Framing 3-manifolds with bare hands

Framing 3-manifolds with bare hands

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Abstract. After surveying existing proofs that every closed, orientable 3-manifold is parallelizable, we give three proofs using minimal background. In particular, our proofs do not rely on spin structures, the theory of Stiefel–Whitney classes, nor the Lickorish–Wallace theorem.

1. Introduction

The aim of this note is to provide three proofs "with bare hands" of the following primary result in 3-dimensional differential topology, originally attributed to Stiefel [Sti] (1936):

Theorem 1.1. *Every orientable, closed 3-manifold is parallelizable.*

We realized by searching the literature that there are at least four modern proofs of the above result, collected in [Gei, FM] in a very clean way. Each of those proofs requires a somewhat robust mathematical background, so we asked ourselves whether there might be a proof which uses minimal background.[1] By asking the use of 'minimal background' we meant that such a proof should (i) satisfy the qualitative constraint of adopting a minimal toolbox (the simplest properties of cohomology and homotopy groups, the basic tools of differential topology and transversality theory such as given, e.g., in [Mil] or [GP] and

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[1] Essentially the same question was asked in the Mathematics StackExchange Forum, see `https://math.stackexchange.com/questions/1107682/ elementary-proof-of-the-fact-that-any-orientable-3-manifold-is-parallelizable`, but the answers given there until July 18, 2018 use the same tools employed in the proofs mentioned above.

a few well-known facts about vector bundles and their Euler classes) and (ii) be as self-contained as possible. Eventually we found three such proofs which, contrary to some of the proofs present nowadays in the literature, do not rely on spin structures, the theory of Stiefel–Whitney classes, nor the Lickorish–Wallace theorem.[2]

Throughout the paper, M denotes an orientable, closed (i.e., compact without boundary) smooth 3-manifold. It is not restrictive to assume that M is connected as well. Recall that a *combing* of M is a nowhere vanishing tangent vector field on M. Moreover, M is *parallelizable* if it admits a *framing*, that is a triple $\mathcal{F} = (w, z, v)$ of pointwise linearly independent combings. The existence of a framing is equivalent to the existence of a *trivialization*

$$\tau_{\mathcal{F}} : M \times \mathbb{R}^3 \to TM$$

of the tangent bundle of M. A framing incorporates an *orientation* of M and, *vice versa*, if M is oriented and parallelizable, then there are framings inducing the given orientation. We will always assume that M is *oriented*, with a fixed auxiliary orientation.

The paper is organized as follows. In Section 2 we briefly recall the four proofs collected in [Gei, FM] and we point out why they do not satisfy our minimal background requirements. In Section 3 we fix some notation and we recall a few well-known bare hands results. In each one of Sections 4, 5 and 6 we give a different bare hands proof of Theorem 1.1. The proof of Section 4 is purely 3-dimensional and could be regarded as a minimalistic version of the available modern proof based on Stiefel–Whitney classes. The proofs provided in Sections 5 and 6 could also be regarded as minimalistic versions of available modern proofs based on even surgery presentations and, respectively, 4-dimensional and purely 3-dimensional considerations. In particular, the proof of Section 6 could be viewed as a simplification of the available modern proof mainly based on spin structures.

2. Available modern proofs of Theorem 1.1

Each of the four modern proofs we survey in this section argues that M admits a *quasi-framing*, that is a framing $\mathcal{F}_0$ of a submanifold M_0 of M of the form

$$M_0 = M \setminus \mathrm{Int}(B),$$

where B is a smooth 3-disk embedded in M. The quasi-framing $\mathcal{F}_0$ can be extended with bare hands to a framing of the whole of M as follows. By the

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[2] Added in proof: After we put this note on arXiv, Durst, Geiges, Gonzalo Pérez and Kegel posted the preprint [DGGK] with two new proofs of Theorem 1.1, both relying on the Lickorish–Wallace theorem.

uniqueness of disks up to ambient isotopy, the choice of B is immaterial. Hence, we can assume that B is contained in a chart of M and looks standard therein. Upon fixing an auxiliary metric on M and a trivialization of TM over B, the restriction of $\mathcal{F}_0$ to $S^2 = \partial B$ is encoded by a smooth map

$$\rho : S^2 \to SO(3).$$

Since the universal covering space of $SO(3) \cong \mathbb{P}^3(\mathbb{R})$ is S^3, we have $\pi_2(SO(3)) = \pi_2(S^3) = 0$, therefore ρ can be extended over B and $\mathcal{F}_0$ to M.

2.1. The three proofs presented in [Gei].

We refer the reader to [Gei, § 4.2] for details. The first and third proofs presented in [Gei] use a certain mixture of the theory of Stiefel–Whitney classes and spin structures to establish the existence of a quasi-framing as follows. The first Stiefel–Whitney class $w_1(M)$ vanishes because M is orientable, and the key point in both proofs is to show that $w_2(M)$ vanishes as well. Using obstruction theory to define Stiefel–Whitney classes one can argue that $w_2(M) = 0$ implies the existence of a spin structure on M, and therefore that M admits a quasi-framing.[3] The first and third proofs differ in the way they establish the vanishing of $w_2(M)$.

The first proof, resting on several properties of Stiefel–Whitney classes, is perhaps the one requiring the most sophisticated background. The so-called *Wu classes* $v_i \in H^i(M; \mathbb{Z}/2\mathbb{Z})$ can be characterized by the property that, for every $x \in H^{3-i}(M; \mathbb{Z}/2\mathbb{Z})$,

$$\langle \mathrm{Sq}^i(x), [M] \rangle = \langle v_i \cup x, [M] \rangle,$$

where $[M]$ denotes the fundamental class of M in $H_3(M; \mathbb{Z}/2\mathbb{Z})$ and Sq^i is the i-th *Steenrod square* operation. It follows that $v_0 = 1$ and, for dimensional reasons, $v_i = 0$ if $i > 3 - i$. Hence, the only potentially nonzero Wu classes are v_0 and v_1. Moreover, Wu classes and Stiefel–Whitney classes are related through *Wu's formula*:

$$w_q(M) = \sum_{i+j=q} \mathrm{Sq}^i(v_j).$$

Since Sq^0 is the identity map and $\mathrm{Sq}^i(x) = 0$ when $i > \deg(x)$, by Wu's formula we have

$$0 = w_1(M) = \mathrm{Sq}^0(v_1) + \mathrm{Sq}^1(v_0) = v_1.$$

By Wu's formula again, the vanishing of v_1 implies $w_2(M) = 0$.

The third proof given in [Gei, § 4.2] goes as follows: First, one shows [Gei, Lemma 4.2.2] that if Σ is a closed, possibly non orientable surface embedded

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[3] According to Pierre de la Harpe [dlHar] these facts were first noticed by André Haefliger [Hae].

in M, then $w_2(E) = 0$ where $E = E(\Sigma)$ is a tubular neighborhood of Σ in M. The proof is elementary modulo the use of the basic *Whitney sum formula* for Stiefel–Whitney classes of vector bundles. The conclusion is entirely based on the theory of spin structures combined with some bare hands reasoning. It is a slight simplification of the proof proposed by R. Kirby in [Kir2]. The argument is by contradiction: if $w_2(M) \neq 0$ then its Poincaré dual in $H_1(M; \mathbb{Z}/2\mathbb{Z})$ is represented by a knot K embedded in M. Then, the assumption implies that:

(a) $M \setminus K$ carries a spin structure **s** which cannot be extended over any embedded 2-disk transverse to K;

(b) there is a compact, closed surface Σ embedded in M intersecting K transversely in a single point x_0.

By the general theory of spin structures, the vanishing of $w_2(E)$ implies that the set of spin structures on $E = E(\Sigma)$ is non-empty, and in fact it is an affine space on

$$H^1(E; \mathbb{Z}/2\mathbb{Z}) \cong H^1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \cong H^1(\Sigma \setminus \{x_0\}; \mathbb{Z}/2\mathbb{Z}).$$

It follows that the restriction of **s** to $E \setminus K$ extends to the whole of E, contradicting (b).

The second proof presented in [Gei, § 4.2] is less standard. It is based on the following non-trivial fact due to Hilden, Montesinos and Thickstun [HTM]: *There exists a branched covering map $\pi : M \to S^3$ such that the branching locus bounds an embedded 2-disk in M.* Using this fact, it is relatively easy to lift a framing of S^3, which can be constructed directly, to a quasi-framing of M. Although this proof is of a geometric-topological nature, clearly it does not use minimal background.

Remark. The three proofs described above are quite demanding from our "bare hands" point of view. The first proof, in particular, comes out of a relatively obscure algebraic machinery – we would have a hard time deducing from such a machinery a heuristic justification for the existence of framings on closed 3-manifolds.

2.2. The proof presented in [FM]. We refer the reader to [FM, §9] for details. The starting point is the *Lickorish–Wallace theorem* [Lic, Wa], stating that the 3-manifold M can be obtained by surgery along a framed link $L \subset S^3$. Equivalently, the statement says that M is the boundary of a 4-manifold W constructed by attaching 4-dimensional 2-handles to the 4-ball. Then, an argument essentially due to Kaplan [Kap] shows that by applying *Kirby moves* to L, it is not restrictive to assume that all the framings of L are even. By using this fact one shows that

the 4-manifold W is parallelizable, hence that $M = \partial W$ is *stably-parallelizable* and eventually admits a quasi framing.

Remark. The proof presented in [FM] satisfies to a large extent the first minimal background requirement from Section 1. In fact: the final portion of the argument, which will be recalled in Section 5, is "bare hands"; Rourke's proof [Rou] of the Lickorish–Wallace theorem is completely elementary and constructive provided one allows the use of *Smale's theorem* [Sma] so that, for example, one can take for granted that the operation of cutting and re-gluing a 3-ball does not change a smooth 3-manifold up to diffeomorphisms; although Kaplan's argument requires the introduction of Kirby calculus, it does *not* use the hard part of Kirby's theorem [Kir1] on the completeness of the calculus. Everything considered, we think that the proof presented in [FM] is not as self-contained as possible and therefore it does not satisfy the second minimal background requirement from Section 1.

3. Some notation and bare hands results

In this section we collect some notation and a few well-known facts that we allow in our minimal toolbox. Let N be a closed, connected manifold of dimension n, and let

$$\xi : B \to N$$

be a vector bundle of rank k, considered up to bundle isomorphisms. According to our bare hands constraints, in this generality the only allowable "characteristic" class of ξ is the Poincaré dual $\mathbf{w}(\xi)$ of the class in $H_{n-k}(N; \mathbb{Z}/2\mathbb{Z})$ carried by the transverse self-intersection of N viewed as the *zero section* of ξ inside B. Thus,

$$\mathbf{w}(\xi) \in H^k(N; \mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(N; \mathbb{Z}/2\mathbb{Z}).$$

The class $\mathbf{w}(\xi)$ actually coincides with the k-th Stiefel–Whitney class $w_k(\xi)$, but we shall not need this fact. Moreover, we will not make use of any other Stiefel–Whitney class. If both N and ξ are oriented, the same construction defines an integral class

$$\mathbf{e}(\xi) \in H^k(N; \mathbb{Z}),$$

sent to $\mathbf{w}(\xi)$ by the natural map $H^k(N; \mathbb{Z}) \to H^k(N; \mathbb{Z}/2\mathbb{Z})$. In both cases we talk about the *Euler class* of ξ, referring to either $\mathbf{w}(\xi)$ or $\mathbf{e}(\xi)$ depending on the context.

We will feel free to use the following facts, all of which are well-known facts about vector bundles and can be established using basic tools of differential topology and transversality theory.

- if $\xi = \xi_1 \oplus \xi_2$ is the Whitney sum of two vector bundles then $\mathbf{w}(\xi) = \mathbf{w}(\xi_1) \cup \mathbf{w}(\xi_2)$;

- a line bundle λ on N has a nowhere vanishing section if and only if $\mathbf{w}(\lambda) = 0$;

- If N is oriented, a rank-2 oriented vector bundle ξ on N has a nowhere vanishing section if and only if $\mathbf{e}(\xi) = 0$;[4]

- if $\xi = \lambda_1 \oplus \lambda_2$ is the Whitney sum of two line bundles and $\eta = \lambda_1 \otimes \lambda_2$ their tensor product, then

$$\mathbf{w}(\det \xi) = \mathbf{w}(\eta) = \mathbf{w}(\lambda_0) + \mathbf{w}(\lambda_1);$$

- $\mathbf{w}(TN) \in H^n(N; \mathbb{Z}/2\mathbb{Z})$ and

$$\langle \mathbf{w}(TN), [N] \rangle = \chi(N) \bmod (2) \in \mathbb{Z}/2\mathbb{Z};$$

- N is orientable if and only if $\mathbf{w}(\det TN) = 0$;

- If N is oriented then

$$\langle \mathbf{e}(TN), [N] \rangle = \chi(N) \in \mathbb{Z};$$

- let M be a closed, oriented 3-manifold and $\beta \in H^2(M; \mathbb{Z})$. Then, there is an oriented, connected, closed 1-submanifold $C \subset M$ which represents the Poincaré dual of β. If $\beta \in H^j(M; \mathbb{Z}/2\mathbb{Z})$ with $0 \leq j \leq 3$, there is a possibly non orientable, connected and closed $(3-j)$-submanifold of M which represents the Poincaré dual of β. Moreover, the cup product of two cohomology classes β_1 and β_2 can be represented by a transverse intersection of submanifolds representing the Poincaré duals of β_1 and β_2;

- any closed 3-manifold M carries a combing.[5]

In the next section we shall use the fact that, given an auxiliary Riemannian metric g on a closed 3-manifold M, by normalization any combing of M can be made of unitary norm, and by the Gram-Schmidt process any framing of M can be turned into a point-wise g-orthonormal framing. A unitary combing v on M determines an oriented distribution of tangent 2-planes

$$F_v = \{F_v(x)\}_{x \in M} \subset TM,$$

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[4] This item follows from the fact that $\mathbb{P}^\infty(\mathbb{C})$ is a $K(\mathbb{Z}, 2)$ space and therefore that real, rank-2 oriented bundles are classified by their Euler classes. Similarly, the previous item follows from the fact that $\mathbb{P}^\infty(\mathbb{R})$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ space.

[5] This fact follows from $\chi(M) = 0$ using the Poincaré-Hopf index theorem, clearly an allowable tool, together with the fact that maps $S^2 \to S^2$ are classified up to homotopy by their $\mathbb{Z}$-degree.

where $F_v(x) \subset T_x M$ is the subspace $g(x)$-orthogonal to $v(x)$. We assume that, for each $x \in M$, $v(x)$ followed by an oriented basis of $F_v(x)$ gives an oriented basis of $T_x M$. The restriction of the projection $TM \to M$ gives rise to an oriented rank-2 real vector bundle $F_v \to M$ whose isomorphism type is independent of the choice of g and depends on v only up to homotopy. We denote by

$$\mathbf{e}(F_v) \in H^2(M; \mathbb{Z})$$

the Euler class of F_v.

4. First bare hands proof of Theorem 1.1

In this section we provide the first bare hands proof of Theorem 1.1, resting neither on the theory of spin structures nor on properties of Stiefel–Whitney classes. Our tools consist of basic properties of cohomology groups, transversality theory, and the facts collected in Section 3. We will also use the notation introduced in Section 3.

The section is organized as follows. In Section 4.1 we give a bare hands proof of the following proposition.

Proposition 4.1. *M is parallelizable if and only if there is a combing v of M such that $\mathbf{w}(F_v) = 0$, in which case $\mathbf{w}(F_v) = 0$ for every combing v.*

Proposition 4.1 reduces the proof of Theorem 1.1 to showing that M carries a combing v such that $\mathbf{w}(F_v) = 0$. Observe that, since every class in $H_2(M; \mathbb{Z}/2\mathbb{Z})$ can be represented by an embedded surface, for each combing v on M the property $\mathbf{w}(F_v) = 0$ is equivalent to the fact that, for every closed, connected, embedded surface $\Sigma \subset M$, we have

$$(4.1) \qquad \langle \mathbf{w}(F_v), [\Sigma] \rangle = \langle \mathbf{w}(F_v|_\Sigma), [\Sigma] \rangle = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

We claim that Equation (4.1) is a consequence of the equation

$$(4.2) \qquad \mathbf{w}(F_v|_\Sigma) = \mathbf{w}(T\Sigma) + \mathbf{w}(\det T\Sigma) \cup \mathbf{w}(\nu_\Sigma),$$

where $\nu_\Sigma \subset TM$ denotes the normal line bundle of Σ. In fact, if Σ is orientable then $\mathbf{w}(\det T\Sigma) = 0$ and by (4.2) we have $\mathbf{w}(F_v|_\Sigma) = \mathbf{w}(T\Sigma)$. Therefore,

$$\langle \mathbf{w}(F_v|_\Sigma), [\Sigma] \rangle = \langle \mathbf{w}(T\Sigma), [\Sigma] \rangle = \chi(\Sigma) \bmod (2) = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

If Σ is non-orientable then Σ is homeomorphic to a connected sum $\#^h \mathbb{P}^2(\mathbb{R})$ of h copies of the projective plane. Observe that $\det TM|_\Sigma = \det(T\Sigma \oplus \nu_\Sigma) \cong \det T\Sigma \otimes \nu_\Sigma$, and since M is orientable $\det TM$ is trivial, therefore

$$0 = \mathbf{w}(\det TM|_\Sigma) = \mathbf{w}(\det T\Sigma \otimes \nu_\Sigma) = \mathbf{w}(T\Sigma) + \mathbf{w}(\nu_\Sigma),$$

which implies that the normal line bundle ν_Σ is isomorphic to the determinant line bundle $\det T\Sigma$. Since $\mathbb{P}^2(\mathbb{R})$ is non-orientable, the class

$$\alpha := \mathbf{w}\big(\det T\mathbb{P}^2(\mathbb{R})\big) \in H^1\big(\mathbb{P}^2(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}\big) \cong \mathbb{Z}/2\mathbb{Z}$$

is the only non-zero element. Thus, $\langle \alpha \cup \alpha, [\Sigma] \rangle = h$ and by Equation (4.2) we have

$$\big\langle \mathbf{w}(F_v|_\Sigma), [\Sigma] \big\rangle = \chi(\Sigma) \bmod (2) + \big\langle \alpha \cup \alpha, [\Sigma] \big\rangle = 2 - h + h \bmod (2) = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

Proposition 4.9 of Section 4.2 below contains a bare hands proof of (4.2), thus concluding our bare hands proof of Theorem 1.1.

Before embarking in the bare hands proofs of Proposition 4.1 and 4.9 it seems worth pointing out the existence of a short argument to prove $w_2(M) = 0$ without spin structures, yielding a simplification of the first and third proofs from [Gei, § 4.2]. More precisely, we prove Proposition 4.2 below using only the existence of Stiefel–Whitney classes and the basic Whitney sum formula.

Proposition 4.2. *Let M be a closed, oriented 3-manifold. Then, $w_2(M) = 0$.*

Proof. Let v be a combing on M and $\Sigma \subset M$ a closed, connected, embedded surface. Then, we have the Whitney sum decompositions

$$TM|_\Sigma = F_v|_\Sigma \oplus \epsilon = T\Sigma \oplus \nu_\Sigma,$$

where ϵ is the trivial line bundle generated by v. By the Whitney sum formula for Stiefel–Whitney classes, the first decomposition gives $\langle w_2(M), [\Sigma] \rangle = \langle w_2(F_v), [\Sigma] \rangle$, hence $w_2(M) = 0$ if and only if $w_2(F_v) = 0$. The second decomposition yields

$$w_2(F_v|_\Sigma) = w_2(\Sigma) + w_1(\Sigma) \cup w_1(\nu_\Sigma),$$

which is analogous to Equation (4.2). An argument similar to the one above showing (4.2) $\Rightarrow$ (4.1) gives $\langle w_2(F_v), [\Sigma] \rangle = 0 \in \mathbb{Z}/2\mathbb{Z}$, therefore we conclude $w_2(F_v) = 0$. $\square$

4.1. Combing and framing 3-manifolds.
Our purpose in this section is to achieve a bare hands proof of Proposition 4.1 above.

Lemma 4.3. *M is parallelizable if and only if $\mathbf{e}(F_v) = 0$ for some unitary combing v.*

Proof. Let v be a unitary combing of M such that $\mathbf{e}(F_v) = 0$. Any nowhere vanishing section of F_v can be normalized with respect to g to a unitary section w of F_v, extended to an oriented orthonormal framing (w, z) of F_v and finally to an oriented orthonormal framing (w, z, v) of M. Conversely, for any orthonormal framing (w, z, v) of M we may view v as a combing of M and w as a section of F_v. $\qquad\square$

The comparison class. We can associate to an ordered pair of unitary combings (v, v') of M a smooth section $v \times v'$ of F_v as follows. At a point $x \in M$ where $v(x) \neq \pm v'(x)$, $v \times v'(x) \in F_v(x) \subset T_x M$ is the "vector product" of $v(x)$ and $v'(x)$, i.e., the only tangent vector such that

- $\|v \times v'(x)\|^2_{g(x)} = 1 - g(v, v')^2$;
- $v \times v'(x)$ is $g(x)$-orthogonal to $v(x)$ and $v'(x)$;
- $(v(x), v'(x), v \times v'(x))$ is an oriented basis of $T_x M$.

At a point $x \in M$ where $v(x) = \pm v'(x)$, we set $v \times v'(x) = 0$.

If the two unitary combings v and v' are generic, the section $v \times v'$ of F_v is transverse to the zero section and the zero locus

$$C := \{x \in M \mid v \times v'(x) = 0\} \subset M$$

is a disjoint collection of simple closed curves. Moreover, $C = C_+ \cup C_-$, where

$$C_+ = \{x \in M \mid v(x) = v'(x)\} \quad \text{and} \quad C_- = \{x \in M \mid v(x) = -v'(x)\}.$$

By the very definition of $\mathbf{e}(F_v)$, C can be oriented to represent the Euler class of F_v. Indeed, let $E(F_v)$ denote the total space of F_v, $M_0 \subset E(F_v)$ the zero-section and $M_1 = v \times v'(M) \subset E(F_v)$. Under the natural identification of M with M_0 the submanifold C is identified with $M_0 \cap M_1$. Let $p_x : T_x E(F_v) \to F_v(x)$ be the natural projection and $N_x(C)$ the fiber of the normal bundle of TC inside $TM|_C$. Since p_x maps $(v \times v')_*\big(N_x(C)\big)$ isomorphically onto $F_v(x)$, the given orientation on $F_v(x)$ can be pulled-back to $N_x(C)$ and, together with the orientation of $T_x M$, it induces an orientation on $T_x C$ in a standard way.

Definitions. An ordered pair of unitary combings (v, v') of M such that $v \times v'$ is a section of F_v transverse to the zero section will be called a *generic pair* of unitary combings. We define the *comparison class* $\alpha(v, v') \in H^2(M; \mathbb{Z})$ of a generic pair of unitary combings as the Poincaré dual of the homology class $[C_-]$, where C_- is oriented as described above.

Lemma 4.4. *Let (v, v') be a generic pair of unitary combings of M. Then,*

$$\alpha(v, v') = -\alpha(v', v) \quad \text{and} \quad \alpha(v, -v') = \alpha(v', -v).$$

Proof. For each $x \in C$ the equality $F_v(x) = F_{v'}(x)$ holds, with the orientations of $F_v(x)$ and $F_{v'}(x)$ being the same or different according to, respectively, whether $x \in C_+$ or $x \in C_-$. We may choose a tubular neighborhood $U = U(C)$ such that the restrictions of the tangent plane fields $F_v|_U$ and $F_{v'}|_U$ are so close that there is a vector bundle isomorphism $\varphi : F_v|_U \xrightarrow{\cong} F_{v'}|_U$ which is the identity map on the intersections $F_v(x) \cap F_{v'}(x)$, $x \in U$, is orientation-preserving near $C_+ = \{x \in M \mid v(x) = v'(x)\}$ and orientation-reversing near $C_- = \{x \in M \mid v(x) = -v'(x)\}$. Since $\varphi \circ (v \times v') = v \times v' = -v' \times v$ and $-v' \times v$ is obtained by composing the section $v' \times v$ with the orientation-preserving automorphism of $F_{v'}$ given by minus the identity on each fiber, the orientation on C_- as part of the zero locus of $v \times v' : M \to F_v$ is the opposite of its orientation as part of the zero locus of $v' \times v = -v \times v' : M \to F_{v'}$. This implies $\alpha(v, v') = -\alpha(v', v)$. Similarly, the orientation on C_+ as part of the zero locus of $v \times (-v') : M \to F_v$ coincides with its orientation as part of the zero locus of $-(v \times (-v')) = v' \times (-v) : M \to F_{v'}$, which implies $\alpha(v, -v') = \alpha(v', -v)$. $\square$

Lemma 4.5. *Let (v, v') be a generic pair of unitary combings of M. Then,*

$$\mathbf{e}(F_v) - \mathbf{e}(F_{v'}) = 2\alpha(v, v').$$

Proof. According to the definitions we have

$$\mathbf{e}(F_v) = \alpha(v, v') + \alpha(v, -v') \quad \text{and} \quad \mathbf{e}(F_{v'}) = \alpha(v', v) + \alpha(v', -v).$$

Taking the difference of the two equations and applying Lemma 4.4 we obtain

$$e(F_v) - e(F_{v'}) = \alpha(v, v') + \alpha(v, -v') - \alpha(v', v) - \alpha(v', -v) = 2\alpha(v, v'). \quad \square$$

Pontryagin surgery. Let v be a unitary combing of M and $C \subset M$ an oriented, simple closed curve such that the positive, unit tangent field along C is equal to $v|_C$ and there is a trivialization

$$j : D^2 \times S^1 \xrightarrow{\cong} U(C)$$

of a tubular neighborhood of C in M such that

$$v \circ j = j_*(\partial/\partial\phi),$$

where ϕ is a periodic coordinate on the S^1-factor of $D^2 \times S^1$. Let (ρ, θ) be polar coordinates on the D^2-factor. Following terminology from [BP], we say that a unitary combing v' is obtained from v by *Pontryagin surgery* along C if, up to homotopy, v' coincides with v on $M \setminus U(C)$ and

$$v' \circ j = j_* \left(-\cos(\pi\rho)\frac{\partial}{\partial\phi} - \sin(\pi\rho)\frac{\partial}{\partial\rho} \right)$$

on $U(C)$.

Remark. A basic fact not used in this paper is that any two combings of M are obtained from each other, up to homotopy, by Pontryagin surgery [BP, Theorem 6.2.2].

Lemma 4.6. *Let v be a unitary combing of M and $\beta \in H^2(M; \mathbb{Z})$. Then, possibly after a homotopy of v, there is a unitary combing v' such that (v, v') is a generic pair of unitary combings and*

$$\alpha(v, v') = \beta.$$

Proof. Let $C \subset M$ be an oriented simple closed curve representing the Poincaré dual of β and let $j : D^2 \times S^1 \to U(C)$ be a trivialization of a neighborhood of C. Without loss of generality we may assume that the pull-back $j^*(g)$ of the auxiliary metric g on M is the standard product metric on $D^2 \times S^1$. After a suitable homotopy of v the assumptions to perform Pontryagin surgery on v along C are satisfied. Consider a normal disc $D_{\phi_0} = j(D^2 \times \{\phi_0\})$ and let $p = D_{\phi_0} \cap C$. Then, $T_p D_{\phi_0}$ coincides, as an oriented 2-plane, with $F_v(p)$ as well as with the $g(p)$-orthogonal subspace of $T_p C$ inside $T_p M$. Let v' be a unitary combing obtained from v by first performing a Pontryagin surgery on $U(C)$ and then applying a small generic perturbation supported on a small neighborhood of $M \setminus U(C)$. Then, (v, v') is a generic pair of unitary combings and $C = \{x \in M \mid v(x) = -v'(x)\}$. By the definition of $\alpha(v, v')$, to prove the statement it suffices to show that the given orientation of C coincides with its orientation as part of the zero set of $v \times v' : M \to F_v$. Near C we have

$$(v \times v') \circ j = j_* \left(-\sin(\pi\rho) \frac{\partial}{\partial\theta} \right) = j_* \left(\frac{\sin(\pi\rho)}{\rho} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right),$$

where $x = \rho \cos\theta$ and $y = \rho \sin\theta$ are rectangular coordinates on the D^2-factor and $\partial/\partial\theta$ is the unit vector field tangent to the circles $\{\rho = \text{const}\}$. Observe that j_* sends the pair $(\partial/\partial x, \partial/\partial y)$ to an oriented framing of F_v. Using the resulting trivialization of F_v we can write locally the restriction of $v \times v'$ to to the disc D_{ϕ_0} followed by projection onto F_v as follows:

$$v \times v'|_{D_{\phi_0}} : (x, y) \mapsto \frac{\sin(\pi\rho)}{\rho} (y, -x) = \pi(y, -x) \; + \; \text{higher order terms}.$$

It is easy to compute that $(v \times v')_* \circ j_*$ sends $\partial/\partial x$ to $-\pi \partial/\partial y$ and $\partial/\partial y$ to $\pi \partial/\partial x$, and since the matrix $\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$ has determinant $\pi^2 > 0$ this shows that the restriction of $(v \times v')_*$ to the normal bundle to C composed with the projection onto F_v is orientation-preserving along C, concluding the proof. $\square$

We shall say that the Euler class $\mathbf{e}(F_v)$ is *even* if there exists $\beta \in H^2(M; \mathbb{Z})$ such that $\mathbf{e}(F_v) = 2\beta$.

Lemma 4.7. *M is parallelizable if and only* $\mathbf{e}(F_v)$ *is even for every unitary combing* v.

Proof. If M is parallelizable, then M has a unitary framing (w, z, v) and the class $\mathbf{e}(F_v) = 0$ is obviously even. Let v' be an arbitrary unitary combing of M. After possibly small perturbations of v' and v which do not change $\mathbf{e}(F_{v'})$ nor $\mathbf{e}(F_v)$, the pair (v', v) becomes a generic pair of unitary combings and by Lemma 4.5 we have

$$\mathbf{e}(F_{v'}) = \mathbf{e}(F_{v'}) - \mathbf{e}(F_v) = 2\alpha(v', v).$$

Therefore, $e(F_{v'})$ is even as well. Conversely, suppose that v is a unitary framing with $\mathbf{e}(F_v) = 2\beta \in H^2(M; \mathbb{Z})$. By Lemma 4.6, possibly after a homotopy of v – which does not change $\mathbf{e}(F_v)$ – there is a unitary framing v' such that (v, v') is a generic pair and $\alpha(v, v') = \beta$. Hence, by Lemma 4.5 we have

$$\mathbf{e}(F_v) - \mathbf{e}(F_{v'}) = 2\alpha(v, v') = 2\beta,$$

which implies $\mathbf{e}(F_{v'}) = 0$, therefore M is parallelizable by Lemma 4.3.   □

Lemma 4.8. *Let v be a unitary combing of M. Then, $\mathbf{e}(F_v)$ is even if and only if $\mathbf{w}(F_v) = 0$.*

Proof. The implication $\mathbf{e}(F_v) = 0 \Rightarrow \mathbf{w}(F_v) = 0$ is trivial. We give two arguments for the other implication. The first argument uses a little bit of homological algebra. The short exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

induces a long exact sequence in cohomology including the segment

$$\cdots \to H^2(M; \mathbb{Z}) \xrightarrow{2\cdot} H^2(M; \mathbb{Z}) \xrightarrow{\varphi} H^2(M; \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

where the map φ is reduction mod 2. Exactness yields the statement.

The second argument is more geometric. The Poincaré dual of $\mathbf{e}(F_v)$ can be represented by an oriented knot $K \subset M$. If $\mathbf{w}(F_v) = 0$ then K bounds an embedded surface $\Sigma \subset M$. If Σ is orientable then $[K] = 0$, hence $\mathbf{e}(F_v) = 0$, which is obviously even. If Σ is non-orientable then there is a collection C of simple closed curves in the interior of Σ such that $\Sigma \smallsetminus C$ is orientable and a tubular neighborhood U of C in Σ is a union of Möbius bands. Orient $\Sigma \smallsetminus \overset{\circ}{U}$ so that K is an oriented boundary component and give ∂U the resulting boundary orientation. Orient the cores of U so that the natural projection $\partial U \to C$ has degree $+2$ on each component. Then, $[K] = [\partial U] = 2[C]$, therefore $\mathbf{e}(F_v)$ is even.   □

Proof of Proposition 4.1. The statement is an immediate consequence of Lemmas 4.7 and 4.8.   □

4.2. Proof of Equation (4.2). The purpose of this section is to give a bare hands proof of Proposition 4.9 below, which establishes Equation (4.2). As explained at the beginning of the present section, this concludes our bare hands proof of Theorem 1.1.

Let v be a unitary combing of M and $\Sigma \subset M$ a closed, embedded surface. At each point $x \in \Sigma$ we have the splittings

$$(4.3) \qquad T_x M = F_v(x) \oplus \epsilon(x) = T_x \Sigma \oplus \nu_\Sigma(x),$$

where $\epsilon(x)$ is the (oriented) line spanned by $v(x)$, while $\nu_\Sigma(x)$ is the (unoriented) line orthogonal to $T_x \Sigma$.

Proposition 4.9. *Let v be a unitary combing of M and $\Sigma \subset M$ a closed, embedded surface. Then,*

$$(4.4) \qquad \mathbf{w}(F_v|_\Sigma) = \mathbf{w}(T\Sigma) + \mathbf{w}(\det T\Sigma) \cup \mathbf{w}(\nu_\Sigma).$$

Proof. Let $s : \Sigma \to F_v|_\Sigma$ be a generic section of the restriction F_v to Σ. For each $x \in \Sigma$, the second splitting from (4.3) induces decompositions

$$s(x) = s_\Sigma(x) + s_\nu(x), \quad v(x) = v_\Sigma(x) + v_\nu(x).$$

By transversality we may assume that:

(i) the zero set $\{s = 0\} \subset \Sigma$ consists of a finite number of points representing $\mathbf{w}(F_v|_\Sigma)$;

(ii) s_ν and v_ν are generic sections of ν_Σ, so that both their zero sets $\{s_\nu = 0\}$ and $\{v_\nu = 0\}$ consist of smooth curves in Σ representing $\mathbf{w}(\nu_\Sigma)$. Moreover, $\{s_\nu = 0\}$ and $\{v_\nu = 0\}$ intersect transversely in Σ, so that the finite set $\{v_\nu = 0\} \cap \{s_\nu = 0\}$ represents $\mathbf{w}(\nu) \cup \mathbf{w}(\nu) = \mathbf{w}(\det T\Sigma) \cup \mathbf{w}(\nu)$;

(iii) $\{s = 0\}$ and $\{v_\nu = 0\}$ are disjoint subsets of Σ;

(iv) s_Σ is a generic section of $T\Sigma$, so that $\{s_\Sigma = 0\}$ consists of a finite number of points representing $\mathbf{w}(T\Sigma)$.

Given a finite set X, denote by $|X|_2 \in \mathbb{Z}/2\mathbb{Z}$ the cardinality of X modulo 2. Then, we have

$$\left\langle \mathbf{w}(F_v|_\Sigma), [\Sigma] \right\rangle = \left| \{s = 0\} \right|_2, \quad \left\langle \mathbf{w}(T\Sigma), [\Sigma] \right\rangle = \left| \{s_\Sigma = 0\} \right|_2,$$
$$\left\langle \mathbf{w}(\det T\Sigma) \cup \mathbf{w}(\nu_\Sigma), [\Sigma] \right\rangle = \left| \{v_\nu = 0\} \cap \{s_\nu = 0\} \right|_2.$$

Therefore Equation (4.4) is equivalent to the following equality:

$$(4.5) \qquad \left| \{s = 0\} \right|_2 = \left| \{s_\Sigma = 0\} \right|_2 + \left| \{v_\nu = 0\} \cap \{s_\nu = 0\} \right|_2.$$

The finite set $\{s_\Sigma = 0\}$ can be tautologically decomposed as a disjoint union:

$$\{s_\Sigma = 0\} = \big(\{v_\nu = 0\} \cap \{s_\Sigma = 0\}\big) \sqcup \big(\{v_\nu \neq 0\} \cap \{s_\Sigma = 0\}\big).$$

We claim that

$$\{v_\nu \neq 0\} \cap \{s_\Sigma = 0\} = \{s = 0\}.$$

In fact, by Assumption (iii) above we have

$$\{s = 0\} = \{v_\nu \neq 0\} \cap \{s = 0\},$$

and clearly

$$\{v_\nu \neq 0\} \cap \{s = 0\} \subset \{v_\nu \neq 0\} \cap \{s_\Sigma = 0\}.$$

On the other hand, if $x \in \{v_\nu \neq 0\} \cap \{s_\Sigma = 0\}$ then $s(x) = 0$ because, since $v_\nu(x) \neq 0$, the projection $F_v(x) \to T_x\Sigma$ is an isomorphism. Thus, the claim is proved. In order to establish Equality (4.5) it is now enough to check that

(4.6) $$\big|\{v_\nu = 0\} \cap \{s_\Sigma = 0\}\big|_2 = \big|\{v_\nu = 0\} \cap \{s_\nu = 0\}\big|_2.$$

Let C be the collection of smooth curves $\{v_\nu = 0\} \subset \Sigma$. At each $x \in C$ we have a splitting

$$F_v(x) = \big(F_v(x) \cap T_x\Sigma\big) \oplus \nu_\Sigma(x),$$

therefore the restriction $F_v|_C$ splits as a sum of line bundles

$$F_v|_C = \lambda \oplus \nu_\Sigma|_C,$$

where $\lambda = \{F_v(x) \cap T_x\Sigma\}_{x \in C}$. We claim that the line bundles λ and $\nu_\Sigma|_C$ are isomorphic. In fact, along each component of C the bundle F_v is trivial because it is oriented, so the two line bundles are either both trivial or both non-trivial. Thus, $\langle \mathbf{w}(\lambda), [C] \rangle = \langle \mathbf{w}(\nu_\Sigma|_C), [C] \rangle$, and Equality (4.6) follows from the observation that the restriction of s_F and s_ν to C are generic sections of, respectively, λ and $\nu_\Sigma|_C$. $\qquad\square$

5. Second bare hands proof of Theorem 1.1

The aim of this section is to provide a genuine proof of Theorem 1.1 using minimal background, employing some of the ideas we summarized in Section 2.2. Let us first outline an elementary proof of the last portion of the proof presented in [FM].

Lemma 5.1. *Let* $N = \chi(S^3, L)$ *be a 3-manifold obtained by surgery along a framed link* $L \subset S^3$ *such that all framings are* even. *Let* W *be the corresponding 4-manifold obtained by attaching 4-dimensional 2-handles to the 4-ball, so that* $N = \partial W$. *Then,* W *is parallelizable.*

Proof. We refer the reader to [FM] for further details. For simplicity, assume that L is a one-component link with even framing n. As we can assume that the attaching tubes of the 2-handles are pairwise disjoint, this is not really restrictive. Let $N(L) \subset \partial D^4$ be the attaching tube of the corresponding 2-handle attached to D^4. Both D^4 and $D^2 \times D^2$ are parallelizable, so we have to show that they carry some framings which match on $N(L)$. Fix a reference framing $\mathcal{F}_0$ on TD^4. Then, the restriction to $N(L)$ of any framing $\mathcal{F}$ on the 2-handle is encoded by a map $\rho : N(L) \to SO(4)$. Viewing S^3 as the group of unit quaternions one can construct a 2-fold covering map $S^3 \times S^2 \to SO(4)$ showing that $\pi_1(SO(4)) = \mathbb{Z}/2\mathbb{Z}$. As the solid torus $N(L)$ retracts onto $L \cong S^1$, ρ determines an element $\overline{\rho} \in \mathbb{Z}/2\mathbb{Z}$ which vanishes if and only if the two framings coincide on $N(L)$. It is easy to see that $\overline{\rho}$ is equal to $n \bmod 2$. $\square$

Corollary 5.2. *If a 4-manifold W is parallelizable, then ∂W is **stably-parallelizable**. In fact, the Whitney sum of the tangent bundle $T\partial W$ with a trivial line bundle ϵ is a product bundle.*

Proof. By the existence of a collar of ∂W in W it is immediate that $T\partial W \oplus \epsilon \cong TW|_{\partial W}$. $\square$

Lemma 5.3. *If a closed, connected, orientable 3-manifold N is stably-parallizable, then it admits a quasi framing, hence it is parallelizable.*

Proof. We reproduce the short bare hands argument of [KM, Lemma 3.4]. With the usual notation, let $N_0 = N \setminus \mathrm{Int}(B)$. Since TN_0 is oriented, a bundle isomorphism $TN_0 \oplus \epsilon \cong \epsilon^4$ gives rise to a map from N_0 to the Grassmannian $Gr(3,4)$ of oriented 3-planes in $\mathbb{R}^4$. Since $Gr(3,4) \cong S^3$ and N_0 has a 2-dimensional spine, by transversality any such map is not surjective up to homotopy, hence it is homotopically trivial, therefore TN_0 is trivial. $\square$

The following lemma is trivial.

Lemma 5.4. *Let M and M' be closed, connected, oriented 3-manifolds. If $M\#M'$ is parallelizable, then both M and M' admit a quasi framing, hence they are parallelizable.*

Proof. Let $N = M\#M'$. Obviously M_0 embeds into N and TM_0 is the restriction of TN to M_0. The same holds for M'. $\square$

Combining Corollary 5.2 with Lemmas 5.3 and 5.4, to complete our second bare hands proof of Theorem 1.1 we are reduced to providing a proof using minimal background of the following proposition.

Proposition 5.5. *For every connected, closed, oriented 3-manifold M, there exists another such 3-manifold M' such that $N = M\#M'$ is of the form $N = \chi(S^3, L)$ for some framed link $L \subset S^3$ such that all framings are even.*

Proof. We use some basic facts about Heegaard splittings of 3-manifolds. Let us start with any Heegaard splitting of M of some genus g. Up to diffeomorphisms, M_0 can be realized as follows. Given a handlebody $\mathfrak{H}_g$ of genus g, the orientable surface $\Sigma_g = \partial \mathfrak{H}_g$ contains a non separating system $C = \{c_1, \ldots, c_g\}$ of g pairwise disjoint smooth circles. A tubular neighbourhood $N(C)$ in Σ_g is formed by a system of pairwise disjoint attaching tubes for 3-dimensional 2-handles, which, when attached to $\mathfrak{H}_g$ give 3-manifold M_0. The closed 3-manifold M is obtained by attaching a further final 3-handle. The union of the above 2- and 3-handles gives the second handlebody $\mathfrak{H}'_g$ of the Heegaard splitting, glued to $\mathfrak{H}_g$ along the common boundary Σ_g. Fix any standard embedding of $\mathfrak{H}_g$ into S^3, so that the closure of $S^3 \setminus \mathfrak{H}_g$ is a handlebody as well. This embedding realizes a genus-g Heegaard splitting of S^3. The collection of curves $C \subset \partial \mathfrak{H}_g \subset S^3$ becomes a link L in S^3, with each component of L framed by a parallel curve in $\partial \mathfrak{H}_g$. Now we can apply the key basic Lemma 1 of [Rou], which has a bare hands proof. In our situation, the lemma implies that

$$\chi(S^3, L) = M\#M'$$

for some 3-manifold M'. It is an immediate consequence of the above description of M_0 and of the definition of surgery along a framed link that $\chi(S^3, L)$ is obtained by gluing M_0 and M'_0 along their spherical boundaries. Applying Smale's theorem [Sma] we can conclude that $\chi(S^3, L) = M\#M'$.

Now fix a complete system $\mathcal{M} = \{m_1, \ldots, m_g\}$ of meridians of $\mathfrak{H}_g$. The curves m_i bound a system of disjoint 2-disks properly embedded into $(\mathfrak{H}_g, \Sigma_g)$. Denote by τ_i the Dehn twist on Σ_g along m_i. Since every τ_i extends to a diffeomorphism $\bar{\tau}_i$ of the whole $\mathfrak{H}_g$, we can modify a given embedding of $\mathfrak{H}_g$ into S^3 by applying any finite sequence of such $\bar{\tau}_i$'s. So we are reduced to show that in this way we can obtain an embedding such that the framing of each c_i determined as above by the embedding in Σ_g is even. This is the content of [BP, Lemma 8.4.1] (proved therein to have a treatment with bare hands of Kaplan's result for the double $D(M) = -M\#M$ of M). The proof of [BP, Lemma 8.4.1] boils down to solving a certain $\mathbb{Z}/2\mathbb{Z}$-linear system.   □

Remarks. (1) In the proof of Proposition 5.5 we refer to Smale's theorem [Sma]. However, that is not really necessary. In fact, the description of $\chi(S^3, L)$ as obtained by gluing together M_0 and M'_0 along their spherical boundaries suffices. Thus, *Smale's theorem can be discarded from the background of our second bare hands proof of Theorem* 1.1.

(2) By [Wa], the Lickorish–Wallace theorem is bare hands equivalent to "$\Omega_3 = 0$". The proof via Rourke's construction is the simplest one modulo Smale's theorem if the construction is performed within the category of smooth 3-manifolds. Alternatively, one could use the so-called Alexander trick and rely on the equivalence between smooth and PL 3-dimensional categories, which is indeed a more demanding fact. A more basic proof that $\Omega_3 = 0$ could probably be concocted by combining a bare hands proof that M is parallelizable with a specialization of the elementary proof of a theorem by Thom given in [BH].

6. Third bare hands proof of Theorem 1.1

We shall make use of Lemma 6.1 below, which could be viewed as a 'ground zero' fact about spin structures. Let N be an oriented 3-manifold, $K \subset N$ an oriented knot and $n : K \to TN|_K$ a unitary normal vector field along K. The orientation of K determines the unitary tangent vector field $t : K \to TN|_K$ and an orthonormal oriented framing $\mathcal{F}_n = (t, n, b)$ of $TN|_K$. Let $F \subset N$ be a smoothly embedded, oriented surface with $\partial F = K$. Since F retracts onto a one-dimensional CW-complex, TF is trivial. Let (a, b) be any oriented, orthonormal framing of TF and (a, b, c) the orthonormal framing of $TN|_F$ obtained by adding the oriented unit normal vector field c to F. From now on, we shall implicitly use the framing (a, b, c) to identify, at any point of F, the set of orthonormal framings of TN with $SO(3)$ and the set of unit vectors of TN with S^2. Define the map $\varphi_n : K \to S^1$ by $\varphi_n(x) = e^{i\theta(x)}$, where $\theta(x)$ is the counterclockwise angle between $c(x)$ and $n(x)$ measured in the oriented normal plane to K at x.

Lemma 6.1. *The framing $\mathcal{F}_n$ of $TN|_K$ extends to a framing of $TN|_F$ if and only if $\deg(\varphi_n)$ is odd.*

Proof. Let $\psi_n : K \to SO(3)$ be the map given by $\psi_n(x) = \mathcal{F}_n(x)$. Clearly $\mathcal{F}_n$ extends to a framing of $TN|_F$ if and only if ψ_n extends to a map $F \to SO(3)$, which happens if and only if the image of $[K] \in H_1(K; \mathbb{Z}/2\mathbb{Z})$ under the induced map $(\psi_n)_* : H_1(K; \mathbb{Z}/2\mathbb{Z}) \to H_1(SO(3); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is trivial. Consider the S^1-fibrations $\pi_1, \pi_2 : SO(3) \to S^2$ given by $\pi_1(a', b', c') = a'$ and $\pi_2(a', b', c') = b'$ and homotope $\mathcal{F}_n$ until there are two disjoint intervals $A, B \subset K$ such that $n = c$ on $K \setminus A$ and $t = a$ on $K \setminus B$. Then, setting $C := \pi_1^{-1}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ and $C' := \pi_2^{-1}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$, it is easy to check that

$$(\psi_n)_*\big([K]\big) = \chi(F)[C] + \deg(\varphi_n)[C'] \in H_1\big(SO(3); \mathbb{Z}/2\mathbb{Z}\big).$$

Since $[C] = [C']$ is a generator of $H_1(SO(3); \mathbb{Z}/2\mathbb{Z})$, we deduce that ψ_n extends to F if and only if $\chi(F) + \deg(\varphi_n)$ is even. But $\chi(F)$ is always odd, therefore the statement holds.   □

Fix a Heegaard splitting $M = \mathfrak{H}_g \cup \mathfrak{H}'_g$ and let $C = \{c_1, \ldots, c_g\} \subset \partial\mathfrak{H}_g = \partial\mathfrak{H}'_g$ be a complete system of meridians for $\mathfrak{H}'_g$. Consider a standard embedding of $\mathfrak{H}_g$ in $\mathbb{R}^3$ and unit vector field n_i along the curves $c_i \subset \partial\mathfrak{H}_g$, normal to $\partial\mathfrak{H}_g$ and pointing towards $\mathfrak{H}_g$. As in the proof of Proposition 5.5, using [BP, Lemma 8.4.1] we can choose the embedding so that each n_i defines an even framing of c_i with respect to the Seifert framing in $\mathbb{R}^3$. Note that, by Lemma 6.1, this is equivalent to saying that the induced framing $\mathcal{F}_{n_i}$ of $T\mathbb{R}^3|_{c_i}$ does not extend to a framing of $T\mathbb{R}^3$ over a Seifert surface. The vector fields n_i coincide with the unit normal vector fields determined by collars of each curve c_i in the corresponding 2-disk D_i properly embedded into $(\mathfrak{H}'_g, \partial\mathfrak{H}'_g)$. Let $B_i \subset M$ be a 3-disk containing D_i. By Lemma 6.1 the framings $\mathcal{F}_{n_i}$, regarded as framings of $TB_i|_{c_i}$, do not extend to framings of $TB_i|_{D_i}$. On the other hand, the restriction of the standard framing $\mathcal{F}$ of $\mathbb{R}^3$ to each c_i is homotopic to a framing $\mathcal{F}_{m_i}$ determined by a unit vector field m_i normal to c_i and defining an odd framing with respect to the Seifert framing. Again by Lemma 6.1, this means that $\mathcal{F}$ can be extended along each D_i, yielding a quasi-framing of M. This concludes the third bare hands proof of Theorem 1.1.

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