in the construction of the smooth structure it is enough to consider smooth coordinate charts that preserve orientations. One key result is Theorem 1.2 in [Sch], which implies that two special atlases satisfying condition (1.1a)–(1.1f) in [Sch] define the same smooth structure on the orbit space, and in our situation we need a similar result for oriented special atlases. One crucial step involves a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^f \times \mathbb{R}^2 & \xrightarrow{\ \psi\ } & \mathbb{R}^f \times \mathbb{R}^2 \\ {\scriptstyle 1\times q}\downarrow & & {\scriptstyle 1\times q}\downarrow \\ \mathbb{R}^f \times \mathbb{R}^2 & \xrightarrow{\ \phi\ } & \mathbb{R}^f \times \mathbb{R}^2 \end{array}$$

where $q(x,y) = (x^2 - y^2, 2xy)$ is the map given by squaring a complex number and $\psi(u,v) = (u,\, \theta(u)v)$ where

$$\theta : \mathbb{R}^f \to \mathrm{O}_2$$

is smooth. In the setting of this result, the maps φ and ψ are orientation preserving, and therefore we can lift θ to a map $\theta' : \mathbb{R}^f \to U_1 = \mathrm{SO}_2$. This yields a refined version of Lemma 1.4 in [Sch], where $\sigma : U_1 \to \mathrm{O}_2$ is the composite of the squaring homomorphism $U_1 \to U_1$ with the embedding $U_1 = \mathrm{SO}_2 \subset \mathrm{O}_2$. This means φ satisfies the identity

$$\varphi\big(u, q(v)\big) = \big(u,\, \sigma \circ \theta'(u) v\big)$$

and hence φ is a diffeomorphism. As in [Sch], this means that if $\mathcal{A}$ and $\mathcal{B}$ are oriented special atlases for $M/\mathbb{Z}_2$, then the identity from $(M/\mathbb{Z}_2, \mathcal{A})$ to $(M/\mathbb{Z}_2, \mathcal{B})$ is a diffeomorphism. One can then proceed as in [Sch] and Section 1 of this paper to show that an orientation preserving equivariant diffeomorphism $M \to N$ yields an orientation preserving diffeomorphism from $M/\mathbb{Z}_2$ to $N/\mathbb{Z}_2$.

If we now take an equivariant connected sum of the conjugation action on $\mathbb{CP}^2$ with two copies of a Dolgachev surface X_k, then the orbit space of the resulting action is $\mathbb{CP}^2/\mathbb{Z}_2 \# X_k \cong S^4 \# X_k \cong X_k$, so there is no orientation preserving equivariant diffeomorphism relating the actions defined using X_k and X_j if $k \neq j$. Furthermore, there cannot be an orientation reversing equivariant diffeomorphism either, for such a map would define an orientation reversing homeomorphism from $\mathbb{CP}^2 \# X_k \# X_k$ to itself. Since the signature of the latter is nonzero, this manifold is not orientation reversingly homeomorphic to itself. Therefore the actions defined using X_k and X_j are smoothly inequivalent if $k \neq j$. On the other hand, one can now use the same arguments as before to show these actions are topologically equivalent and the associated eqivariant linearizations of the equivariant topological tangent bundle are also equivalent.

To conclude the proof, we need to show that $\mathbb{CP}^2 \# X_k \# X_k$ splits smoothly into a connected sum of copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$. This follows from two applications of Corollary 9 in [Gom3] (the result is due to R. Mandelbaum [Man1] and B. Moishezon [Moi]; see also the survey article [Man2]). This result implies that $\mathbb{CP}^2 \# X_k$ splits as a connected sum $2\,\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}^2}$, and another application yields the desired splitting:

$$\mathbb{CP}^2 \# X_k \# X_k \;\cong\; 2\,\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}^2} \# X_k \;\cong\; 3\,\mathbb{CP}^2 \# 18\,\overline{\mathbb{CP}^2} \qquad \square$$

2.3. Minimal examples. One fundamental question about exotic smoothings of simply connected 4-manifolds is to find examples for which $H_2(M;\mathbb{Z})$ is as small as possible. Much work has been done on this problem since [Don2] and [FM1]–[FM4] (e.g., see Fintushel and Stern [FS]), and one can combine these advances with the methods of this section to construct exotic smooth actions on $n\,\mathbb{CP}^2 \# n\,\overline{\mathbb{CP}^2}$ and $3\,\mathbb{CP}^2 \# m\,\overline{\mathbb{CP}^2}$ where $n < 10$ and $m < 18$. We have focused on Dolgachev surfaces to simplify the discussion and to leave space for further advances in an active research area; new results on minimality questions will surely figure in determining the least possible values of m and n for which such exotic families exist.

Linear actions on S^4 are another fundamentally important class of minimal examples. It is well known that two linear actions of finite groups on S^4 are topologically equivalent if and only if they are linearly equivalent (compare [dRh]), and it would be particularly illuminating to know if there are smooth actions of finite groups on S^4 which are topologically but not smoothly linear. In higher dimensions there are are many such examples and extensive literature beginning with [CMY] and [Son] (see [RS] for a definitive account of the latter).

3. Actions on exotic 4-spaces

Theorem 1.1 implies that all locally linear actions of odd order groups on noncompact connected 4-manifolds are smoothable, and the same conclusion is valid for a large class of even order group actions. In view of this, henceforth we shall limit the discussion of the noncompact case to constructing examples of nondiffeomorphic smoothings of a locally linear action which define equivalent linearizations of the equivariant topological tangent bundle. In fact, we shall specialize even further to locally linear actions on $\mathbb{R}^4$ which are equivariantly contractible; note that the latter property implies there is a unique equivalence class of linearizations for the equivariant topological tangent bundle (cf. tom Dieck [tDie] or Lashof [Las]). Orthogonal actions of finite groups on $\mathbb{R}^4$ clearly satisfy this condition, and additional examples are given by deleting a fixed point from a smooth semifree cyclic group action on S^4 for which the fixed point set is a knotted 2-sphere (see Giffen [Gif], Gordon [Gor] or Sumners [Sum] for examples and additional background information).

Shortly after the discovery of exotic smooth structures on $\mathbb{R}^4$, it became clear that there were examples which supported smooth actions which are topologically equivalent to the previously mentioned types of equivariantly contractible smooth actions on the standard $\mathbb{R}^4$. In particular, if Ω^4 denotes the "universal" smoothing structure on $\mathbb{R}^4$ constructed by M. Freedman and L. Taylor [FT], the following result was well known when that paper appeared in print:

Proposition 3.1. *Let G be a finite group, and let Φ be a smooth effective orientation preserving action of G on $\mathbb{R}^4$ (with the standard smooth structure) such that Φ is equivariantly contractible. Then there is a smooth action Φ_Ω of G on Ω^4 which is topologically equivalent to Φ.*

These examples can be viewed as special cases of a more general construction which we shall use. If $\mathcal{R}$ is the set of oriented diffeomorphism classes of smooth manifolds homeomorphic to $\mathbb{R}^4$, then the Gompf end connected sum construction $(U, V) \to U \natural V$ of [Gom1] and [GS] (see Definition 9.4.6 and the accompanying discussion on pp. 368–369) makes $\mathcal{R}$ into a commutative monoid with involution (given by reversing the orientation); the identity is the class of the standard smooth structure on $\mathbb{R}^4$, and the class of the Freedman-Taylor manifold Ω^4 is a null element by the results of [FT] (*i.e.*, for every oriented representative V^4 of a class in $\mathcal{R}$ we have $V^4 \natural \Omega^4 \cong \Omega^4$ as oriented smooth manifolds). Furthermore, $\mathcal{R}$ has an infinite end sum operation $\natural_{i=1}^{\infty} U_i$ associated to a countably infinite sequence $\{U_i\}$ of representatives for classes in $\mathcal{R}$, and this infinite operation has good commutativity and associativity properties. As noted in [GS], this implies that $U \natural V \cong \mathbb{R}^4$ (smoothly) if and only if U and V are diffeomorphic to $\mathbb{R}^4$ and that for every exotic oriented smoothing V of $\mathbb{R}^4$ the iterated sums $\natural^m V$ $(m \geq 1)$ are never diffeomorphic to $\mathbb{R}^4$. In particular, the commutative monoid $\mathcal{R}$ has no invertible elements aside from the identity.

As in Section 2, if W is a connected oriented manifold, then $\overline{W}$ will denote W with the opposite orientation.

We can now place Proposition 3.1 into the desired general context:

Proposition 3.2. *Let G be a finite group and let Φ be an effective smooth action of G on $\mathbb{R}^4$, and let V be a smooth oriented manifold which is homeomorphic to $\mathbb{R}^4$.*

(i) *If G acts orientation preservingly then there is a smooth action Φ_V of G on $\natural^{|G|} V$ (where $|G|$ is the order of G) such that Φ_V is topologically equivalent to Φ.*

(ii) *If $G = \mathbb{Z}_2$ and the action reverses the orientation then there is a smooth action Φ_V of $G = \mathbb{Z}_2$ on $V \natural \overline{V}$ such that Φ_V is topologically equivalent to Φ.*

Derivation of Proposition 3.1 *from Proposition* 3.2. The first result follows immediately from the second if we take $V = \Omega^4$, for in this case the absorption property implies that $\natural^{|G|} \Omega^4 \cong \Omega^4$ and $\Omega^4 \natural \overline{\Omega^4} \cong \Omega^4$. □

Comments on the proof of Proposition 3.2. The first part of the result is stated informally on page 202 of Gompf's exotic menagerie paper [Gom4], and the idea of the proof is fairly simple. In particular, if $x \in \mathbb{R}^n$ has a trivial isotropy subgroup (such points form an open dense subset), we can construct the end sum $\natural^{|G|} V$ by means of a smooth proper equivariant embedding $\gamma : G \times [0, \infty) \to \mathbb{R}^4$ such that $\gamma(1, 0) = x$ and an extension of γ to a smooth invariant tubular neighborhood. In the orientation reversing case there is a similar construction but the ambient manifold for Φ_V is given by $V \natural \overline{V}$. □

Section 4 of [Gom4] describes some considerably more complicated examples of finite group actions involving exotic 4-spaces. We shall proceed in a more elementary direction and consider the following basic question:

Question 1. Let G be a finite group which acts smoothly and semifreely on $\mathbb{R}^4$. Is there an uncountable family of pairwise nondiffeomorphic smooth actions W_d which are topologically equivalent to the given action?

We shall be particularly interested in constructing examples with the following additional properties:

Question 2. Let G be as above, and let V be an exotic smoothing of an action on $\mathbb{R}^4$ obtained by deleting a fixed point from a smooth action on S^4. Is there an uncountable family of invariant open subsets $W_d \subset V$ such that the induced smooth actions on W_d are pairwise nondiffeomorphic but the actions W_d are all topologically equivalent to the original action?

If $G = \mathbb{Z}_2$ and the action reverses orientation, then a positive answer to the first question is implicit in the results of [Gom2] and [Gom4]:

Proposition 3.3. *Suppose that $\mathbb{Z}_2$ acts effectively, smoothly and orientation reversingly on $\mathbb{R}^4$, and denote the action by Φ. Then there is an uncountable family $V(t)$ of exotic 4-spaces such that the associated action $\Phi_{V(t)}$ are pairwise smoothly inequivalent.*

Comment on the proof. This follows immediately from the construction of [Gom2] and [Gom4] which gives a family of $2^{\aleph_0}$ mutually inequivalent exotic 4-spaces V_t such that no two of the end sums $V_t \natural \overline{V_t}$ are diffeomorphic. □

Examples of continuous families of orientation preserving actions on $\mathbb{R}^4$ are not so apparent in the existing literature, so we shall explain some ways of constructing such families. If we knew there was continuum of exotic 4-spaces V_t such that no two end sums $\natural^m V_t$ were orientation preservingly diffeomorphic for some $m \geq 2$, then the first part of Proposition 3.2 would yield the desired families of actions for all groups G of order m, but results of this type do not seem to be in the literature.

We shall answer Question 2 positively in many important cases. The idea is to construct continua of actions starting with the initial examples of exotic 4-spaces arising in the work of Donaldson and Freedman (cf. Gompf's description in [Gom1]). Strictly speaking, we shall need modified versions of those examples, and it will be convenient to work in an axiomatic setting modeled on results in [Gom1].

Our construction will use the following standard observation involving the local equivalence of coordinate charts in a topological manifold; in the 4-dimensional case it is a consequence of the Annulus Theorem in [Qui] and the Topological Isotopy Extension Theorem.

Lemma 3.4. *Let A be an oriented smooth 4-manifold, let $h : S^3 \times (-1,1) \to A$ be a homeomorphism, and let $U \subset S^3$ be an open 3-disk whose closure $\overline{U}$ is a locally flat closed 3-disk in S^3. Then there is an isotopy of h (through homeomorphisms) to a homeomorphism h' and a concentric subdisk $U_0 \subset \overline{U_0} \subset U$ such that h' is a* smooth *embedding on an open neighborhood of $\overline{U_0} \times \{0\}$ in $S^3 \times (-1,1)$.*

We shall be working with iterated connected sums $\#^m K$ of the Kummer **K3** surface K. As noted in [FQ], this smooth (in fact, complex) manifold is topologically a connected sum $\#^m\ (N \# P)$, where P is a connected sum of 3 copies of $S^2 \times S^2$ and N is a simply connected 4-manifold whose intersection form is two copies of the E_8 matrix [MH]. For each positive integer m, it follows that $\#^m K$ contains a region A which is homeomorphic (but not diffeomorphic) to $S^3 \times (-1,1)$ and has the following additional properties:

(1) If $B \subset A$ corresponds to $S^3 \times \{0\}$, then $\#^m K - B$ has two components V^- and V^+ whose closures F^- and F^+ are homeomorphic to $(\#^m N)^{\text{o}}$ and $(\#^m P)^{\text{o}}$, where X^{o} is the compact manifold with boundary formed by deleting the interior of a suitably embedded closed disk.

(2) If A^- and A^+ are the subsets of A corresponding to $S^3 \times (-1,0]$ and $S^3 \times [0,1)$, then $F^- \cap A = A^-$ and $F^+ \cap A = A^+$.

Specific choices of objects in the preceding discussion will be called *Donaldson–Freedman splitting data* for $\#^m\ K$.

Definition. A *Donaldson–Freedman system* for $\#^m\ K$ will consist of the following data:

(1) A set of Donaldson–Freedman splitting data such that the homeomorphism $h : S^3 \times (-1,1) \to A$ satisfies the partial smoothness condition in the conclusion of Lemma 3.4

(2) A pair of spaces (W, C) such that W is a smooth manifold homeomorphic to $\mathbb{R}^4_+$ and $C \subset W$ is compact, together with a diffeomorphism $k : W - C \to A$.

If we are given a Donaldson–Freedman system as above, then we set W^- equal to $W - h^{-1}[A_+]$. It follows that W^- is homeomorphic to $\mathbb{R}^4$, but it is not diffeomorphic to $\mathbb{R}^4$. Although W^- is not quite the same as a Donaldson–Freedman exotic 4-space in [Goml], it has similar properties.

Proposition 3.5. *In the setting above, the smooth manifold W is not diffeomorphic to $\mathbb{R}^4$. Furthermore, if $\varphi : W^- \to \mathbb{R}^4$ is a homeomorphism, and $U_t \subset \mathbb{R}^4$ is the open disk of radius t centered at the origin, then there is some $\lambda > 0$ such that $s, t > \lambda$ implies that $\varphi^{-1}[U_s]$ and $\varphi^{-1}[U_t]$ are not orientation preservingly diffeomorphic.*

Proof. This is analogous to the proof of the corresponding result for the Donaldson–Freedman exotic 4-space (see [Gom1], [Gom2] and [Tau]). If W^- were diffeomorphic to $\mathbb{R}^4$, then one could find a smoothly embedded 3-sphere in $A^- - B$, and one could use this to define a smooth structure on $\#^m N$ as in [Gom1]; as with the latter, Donaldson's results imply that $\#^m N$ is not smoothable, so no smooth embedding can exist. Similarly, if there is a homeomorphism φ such that $\varphi[U_s]$ and $\varphi[U_t]$ are diffeomorphic for sufficiently large values of $s \neq t$, then as in Gompf [Gom2] and Taubes [Tau] the manifold $\#^m K - (\#^m P)^{\mathrm{O}}$, which is homeomorphic to $\#^m N - \{\text{point}\}$, would have a smooth structure with a periodic end. The results of [Tau] imply that $\#^m N - \{\text{point}\}$ has no such smooth structure, and therefore it follows that no two of the smooth manifolds $\varphi[U_t]$, $\varphi[U_s]$ are orientation preservingly diffeomorphic if s and t are sufficiently large. □

A crucial step in our construction is the following relationship between Donaldson–Freedman systems and connected sums.

Proposition 3.6. *Let p and q be positive integers, and suppose that we have Donaldson–Freedman Systems $(W_p, C_p; A_p, etc.)$ and $(W_q, C_q; A_q, etc.)$ for $\#^p K$ and $\#^q K$ respectively. Then one can construct a Donaldson–Freedman system $(W_{p+q}, C_{p+q}; A_{p+q}, etc.)$ for $\#^{p+q} K$ such that $W^-_{p+q} \cong W^-_p \natural W^-_q$.*

Proof. The idea is to construct a smooth connected sum of $\#^p K$ with $\#^q K$ in a manner compatible with all the data in the Donaldson–Freedman systems. This can be done by choosing smooth coordinate neighborhoods E'_p and E'_q at smooth points of the 3-sphere B_p and B_q; more precisely, we want smooth round closed disks E_p and E_q in product neighborhoods of the form $\Lambda \times (-\varepsilon, \varepsilon)$ such that $\Lambda \times \{0\}$ corresponds to the points in B_p or B_q, $\Lambda \times (-\varepsilon, 0)$ corresponds to the points in W^-_p or V^-_q, and $\Lambda \times (0, \varepsilon)$ corresponds to the points in W^+_p or V^+_q. This yields a bicollared topological embedding of $B_p \# B_q$ in $(\#^p K)\#(\#^q K)$ which is a smooth embedding around some point, The complement of $B_p \# B_q$ splits into two components, and the closure of these components are homeomorphic to $(\#^{p+q} N)^{\mathrm{O}}$ and $(\#^{p+q} P)^{\mathrm{O}}$. In fact, we can say even more. Let $E^-_p \subset E_p$ and $E^-_q \subset E_q$ correspond to the points $\Delta_- \subset D^4$ whose last coordinates are negative. Then the compact manifold with boundary Δ_- is a closed tubular neighborhood of the curve $[1, \infty) \to \operatorname{Int} \Delta_-$ defined along the fourth coordinate axis by $(0, 0, 0, w(t))$, where $w(t)$ is a smooth function such that $w(1) = -\frac{1}{2}$, $w' > 0$ everywhere, and $\lim_{t \to \infty} w(t) = 0$, and therefore E^-_p and E^-_q are subsets which can be used to construct the end connected sums $V_p \natural, V_q$ and $W_p \natural W_q$. If V^-_{p+q} is the connected component of $\#^{p+q} K$ which is homeomorphic to $\operatorname{Int}(\#^{p+q} K)^{\mathrm{O}}$, then it follows that V^-_{p+q} is orientation preservingly diffeomorphic to $V^-_p \natural V^-_q$. Now consider the manifold

$W_p^- \natural W_q^- \cong (k_p[A_p - A_p^+] \cup C_p)\#(k_q[A_q - A_q^+] \cup C_q)$. By construction the ends of $V_p^- \natural V_q^-$ and $W_p^- \natural W_q^-$ have diffeomorphic (deleted) neighborhoods, so it follows that this end sum is a good candidate for the manifold W_{p+q} in suitably defined Donaldson–Freedman data for $\#^{p+q}K$. If we now choose A_{p+q} to be a small open bicollar neighborhood of $B_p\#B_q$ which is contained in $A_p\#A_q$, then the remaining data for a Donaldson–Freedman system on $\#^{p+q}K$ can be constructed in a straightforward manner, and therefore we have a choice of data for $\#^{p+q}K$ such that W_{p+q}^- is orientation preservingly diffeomorphic to $W_p^- \natural W_q$. □

Corollary 3.7. *Let $m \geq 2$ be an integer and suppose that we have a Donaldson–Freedman system $(W_1, C_1; A_1, etc.)$ for K. Then there is a Donaldson–Freedman system $(W_m, C_m; A_m, etc.)$ for $\#^m K$ such that $W_m^- \cong \natural^m\ W_1$.*

Proof. This follows by repeated application of Proposition 3.6. □

Remark. The existence of a Donaldson–Freedman system for K follows directly from [Gom1] and Lemma 3.4, so Corollary 3.7 implies the existence of Donaldson–Freedman systems for each connected sum $\#^m\ K$.

3.1. Construction of exotic smooth actions. We are almost ready to state and prove the main result of this section for smooth orientation preserving actions of finite groups on $\mathbb{R}^4$. Our proof will use the following elementary consequence of local linearity at fixed points of smooth actions.

Lemma 3.8. *Let G be a finite group, let Ψ be an effective, smooth and orientation preserving action on S^n, and let Φ be the smooth action on $\mathbb{R}^n$ obtained by deleting one fixed point p_0. For each $t > 0$, let $U_t \subset \mathbb{R}^n$ denote the open disk with radius t which is centered at the origin. Then Φ is smoothly equivalent to an action Φ' of $\mathbb{R}^n$ with the following property:*

There is a positive constant λ_0 such that U_{λ_0} is Φ'-invariant and the restriction of this action to $\mathbb{R}^n - U_{\lambda_0} \cong S^{n-1} \times [\lambda_0, \infty)$ is the product of an orthogonal action on S^{n-1} with the trivial action on $[\lambda_0, \infty)$.

In particular, it follows that if $t > \lambda_0$ then U_t is Φ'-invariant and $\Phi'|_{U_t}$ is smoothly equivalent to Φ'.

Lemma 3.8 allows us to simplify the formulation of the main result.

Theorem 3.9. *Let G be a finite group, and let Φ' be a smooth action on $\mathbb{R}^4$ satisfying the conditions in Lemma* 3.8*. Given an exotic 4-space W, let Φ_W be the topologically equivalent smooth action on $\natural^{|G|} W$ constructed in Proposition* 3.2*, and let U_t and λ_0 be as in Lemma* 3.8*, and let θ be an equivariant homeomorphism from $(\natural^{|G|} W, \Phi_W)$ to $(\mathbb{R}^4, \Phi)$. Then for at least one choice of W the restricted actions on the G-invariant open sets $\theta^{-1}[U_t], t > \lambda_0$ are all topologically equivalent to $(\mathbb{R}^4, \Phi)$, but there is some $\lambda_1 > \lambda_0$ such that if $t, s > \lambda_1$ and $t \neq s$ then $\theta^{-1}[U_t]$ and $\theta^{-1}[U_s]$ are not even nonequivariantly orientation preservingly diffeomorphic.*

Since the hypotheses (hence also the conclusions) of Lemma 3.8 hold for orientation preserving orthogonal actions of a finite group G on S^4 with fixed points and their associated actions on $\mathbb{R}^4$, it follows that every such action is topologically equivalent to an action on an exotic 4-space which contains an uncountable family of symmetric exotic 4-spaces as invariant open subsets where each restricted action is topologically equivalent to an orthogonal action and no two of the open subsets are orientation preservingly diffeomorphic to each other.

One can also state a version of Theorem 3.9 which does not require the conditions in Lemma 3.8.

Corollary 3.10. *If the finite group G acts smoothly and orientation preservingly on S^4 with* (*at least*) *one fixed point p_0, then there is a continuum of smooth G-actions $\{\Phi_\alpha\}$ on exotic 4-spaces $\{V_\alpha\}$ such that each action is topologically equivalent to the given action on $S^4 - \{p_0\}$ but the underlying exotic 4-spaces V_α and V_β are not orientation preservingly diffeomorphic if $\alpha \neq \beta$.*

Proof of Corollary 3.10. This follows immediately from Lemma 3.8 and Theorem 3.9. □

Remark. In the preceding two results we have constructed families of $2^{\aleph_0}$ smoothly inequivalent actions which are topologically equivalent to a given example. More generally, if we are given a smooth finite group action on a second countable manifold, then by the Mostow-Palais equivariant embedding theorem (see Mostow [Mos] and Palais [Pal]) there are at most $2^{\aleph_0}$ smoothly inequivalent actions which are topologically equivalent to the given one because there are only $2^{\aleph_0}$ locally closed subsets in the spaces $\mathbb{R}^n$, where n runs through all positive integers.

Proof of Theorem 3.9. Let $(W_1, \mathcal{C}_1; A_1, etc.)$ be a Donaldson–Freedman system for K, and let $(W_{|G|} \cong \natural^{|G|} W_1, \mathcal{C}_{|G|}; A_{|G|}, \text{etc})$ be the system for $\#^{|G|} K$ described in Corollary 3.7. By Proposition 3.2 there is an equivariant homeomorphism

$$\theta : (\natural^{|G|} W_1^-, \Phi_W) \to (\mathbb{R}^4, \Phi)$$

and if t is sufficiently large then $\theta^{-1}[U_t]$ is a smooth G-manifold which is topologically equivalent to U_t and hence is also topologically equivalent to the original action on $\mathbb{R}^4$. On the other hand, by Proposition 3.5 if s and t are sufficiently large and $s \neq t$, then the underlying smooth manifolds $\theta[U_t]$ and $\theta[U_s]$ are not even nonequivariantly diffeomorphic as oriented manifolds. □

It is not difficult to state many further questions about group actions on exotic 4-spaces, but often it is unclear whether these questions can be studied successfully. We shall conclude this section with one easily stated example which is motivated by the results of DiMichelis and Freedman [dMF] on exotic 4-spaces that are open subsets of $\mathbb{R}^4$:

Question. Let G act orthogonally on $\mathbb{R}^4$. Is there a continuum of invariant open subsets $U_\alpha \subset \mathbb{R}^4$ such that the restricted smooth actions are all topologically equivalent to the given action but the sets U_α and U_β are not even nonequivariantly diffeomorphic if $\alpha \neq \beta$?

Here is a statement (without proof) of a partial result:

Theorem 3.11. *Suppose that G acts orthogonally and semifreely on $\mathbb{R}^4$ with a 2-dimensional fixed point set. Then there is a continuum of invariant open subsets $U_\alpha \subset \mathbb{R}^4$ such that the restricted smooth actions are all topologically linear but the restricted actions on U_α and U_β are not equivariantly diffeomorphic if $\alpha \neq \beta$.*

For the family of examples in this result, the canonical smooth structures on the orbit spaces U_α/G and U_β/G are not diffeomorphic if $\alpha \neq \beta$.

4. Higher dimensions

If G is a finite group, then the results of [LR] on G-smoothings assume that, for each subgroup $H \subset G$, the fixed point set M^H has no 4-dimensional connected components. It is not difficult to construct counterexamples to the conclusions of [LR] if this condition is not met. For example, suppose that V^n is an positive dimensional orthogonal representation of G such that G acts freely on $V^n - \{0\}$ (*i.e.*, a free linear representation). Then for each smooth manifold W^4 that is homeomorphic but not diffeomorphic to $\mathbb{R}^4$ the G-action product manifold $W^4 \times V^n$ is topologically equivalent to the linear action of G on $\mathbb{R}^4 \oplus V^n$ and the manifold $W^4 \times V^n$ is nonequivariantly diffeomorphic to $\mathbb{R}^{4+n}$

(since homeomorphic to $\mathbb{R}^q$ implies diffeomorphic to $\mathbb{R}^q$ if $q \neq 4$), but $W^4 \times V^n$ is not equivariantly diffeomorphic to $\mathbb{R}^4 \oplus V^n$ because the fixed point sets are not diffeomorphic. Furthermore, since $\mathbb{R}^4 \oplus V^n$ is equivariantly contractable it follows that the linearization of the equivariant tangent bundle for the smoothing $W^4 \times V^n$ is equivalent to the linearization for the orthogonal action on $\mathbb{R}^4 \oplus V^n$. Our main objective in this section is to describe *compact* counterexamples to the conclusions of [LR] for locally linear group actions with 4-dimensional fixed point sets. As before, there are examples of two types; the first involves nonexistence of equivariant smoothings, and the second concerns nonuniqueness of equivariant smoothings which determine the same linearization of the equivariant topological tangent bundle.

Theorem 4.1. *Let G be a finite group, and let V be an orientation preserving free orthogonal representation of G such that $n = \dim V \geq 2$. Then there is an infinite family of locally linear orientation preserving, semifree G-actions on connected oriented $(n+4)$-manifolds M_j (where j is a positive integer) with the following properties:*

(1) *The fixed point sets F_j of the G-manifolds are closed, simply connected 4-manifolds whose signatures satisfy $|\mathrm{sgn} F_{j(1)}| \neq |\mathrm{sgn} F_{j(2)}|$ if $j(1) \neq j(2)$.*
(2) *The equivariant tangent bundles of the G-manifolds M_j come from G-vector bundles.*
(3) *The actions are not equivalently smoothable; in fact, their fixed point sets are not smoothable.*

Theorem 4.2. *Let G and V be as in the preceding theorem. Then there is an infinite family of smooth, orientation preserving, semifree G-actions connected manifolds $M_{j,h}$ (where j runs through all sufficiently large positive integers and h runs through all positive integers) with the following properties:*

(1) *The fixed point sets of the actions are closed, simply connected, oriented 4-manifolds $F_{j,h}$ such that $|\mathrm{sgn} F_{j(1),h(1)}| = |\mathrm{sgn} F_{j(2),h(2)}|$ if and only if $j(1) = j(2)$.*
(2) *For each j, the smooth G-manifolds $M_{j,h(1)}$ and $M_{j,h(2)}$ are orientably topologically equivalent, and the equivariant linearizations of their equivariant tangent bundles are also equivalent.*
(3) *For each j, the smooth G-manifolds $M_{j,h(1)}$ and $M_{j,h(2)}$ are not equivariantly diffeomorphic if $h(1) \neq h(2)$. However, the manifolds $M_{j,h(1)}$ and $M_{j,h(2)}$ are nonequivariantly orientation preservingly diffeomorphic for all $h(1)$ and $h(2)$, and $M_{j,h(1)} - F_{j,h(1)}$ is orientation preservingly equivariantly diffeomorphic to $M_{j,h(2)} - F_{j,h(2)}$ for all $h(1)$ and $h(2)$.*

The final conclusions in part (3) of the second theorem are analogs of the replacement theorems for fixed point sets due to S. Cappell, S. Weinberger and M. Yan (see [CW] and [CWY]). A more general statement of the replacement principle for 4-dimensional fixed point sets is given in Proposition 4.3 below.

Proof of Theorem 4.1. We begin with an alternate description of the smooth oriented bordism homology theory $\Omega_*^{\mathrm{SO}}(X)$ considered by P. E. Conner and E. E. Floyd (see [Con], Section 1.4, and [Sto], Example 6, p. 43). Chapter II of [Sto] discusses bordism theories for smooth manifolds with an extra (B, f) structure arising from a suitable map $f : B \to BO$. The structures may be viewed as suitably defined equivalence classes of liftings for the diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ M & \xrightarrow{N_M} & B_0 \end{array}$$

where N_M is the classifying map for the stable normal bundle of M, where M is embedded in some large $\mathbb{R}^q$. There is an analogous theory for topological manifolds because (i) embeddings of a topological manifold M in some large $\mathbb{R}^q$ have topological tubular neighborhoods (see [KiS]), (ii) there are topological transversality theorems analogous to the usual smooth transversality results (see [KiS] for dimensions $\neq 4$ and [FQ] for the 4-dimensional case). It follows that the oriented bordism homology groups $\Omega_*^{\mathrm{SO}}(X)$ are isomorphic to the topological bordism groups $\Omega_*^{\mathrm{TOP}}(B, f)$, where $f : B \to B\mathrm{Top}$ is the composite

$$X \times B\mathrm{SO} \underset{\text{proj}}{\overset{\text{coord}}{\longrightarrow}} B\mathrm{SO} \xrightarrow{B_i} B\mathrm{Top}$$

and B_i is the map of classifying spaces corresponding to the standard homeomorphism $i : \mathrm{SO} \to \mathrm{Top}$. Since a smooth structure or a manifold M defines a unique equivalence class of vector bundle structures on the topological tangent bundle, it follows that each smooth representative of a class in $\Omega_*^{\mathrm{SO}}(X)$ in the sense of [Con] determines a unique class in the topologically defined bordism group $\Omega_*^{\mathrm{TOP}}(B, f)$, and in dimensions$\neq 4$ the resulting bijection of bordism groups reflects the fact that equivalence classes of smoothings correspond to equivalence classes of stable tangent bundle linearizations in such cases (see [KiS]).

Our reason for interest in this alternate description of $\Omega_*^{\mathrm{SO}}(X)$ is that it allows one to view classes in $\Omega_*^{\mathrm{SO}}(X)$ as representable by data $(M^4, \lambda, h : M^4 \to X)$, where λ is an oriented vector bundle structure on the tangent bundle for a topological 4-manifold and $f : M^4 \to X$ is a continuous map as in the Conner-Floyd definition. If X is a point, then $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$ is detected by the signature, so the latter determines the class of (M^4, λ) where M^4 and λ are given as above.

With the preceding observations, the proof of Theorem 4.1 becomes fairly straightforward. Let $S(V)$ denote the unit sphere in V with its associated free orthogonal action of G, let $L(V) = S(V)/G$, and let $h : L(V) \to BG$ be the classifying map for the principal G-bundle $S(V) \to L(V)$. Consider the class in $\Omega^{\mathrm{SO}}_{n-1}(BG)$ represented by $(L(V), h)$. Since G acts orientation preservingly, n must be even and therefore the results of [Con] imply that the class $[L(V), h]$ has a finite order that we shall denote by m. For each $j > 0$ let F^4_j be the oriented, simply connected 4-manifold which is a connected sum of $2mj$ copies of the simply connected E_8 manifold (hence the tangent bundle of F^4_j can be linearized). It follows that $F^4_j \times [L(V), h]$ represents the trivial class in $\Omega^{\mathrm{SO}}_{4+n}(BG)$. A null bordism of the class corresponds to a map $k : W^* \to BG$ such that $k|\partial W^*$ is the composite of h and the coordinate projection $F^4_j \times L(V) \to L(V)$. Since $F \times L(V)$ is connected, we might as well assume the same for W^*. Take $W \to W^*$ to be the principal G-bundle classified by k, and form the G-manifold

$$M^{4+n}_j = \left(F^4_j \times D(V)\right) \cup_\partial W^{4+n}$$

where $D(V) \subset V$ is the unit disk. It follows that M^{4+n}_j is a closed, connected, semifree and locally linear G-manifold with a linearization of its equivariant topological tangent bundle. However, the action is not smoothable; if it were, then F^4_j would be smoothable, and by [Don1] we know this is not the case. □

Before beginning the proof of Theorem 4.2, we shall formulate a generalization of one step in the argument.

Proposition 4.3. (Replacement Principle for fixed point sets) *Suppose that the finite group G acts smoothly and semifreely on a closed manifold M, let F be a connected component of the fixed point set M^G, and let $(N; F, F')$ be a smooth s-cobordism which is topologically trivial. Assume that* $\dim M \geq 5$ *and* $\dim F \geq 4$. *Then there is a smooth semifree G-action on $M \times [0,1]$ with the following properties:*

1. *The restriction to $M \times \{0\}$ is smoothly equivalent to the original action.*
2. *The action on $M \times [0,1]$ is topologically equivalent to the product of the original action with the trivial action on $[0,1]$.*
3. *If $M' = M \times \{1\}$, then the fixed point set of M' is diffeomorphic to $(M^G - F) \amalg F'$.*
4. *The restrictions of the group actions to $M - F$ and $M' - F'$ are equivariantly diffeomorphic.*

(5) *The equivariant smoothing of M' given by the associated G-homeomorphism $M' \to M$ determines a linearization of the equivariant tangent bundle of M which is equivalent to the usual one given by the smooth structure on the G-manifold M.*

More concisely, we can form a topologically equivalent smooth action in which the fixed point component F is replaced by F'.

Proof of Proposition 4.3. The ideas are fairly standard, so we shall only sketch the argument. Start with a closed collar neighborhood $F \times [0, \varepsilon]$ of $F \times \{0\} \cong F$ in the s-cobordism N. Let ν be the equivariant normal bundle of F in M, let $p : N \to F$ be a homotopy inverse to the inclusion $F \subset \partial N \subset N$, and let $D(p^*\nu)$ denote the disk bundle for the pullback of ν. Then the restriction of $p^*\nu$ to the closed collar neighborhood $F \times [0, \varepsilon] \subset N$ is a product $D(\nu) \times [0, \varepsilon]$. Form a smooth G-manifold from

$$W = M \times [0, \varepsilon] \ \cup_{D(\nu)\ \times [0,\varepsilon]} D(p^*\nu)$$

by rounding the corners at $S(\nu) \times \{\varepsilon\}$ and $S(\nu) \times \{1\}$ equivariantly. It follows that W is equivariantly homeomorphic to $M \times [0, 1]$, and the fixed point set of the induced action on the upper component M' of W is diffeomorphic to $(M^G - F) \amalg F'$. The smoothness of the action on W and the homeomorphism $W \cong M \times [0, 1]$ imply that the equivariant linearization of the tangent bundle to M' given by the G-homeomorphism $M' \to M$ corresponds to the usual linearization coming from the equivariant smooth structure on M.

Since $W \cong M \times [0, 1]$ topologically, it follows that $(W, M \times \{0\})$ is (nonequivariantly) an s-cobordism, and since $\dim M \geq 5$ the smooth s- cobordism Theorem implies that W is nonequivariantly diffeomorphic to $M \times [0, 1]$. Therefore M' must be nonequivariantly diffeomorphic to M. Finally, we need to show that $M - F$ and $M' - F'$ are equivariantly diffeomorphic. Let ν' be the pullback of ν with respect to the composite

$$F' \subset N \to F$$

where the second map is the homotopy inverse to $F \subset \partial N \subset N$. Then $M - F$ is equivariantly diffeomorphic to $(M - \mathrm{Int}D(\nu)) \cup_{S(\nu)} (D(\nu) - F)$ and $M' - F'$ is equivariantly diffeomorphic to $(M - \mathrm{Int}D(\nu)) \cup_{S(\nu)} S(p^*\nu) \cup_\partial D(\nu') - F'$ and since $D(\nu') - F'$ is an open collar neighborhood of $S(\nu')$ in $D(\nu')$ it follows that $M' - F'$ is equivariantly diffeomorphic to

$$\big(M - \mathrm{Int}D(\nu')\big) \cup S(p^*\nu) \cup \text{open collar}$$

and $M - F$ is equivariantly diffeomorphic to

$$\big(M - \mathrm{Int}D(\nu)\big) \cup S(\nu) \times [0,1) \cong M - D(\nu)$$

Note that $S(p^*\nu) -$ open collar is just $S(p^*\nu) - S(\nu')$. By construction $S(p^*\nu)/G$ is an s-cobordism, and therefore the Half Open h-cobordism Theorem implies that $S(p^*\nu)/G - S(\nu')/G$ is diffeomorphic to $S(\nu)/G \times [0,1)$ (see Hudson [Hud], Theorem 7.11, p. 171, for the piecewise linear case, and extend it to the smooth case using the methods and results of [HM]). Therefore it follows that $M' - F'$ is also equivariantly diffeomorphic to $M - D(\nu) \cong M - \mathrm{Int}D(\nu) \cup$ open collar, which means that $M - F$ and $M' - F'$ are equivariantly diffeomorphic. □

Proof of Theorem 4.2. Let V be given as in the statement and proof of Theorem 4.1, and as in the proof of that result, let m denote the order of the oriented bordism class $[L(V), h] \in \Omega^{\mathrm{SO}}_{n-1}(BG)$.

As noted in Theorem 3.6 in Subsection 7.3.2 of [FM4], if q is sufficiently large then the manifold

$$B(q) = \mathbb{CP}^2 \# q\, \overline{\mathbb{CP}^2}$$

has infinitely many smooth structures, and for a fixed value of q the signatures of these manifolds are all equal to $1-q$. Let q^* be the least positive integer such that q^* is sufficiently large and $q^* - 1 \equiv 0 \bmod m$, and let $F_{j,0} = B(q^* + jm)$, where j runs through the positive integers. Then for each j the cited theorem in [FM4] yields an infinite family of pairwise nondiffeomorphic manifolds $F_{j,k}$ which are homeomorphic to $F_{j,0}$. By construction $|\mathrm{sgn}\, F_{j(1),h(1)}| = |\mathrm{sgn}\, F_{j(2),h(2)}|$ if and only if $j(1) = j(2)$, and these signatures are all nonzero multiples of m.

We can now construct smooth actions of G on smooth manifolds M_j with fixed point sets $F_{j,0}$ as in the proof of Theorem 4.1; in the present setting, the actions are smoothable because we are given smooth structures on the manifold $F_{j,0}$. For each k, there is smooth s-cobordism from $F_{j,0}$ to $F_{j,k}$ by [Wall] (recall that h-cobordism and s-cobordism are equivalent in the simply connected case). Therefore, if we fix j, then Proposition 4.3 yields an infinite family of nondiffeomorphic smooth G-manifolds $M_{j,k}$ such that the fixed point sets are given by the nondiffeomorphic 4-manifolds $F_{j,k}$, the manifolds $M_{j,k}$ are nonequivariantly diffeomorphic to each other, and the remaining conditions in the conclusion of the theorem are satisfied. □

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