The Magnetic Exchange Moments for H3 and He3

The Magnetic Exchange Moments for H_3 and He_3

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Zusammenfassung. Es wird gezeigt, dass im Grundzustand des 3-Nukleon-Systemes der Ladungsaustausch einen Beitrag zum magnetischen Moment gibt. Die Rechnung wird auf Grund der symmetrischen Pseudoskalartheorie durchgeführt. Eine rohe Abschätzung ergibt für den Betrag des magnetischen Austauschmomentes $\sim {}^1/_3$ Kernmagneton, ferner positives Vorzeichen für H_3, negatives für He_3. Dieser Befund erlaubt eine Interpretation der Messung von μ_{H_3} durch Bloch und Anderson.

Summary. Charge exchange is shown to give a contribution to the magnetic moment of the three-body system in its ground state. A calculation, carried through for the case of the symmetrical pseudoscalar meson theory gives a value of $\sim {}^1/_3$ of a nuclear magneton for this exchange moment. The sign is positive for H_3, negative for He_3. This result furnishes an interpretation of the value of μ_{H_3} measured by Bloch and Anderson.

§ 1. Introduction.

Recently, the ratio of the magnetic moments of the H_3-nucleus and the proton has been measured[1]) by the nuclear induction method, with the result that

$$\mu_{H_3} = 1.0666\ \mu_P,$$

or, taking a value of 2.789 n. m. for the proton moment:

$$\mu_{H_3} = \mu_P + 0.186 \text{ n.m.}.$$

A value $\mu_{H_3} = \mu_P$ was to be expected if the H_3-ground state were a pure ^2S-state. This is not quite true, since the spin-orbit coupling (tensor force) gives rise to small admixtures of higher states (^2P, ^4P, ^4D). But these admixtures can hardly be made responsible for the excess $\mu_{H_3} - \mu_P$, on the contrary, as was shown by Sachs and Schwinger[2]), they even reduce the magnetic moment*).

*) Recently R. G. Sachs[3]) has discussed the conditions under which the admixtures would be able to give the correction required by the experiment. This correction can only be obtained with the help of the $^2P - {}^4P$ interference term, but this requires that the ^2P and ^4P admixtures are relatively strong (20% and 8%, respectively). However, such an assumption can hardly be justified.

However, SCHWINGER's phenomenological formalism[4]) is not fitted to the description of the charge exchange phenomena connected with the interactions of nucleons. On the other hand, these phenomena are known to give rise to exchange moments[5]). In the case of the Deuteron, the magnetic exchange moment vanishes on account of the symmetry properties of the Deuteron ground-state. It will be shown that this is not the case for the ground state of the three body system.

The following investigation will be carried through for the case of the symmetrical pseudoscalar meson theory which, apart from the fundamental r^{-3} difficulty in the tensor force potential, is known to give the best qualitative agreement with experiment.[6]) Below, we shall presuppose that the influence of the spin-orbit coupling is small and does not mix up the ordering of the energy levels as given by the central force approximation. It may, however, affect to a certain degree the expectation value of the exchange moment, since the diagonal element $({}^2S\,|M_{\text{exch}}|\,{}^2S)$ will prove to be rather small; but we think that at present our still incomplete knowledge of the H_3 and He_3 ground-state eigenfunction does not justify a more detailed investigation. The result is therefore merely supposed to give the sign and the order of magnitude of the effect to be expected.

§ 2. The Magnetic Exchange Moment in the Pseudoscalar Theory.

We start with the well known Hamiltonian of the symmetrical pseudoscalar theory[7]):

$$H = \frac{1}{2}\sum_{\alpha=1}^{3}\int d^3x\,\{\pi_\alpha^2(x) + |\text{grad}\,\varphi_\alpha(x)|^2 + \mu^2\varphi_\alpha^2(x)\} \\ + \sqrt{4\pi}\,f\sum_{A,\alpha}\tau_\alpha^A\,(\sigma^A\cdot\text{grad}\,\varphi_\alpha(z^A)). \tag{1}$$

z^A is the position of the nucleon A, φ_1 and φ_2 are the charged fields and the connection with the fields φ and φ^* of the charged theory is given by

$$\varphi = 1/\sqrt{2}\cdot(\varphi_1 - i\,\varphi_2) \qquad \varphi^* = 1/\sqrt{2}\cdot(\varphi_1 + i\,\varphi_2)\,.$$

The charge and current densities due to the fields $\varphi_1\,\varphi_2$ are:

$$\varrho_M = e\cdot(\dot\varphi_2\,\varphi_1 - \dot\varphi_1\,\varphi_2)$$
$$s_M = e\cdot\{(\varphi_2\,\text{grad}\,\varphi_1 - \varphi_1\,\text{grad}\,\varphi_2) - \sqrt{4\pi}\,f\sum_A\sigma^A\delta(x-z^A)\cdot(\varphi_1\tau_2^A - \varphi_2\tau_1^A).$$

In order to satisfy the continuity equation

$$\dot\varrho + \text{div}\,s = 0 \tag{2}$$

we must add to ϱ_M and s_M the corresponding expressions due to the charge of the nucleons:

$$\begin{aligned} \varrho_N &= e \sum_A \delta(x - z^A) \frac{1+\tau_3^A}{2} \\ s_N &= e \sum_A \delta(x - z^A) \cdot \dot{z}^A \frac{1+\tau_3^A}{2}. \end{aligned} \tag{3}$$

Equation (2) is then easily verified with the help of the relation

$$\dot{\tau}_3^A = i[H, \tau_3^A] = 2\sqrt{4\pi}\, f \cdot \sigma^A \{\text{grad}\, \varphi_1(z^A)\, \tau_2^A - \text{grad}\, \varphi_2(z^A)\, \tau_1^A\}. \tag{4}$$

The magnetic moment operator is given by

$$M = \frac{1}{2} \int dv \cdot (x \times s) = \frac{1}{2} \int dv \cdot (x \times s_M + s_N) = M_M + M_N. \tag{5}$$

This expression is of course not translationly invariant; it can, however, be split into an invariant part (viz. a function of the relative position of the nucleons alone) and a part due to the motion of the center of mass of the system; this latter part is of no interest to us. The evaluation of M_M is most easily accomplished in the momentum space, where the Hamiltonian (1) reads:

$$\begin{aligned} H &= \frac{1}{2} \sum_\alpha \int d^3k \{p_\alpha(k) \cdot p_\alpha(-k) + k_0^2\, q_\alpha(k) \cdot q_\alpha(-k)\} \\ &+ \frac{if}{\pi\sqrt{2}} \int d^3k \sum_{A,\alpha} \tau_\alpha^A (\sigma^A \cdot k) \cdot e^{ikz^A} q_\alpha(k); \quad k_0^2 = \mu^2 + k^2. \end{aligned} \tag{6}$$

A canonical transformation is then performed:

$$H' = e^{if \cdot W} H e^{-if \cdot W} = H + if[W, H] - \frac{1}{2} f^2 [W[W, H]] + \cdots \tag{7}$$

which, in a first approximation (up to terms in f^2) gives a separation of H into a free-meson part and an interaction energy between the nucleons. The transformation function W is

$$W = \frac{i}{\pi\sqrt{2}} \sum_{A,\alpha} \int d^3k \frac{p_\alpha(k)}{k_0^2} e^{-ikz^A} \tau_\alpha^A (\sigma^A \cdot k). \tag{8}$$

The transformation (7), (8) will likewise be applied to M_M, which is of the form $M_M = M^{(0)} + f.\, M^{(1)}$. Arranging M'_M according to powers of f, we have

$$\begin{aligned} M'_M &= M^{(0)} + f\{i[W, M^{(0)}] + M^{(1)}\} + f^2 \left\{\frac{-1}{2} [W[W, M^{(0)}]] + i[W, M^{(1)}]\right\} \\ &= M'^{(0)} + f \cdot M'^{(1)} + f^2 \cdot M'^{(2)}. \end{aligned} \tag{9}$$

$M'^{(0)}$ is the contribution of the free mesons and will therefore be neglected; the expectation value of $M'^{(1)}$ is zero since this expression is linear in the field variables; $M'^{(2)}$ is a function of the nucleon coordinates alone and represents the so-called exchange moment. Its evaluation is straightforward and yields

$$M_{\text{exch}} = -\frac{e\,(f\mu)^2}{2} \sum_{A<B} (\tau^A \times \tau^B)_3 \left\{ \frac{1}{\mu} \left[\frac{z^{AB}\,(z^{AB},\, \sigma^A \times \sigma^B)}{r_{AB}^2} \left(1 + \frac{1}{\mu r_{AB}}\right) - (\sigma^A \times \sigma^B) \right] e^{-\mu r_{AB}} + 1/2\,(z^A + z^B \times z^{AB}) \cdot V(AB) \right\},$$

where $z^{AB} = z^A - z^B$, $r_{AB} = |z^{AB}|$ and $V(AB)$ is the interaction energy of the pseudoscalar theory

$$V(AB) = \frac{1}{3}(\sigma^A \cdot \sigma^B) + \left(\frac{1}{3} + \frac{1}{\mu r_{AB}} + \frac{1}{(\mu r_{AB})^2}\right) \cdot T_{AB} \frac{e^{-\mu r_{AB}}}{r_{AB}},$$

$$T_{AB} = 3\,\frac{(\sigma^A \cdot z^{AB})\,(\sigma^B \cdot z^{AB})}{r_{AB}^2} - (\sigma^A \cdot \sigma^B).$$

With the help of the expression for the total current S_M:

$$S_M = -e(f\mu)^2 \sum_{A<B} (\tau^A \times \tau^B)_3\, z^{AB} \cdot V(AB) = e \sum_A z^A \cdot \frac{\dot{\tau}_3^A}{2}$$

and with the notation $z^A = Z + \zeta^A$, (Z = center of mass of the system) we can split from M_M the part

$$\frac{1}{2}(Z \times S_M) = \frac{e}{2} \sum_A (Z \times \zeta^A)\,\frac{\dot{\tau}_3^A}{2}. \qquad (10)^*)$$

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*) It may be noted that the expectation value of (10) is cancelled by a contribution from the orbital part of the magnetic moment

$$M_{\text{orb}} = \frac{e}{2} \sum_A (z^A \times \dot{z}^A)\,\frac{1+\tau_3^A}{2}$$

$$= \frac{e}{2}\left(Z \times \sum_A \dot{\zeta}^A\,\frac{1+\tau_3^A}{2}\right) + \frac{e}{2}\sum_A (\zeta^A \times \dot{\zeta}^A)\,\frac{1+\tau_3^A}{2} + \frac{e}{2}\left(\sum_A \frac{1+\tau^A}{2}\, z^A \times \dot{Z}\right).$$

The first term gives, together with $\tfrac{1}{2}(Z \times S_M)$, the expression

$$\frac{e}{2}\left(Z \times \frac{d}{dt}\left(\sum_A \zeta^A\,\frac{1+\tau_3^A}{2}\right)\right),$$

whose expectation value vanishes; the second term represents the intrinsic orbital moment and the third the moment due to the motion of the center of mass.

The remainder depends only on the relative positions of the nucleons and it is this part which will be called exchange moment below. Thus finally we may write

$$M_{\mathrm{exch}} = -\frac{e\,(f\,\mu)^2}{2} \sum_{A<B} (\tau^A \times \tau^B)_3 \left\{ \frac{1}{\mu} \left[\frac{z^{AB}(z^{AB},\,(\sigma^A \times \sigma^B))}{r_{AB}^2} \left(1 + \frac{1}{\mu\, r_{AB}}\right) \right.\right.$$
$$\left.\left. - (\sigma^A \times \sigma^B) \right] e^{-\mu r_{AB}} + 1/2\, (\zeta^A + \zeta^B \times z^{AB}) \cdot V(AB) \right\}, \tag{11}$$

in accordance with MØLLER-ROSENFELD's result*).

§ 3. The Eigenstates of a System of Nucleons in the Central Force Approximation.

In this approximation, the tensor force is entirely neglected; the interaction energy takes then the form

$$V = \sum_{A<B} (\tau^A \cdot \tau^B) \cdot (\sigma^A \cdot \sigma^B) \cdot U(AB)\,, \tag{12}$$

in which, for not too small distances r_{AB}, $U(AB)$ is YUKAVA's potential function const. $\exp(-\mu r_{AB}) \cdot r_{AB}^{-1}$.

We shall first discuss in a somewhat more general manner the properties of the eigenstates of a system with an interaction given by (12). The Hamiltonian

$$H = \sum_A \frac{1}{2\,M}\, p_A^2 + V$$

of the system is invariant under simultaneous rotations of all spin or all isotopic spin vectors. The quantities

$$S_k = 1/2 \sum_A \sigma_k^A \qquad S^2 = \sum_K S_k^2$$

and

$$T_\alpha = 1/2 \sum_A \tau_\alpha^A \qquad T^2 = \sum_\alpha T_\alpha^2$$

are therefore integrals of motion and define the quantum numbers S, M and T, N, M and N being the eigenvalues of S_3 and T_3 respectively. The corresponding eigenfunctions of S^2, S_3 and of T^2, T_3 will be written as

$$\xi^{S,M} \quad \text{and} \quad \eta^{T,N} \quad \text{respectively.}$$

*) Note that in MØLLER-ROSENFELD's paper[4]) a different definition of $\vec{\varphi}$ $(\varphi_1, \varphi_2, \varphi_3)$ and τ is utilized. Our vectors $\vec{\varphi}$ and τ are obtained from M-R's by a rotation of π around the 1-axis: $\varphi_1, \varphi_2, \varphi_3 \to \varphi_1, -\varphi_2, -\varphi_3$ and $\tau_1\,\tau_2\,\tau_3 \to \tau_1, -\tau_2, -\tau_3$. Accordingly, in M-R's paper the nucleon charge is represented by $\frac{1}{2} \cdot (1-\tau_3^A)$.

As is well known[8]) there exists for any pair of quantum numbers S, M or T, N a set of functions ξ_α ($\alpha = 1 \ldots h_S$) and η_μ ($\mu = 1 \ldots h_T$) which are transformed into each other under permutations of the spin or isotopic variables and which generate an irreducible representation of the permutation group. Let P be the permutation which transforms the set of variables $1, 2, \ldots n$ into $p_1, p_2, \ldots p_n$; then we have

$$P\,\xi_\alpha(1 \ldots n) \equiv \xi_\alpha(p_1\, p_2 \ldots p_n) = \sum_\beta D^S_{\beta\alpha}(P)\,\xi_\beta(1 \ldots n) \tag{13}$$

and analogously for η. The representations D^S and D^T are uniquely determined by S, T and the number n of particles. (Equivalent representations are considered as equal.)

The $h_S \cdot h_T$ products $\xi_\alpha \cdot \eta_\mu$ are likewise transformed into each other under permutations simultaneously applied both to ξ and η. The representation $D^S \times D^T$ thus induced is in general a reducible one. Its decomposition into irreducible parts

$$D^S \times D^T = \sum_\Gamma \alpha_\Gamma \cdot D^\Gamma \tag{14}$$

is obtained by means of an orthogonal transformation $U^\Gamma_{\alpha\mu,\varepsilon}$, which in the $h_S \cdot h_T$-dimensional vector space of the $\xi_\alpha\,\eta_\mu$ sets up a new basis

$$\Theta^\Gamma_\varepsilon = \sum_{\alpha\mu} U^\Gamma_{\alpha\mu\varepsilon}\,\xi_\alpha \cdot \eta_\mu. \tag{15}$$

The α_Γ give the number of irreducible representations D^Γ contained in $D^S \times D^T$ and are expressed by the formula

$$\alpha_\Gamma = \frac{1}{n!}\sum_{\mathfrak{c}} c\,\chi^S(\mathfrak{c})\,\chi^T(\mathfrak{c})\,\chi^\Gamma(\mathfrak{c}) \tag{16}$$

in which $\chi(\mathfrak{c})$ are the group characters of the representations D^S, D^T and D^Γ, $\mathfrak{c}$ the class of permutations and c the number of elements in $\mathfrak{c}$[9]).

Let $\hat{\Gamma}$ be the symmetry class reciprocal to Γ, defined as follows:

$$D^{\hat{\Gamma}}_{\chi\beta}(P) = \delta_P\, D^\Gamma_{\alpha\beta}(P); \qquad \delta_P = \pm 1 \text{ for } \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} \text{ permutations.}$$

Let then $F^{\hat{\Gamma}}_\varepsilon(1 \ldots n)$ be a set of functions transforming under permutations according to $D^{\hat{\Gamma}}$:

$$P\,F^{\hat{\Gamma}}_\varepsilon(1 \ldots n) = F^{\hat{\Gamma}}_\varepsilon(p\,p \ldots p) = \sum_\zeta D^{\hat{\Gamma}}_{\zeta\varepsilon}(P) \cdot F^{\hat{\Gamma}}_\zeta(1 \ldots n). \tag{17}$$

Then it is easily verified that the sum of products

$$\Psi = \sum_\varepsilon F^{\hat{\Gamma}}_\varepsilon(1 \ldots n) \cdot \Theta^\Gamma_\varepsilon(1 \ldots n)$$

is antisymmetrical under simultaneous permutations of the arguments of F and Θ.

In application to our problem let F be a function of the space coordinates alone: $F^{\hat{\Gamma}} = F^{\hat{\Gamma}}(x_1\, x_2 \ldots x_n)$; then

$$\psi^{\Gamma} = \sum_{\varepsilon} F_{\varepsilon}^{\hat{\Gamma}}(x_1\, x_2 \ldots x_n) \cdot \Theta_{\varepsilon}^{\Gamma}$$

is antisymmetrical and the most general function satisfying the exclusion principle is therefore

$$\Psi_{\text{antis.}} = \sum_{\Gamma,\,\varepsilon} F_{\varepsilon}^{\hat{\Gamma}}(x_1\, x_2 \ldots x_n) \cdot \Theta_{\varepsilon}^{\Gamma}(1 \ldots n)\,. \tag{18}$$

If we introduce the matrix elements of H with respect to the variables Γ and ε:

$$H\,\Theta_{\varepsilon}^{\Gamma} = \sum_{\Gamma'\,\varepsilon'} (\Gamma'\,\varepsilon' \,|\, H \,|\, \Gamma\,\varepsilon) \cdot \Theta_{\varepsilon'}^{\Gamma'},$$

we obtain the following Schroedinger equation for the F:

$$\sum_{\Gamma,\,\varepsilon} \left\{ (\Gamma'\,\varepsilon' \,|\, H \,|\, \Gamma\,\varepsilon) - (\Gamma'\,\varepsilon' \,|\, 1 \,|\, \Gamma\,\varepsilon) \cdot E \right\} F_{\varepsilon}^{\hat{\Gamma}}(x_1\, x_2 \ldots x_n) = 0\,. \tag{19}$$

The matrix elements $(\Gamma'\varepsilon' \mid H \mid \Gamma\varepsilon)$ are easily evaluated with the help of the relations

$$\begin{aligned} (\sigma^A \cdot \sigma^B) &= 2\, P_{AB}^S - 1 \\ (\tau^A \cdot \tau^B) &= 2\, P_{AB}^T - 1 \end{aligned} \tag{20}$$

where $P_{AB}^{S(T)}$ indicates the transposition $(1 \ldots A \ldots B \ldots n) \to (1 \ldots B \ldots A \ldots n)$, applied to the variables of ξ (η). Employing (15), we obtain

$$(\Gamma'\,\varepsilon' \,|\, H \,|\, \Gamma\,\varepsilon) = \sum_{A} \frac{p_A^2}{2\,M} (\Gamma'\,\varepsilon' \,|\, 1 \,|\, \Gamma\,\varepsilon)$$

$$+ \sum_{A<B} U(AB) \left\{ \sum_{\alpha\,\beta\,\mu\,\nu} U_{\alpha\,\mu,\,\varepsilon}^{\Gamma}\, U_{\beta\,\nu,\,\varepsilon'}^{\Gamma'} (2\, D_{\beta\,\alpha}^S(P_{AB}) - \delta_{\beta\,\alpha}) \cdot (2\, D_{\nu\mu}^T(P_{AB}) - \delta_{\nu\,\mu}) \right\}.$$

This expression can be simplified into

$$\sum_{A} \frac{p_A^2}{2\,M} (\Gamma'\,\varepsilon' \,|\, 1 \,|\, \Gamma\,\varepsilon) + \sum_{A<B} U(AB) \Big\{ (4\, D_{\varepsilon'\,\varepsilon}^{\Gamma}(P_{AB}) + \delta_{\varepsilon'\,\varepsilon}) \cdot (\Gamma' \,|\, 1 \,|\, \Gamma)$$

$$-2 \left\{ \sum_{\alpha\,\beta\,\mu} U_{\alpha\,\mu,\,\varepsilon}^{\Gamma}\, U_{\beta\,\mu,\,\varepsilon'}^{\Gamma'}\, D_{\beta\,\alpha}^S(P_{AB}) + \sum_{\alpha\,\mu\,\nu} U_{\alpha\,\mu,\,\varepsilon}^{\Gamma}\, U_{\alpha\,\nu,\,\varepsilon'}^{\Gamma'}\, D_{\nu\,\mu}^T(P_{AB}) \right\} \Big\}. \tag{21}$$

Provisional information concerning the ordering of the energy levels may be obtained from (21) by forming the mean values of the diagonal elements of the potential energy V for a definite class:

$\overline{(\Gamma \mid V \mid \Gamma)} = 1/h_\Gamma \cdot \sum_{\varepsilon} (\Gamma\varepsilon \mid V \mid \Gamma\varepsilon)$. We obtain

$$\overline{(\Gamma \mid V \mid \Gamma)} = \sum_{A<B} U(AB).$$
$$\left\{1 + \frac{4}{h_\Gamma}\chi^\Gamma(c_2) - \frac{2}{n!}\sum_P \chi^\Gamma(P)[\chi^S(P)\chi^T(P\cdot P_{AB}) + \chi^S(P\cdot P_{AB})\chi^T(P)]\right\}. \tag{22*)}$$

If $h_\Gamma = 1$, formula (22) may replace (21) for calculating the diagonal elements of V.

§ 4. Application to the Three-Body System.

The quantum number S (T) may take the values 1/2 or 3/2. The corresponding states will be designated as spin (charge) doublets or quartets. The latter are symmetrical states, viz. invariant under permutations of the variables, whereas the doublet states induce the two-dimensional representation of the permutation group $\mathfrak{S}_3$. The three irreducible representations of this group will be denoted as follows:

I: the symmetrical,
A: the antisymmetrical,
Δ: the two-dimensional representation.

In table I we give a list of the corresponding characters χ:[11])

Table I.

Class	c	$\chi^I(\mathfrak{c})$	$\chi^A(\mathfrak{c})$	$\chi^\Delta(\mathfrak{c})$
c_1: identity . . .	1	1	1	2
c_2: (12) (13) (23) .	3	1	−1	0
c_3: (123) (132) . .	2	1	1	−1

*) Another formula for a rough evaluation of $(\Gamma|V|\Gamma)$ is obtained from replacing $U(AB)$ by $U(O)$. (Long-range approximation, see e. g. Feenberg and Phillips[10]).) In this case, the formula

$$\sum_{P\varepsilon\mathfrak{c}} D^\Gamma_{\beta\alpha}(P) = \delta_{\beta\alpha}\chi^\Gamma(\mathfrak{c})\cdot c/h_\Gamma$$

may be applied to (21) with $\mathfrak{c} = c_2 = (P_{AB})$; we thus obtain

$$(\Gamma'\varepsilon' \mid V_0 \mid \Gamma\varepsilon) = (\Gamma'\varepsilon' \mid 1 \mid \Gamma\varepsilon)\cdot U(0)\frac{n(n-1)}{2}\left\{1 + 4\frac{\chi^\Gamma(c_2)}{h_\Gamma} - 2\frac{\chi^S(c_2)}{h_S} - 2\frac{\chi^T(c_2)}{h_T}\right\}.$$

The reduction of the product (14) with the help of (16) gives for $S = T = 1/2$ $(D^S = D^T = \varDelta)$: $\varDelta \times \varDelta = I + A + \varDelta$. The following six cases will therefore be considered:

Table II.

Case:	a_1	a_2	a_3	b_1	b_2	c
Γ	I	A	$\varDelta$	$\varDelta$	$\varDelta$	I
$\hat{\Gamma}$	A	I	$\hat{\varDelta}$	$\hat{\varDelta}$	$\hat{\varDelta}$	A
S	1/2	1/2	1/2	1/2	3/2	3/2
T	1/2	1/2	1/2	3/2	1/2	3/2

($\varDelta$ is self-reciprocal, viz. $\hat{\varDelta}$ is equivalent to $\varDelta$.)

Let us now evaluate, for the six cases mentioned in table II, the mean values $\overline{(\Gamma \,|\, V \,|\, \Gamma)}$ according to (22):

$$\left.\begin{aligned}
a_1)&: \ (I \,|\, V \,|\, I) &&= +5 \cdot \sum_{A<B} U(AB) \\
a_2)&: \ (A \,|\, V \,|\, A) &&= -3 \cdot \sum_{A<B} U(AB) \\
a_3)&: \ \overline{(\varDelta \,|\, V \,|\, \varDelta)} &&= +1 \cdot \sum_{A<B} U(AB) \\
b_{1,2}&: \ \overline{(\varDelta \,|\, V \,|\, \varDelta)} &&= -1 \cdot \sum_{A<B} U(AB) \\
c)&: \ (I \,|\, V \,|\, I) &&= +1 \cdot \sum_{A<B} U(AB)
\end{aligned}\right\} \qquad (24)$$

A more detailed investigation requires a special choice of the ξ and η. Expressed in terms of the eigenvectors $\alpha(A)$ and $\beta(A)$ of σ_3^A [12]) and of the rotational invariant $a(AB) = \alpha(A)\,\beta(B) - \beta(A)\,\alpha(B)$, they may be written as follows:

$$\left.\begin{aligned}
S = \frac{3}{2}: \ \xi^{3/2\,3/2} &= \alpha(1)\,\alpha(2)\,\alpha(3) \\
\xi^{3/2\,1/2} &= 1/\sqrt{3} \cdot [\alpha(1)\,\alpha(2)\,\beta(3) + \alpha(1)\,\beta(2)\,\alpha(3) \\
&\quad + \beta(1)\,\alpha(2)\,\alpha(3)], \text{ etc.} \\
S = \frac{1}{2}: \ \xi_1^{1/2\,1/2} &= 1/\sqrt{6} \cdot (a(23)\,\alpha(1) + a(13)\,\alpha(2)) \\
\xi_2^{1/2\,1/2} &= 1/\sqrt{2} \cdot a(12)\,\alpha(3) \\
\xi_1^{1/2\,-1/2} &= 1/\sqrt{6} \cdot (a(23)\,\beta(1) + a(13)\,\beta(2)) \\
\xi_2^{1/2\,-1/2} &= 1/\sqrt{2} \cdot a(12)\,\beta(3)\,.
\end{aligned}\right\} \qquad (25)$$

The isotopic spin functions η are built quite analogously.

The matrix $U^{\Gamma}_{\alpha\mu,\varepsilon}$ can now be determined: U is the unit matrix except for $S = T = 1/2$. We shall give a list of the new basis vectors for this special case:

$$\left.\begin{aligned} \Theta^{\mathrm{I}} &= 1/\sqrt{2}\cdot(\xi_1\,\eta_1 + \xi_2\,\eta_2) & \Theta^{\Delta}_1 &= 1/\sqrt{2}\cdot(-\xi_1\,\eta_1 + \xi_2\,\eta_2) \\ \Theta^{A} &= 1/\sqrt{2}\cdot(\xi_1\,\eta_2 - \xi_2\,\eta_1) & \Theta^{\Delta}_2 &= 1/\sqrt{2}\cdot(\xi_1\,\eta_2 + \xi_2\,\eta_1) \end{aligned}\right\} \quad (26)$$

With the help of (26) all matrix elements of V can be determined. We need not go into the details of this investigation; we wish, however, to emphasize the following important result: As may be seen from table I, the character $\chi^{\Delta}(c_2)$ vanishes, a fact which immediately follows from the self-reciprocity of Δ (compare (17)). On account of the orthogonality of the representation matrices $D_{\alpha\beta}(P)$, this is equivalent to the statement

$$\left.\begin{aligned} D^{\Delta}_{11}(P_{AB}) &= -D^{\Delta}_{22}(P_{AB}) \\ D^{\Delta}_{12}(P_{AB}) &= +D^{\Delta}_{21}(P_{AB})\,. \end{aligned}\right\} \quad (27)$$

With the help of (27) we easily verify that the vector Θ^{A} (26) is an eigenvector of the operator $(\sigma^A\cdot\sigma^B)\,(\tau^A\cdot\tau^B)$:

$$(\sigma^A\cdot\sigma^B)\cdot(\tau^A\cdot\tau^B)\;\Theta^A = -3\;\Theta^A.$$

The antisymmetric state a_2 is therefore an eigenstate of the system. According to (18) the corresponding space function $F(x_1\,x_2\,x_3)$ is symmetrical in 1, 2, 3. Because the kinetic energy takes its minimum value for space symmetrical states and because of the low value of the potential energy (24) for a_2, we conclude that a_2 represents the ground-state of the three body system.

This assumption is strongly supported by the values of the spin contribution to the magnetic moment of the system:

$$\begin{aligned} M_{\mathrm{spin}} &= \sum_A \sigma^A\,(1/2\cdot(\mu_P+\mu_N) + 1/2\cdot(\mu_P-\mu_N)\,\tau^A_3) \\ &= (\mu_P+\mu_N)\cdot\tilde{S} + 1/2\cdot(\mu_P-\mu_N)\sum_A \sigma^A\,\tau^A_3\,. \end{aligned} \quad (28)$$

In order to evaluate the diagonal elements of M_{spin} with respect to S and T, we note that

$$(S\,|\,\sigma^A\,|\,S) = (\alpha\,|\,\varrho^A(S)\,|\,\alpha')\cdot\tilde{S}$$
$$\text{and}\;\;(T\,|\,\tau^A_3\,|\,T) = (\mu\,|\,\varrho^A(T)\,|\,\mu')\cdot T_3\;{}^{13})\,.$$

The matrices ϱ^A are calculated with the help of (26). We then obtain

$$(S T \alpha \mu \mid \sigma^A \tau_3^A \mid S T \alpha' \mu') = (\alpha \mid \varrho^A(S) \mid \alpha') \cdot (\mu \mid \varrho^A(T) \mid \mu') \cdot \vec{S}\, T_3,$$

and from this

$$(S T \Gamma \varepsilon \mid \sum_A \sigma^A \tau_3^A \mid S T \Gamma \varepsilon)$$
$$= \vec{S}\, T_3 \sum_{\alpha\alpha'} \sum_{\mu\mu'} U^{\Gamma}_{\alpha\mu,\varepsilon}\, U^{\Gamma}_{\alpha'\mu',\varepsilon}\, (S T \alpha \mu \mid \sum_A \sigma^A \tau_3^A \mid S T \alpha' \mu').$$

The results are given in table III: (Supposed $L = 0$, $N = -1/2$.)

Table III.

Case	$\sum_A \sigma^A \tau_3^A$	Magn. Moment	Numer. value (n. m.)
a_1	$20/3 \cdot S\, T_3$	$-1/3\, \mu_P + 4/3\, \mu_N$	-3.478
a_2	$-4 \cdot S\, T_3$	μ_P	$+2.789$
a_3	$4/3 \cdot S\, T_3$	$1/3\, \mu_P + 2/3\, \mu_N$	-0.344
b_1	$4/3 \cdot S\, T_3$	$1/3\, \mu_P + 2/3\, \mu_N$	-0.344
b_2	$4/3 \cdot S\, T_3$	$\mu_P + 2\, \mu_N$	-1.033
c	$4/3 \cdot S\, T_3$	$\mu_P + 2\, \mu_N$	-1.033

From table III we see that all cases besides a_2 may definitely be ruled out.

§ 5. The Expectation Value of the Exchange Moment.

The magnetic exchange moment (11) consists of two parts with different symmetry. If we take into account the results of § 4, we see that the expectation value of the part containing $(\zeta^A + \zeta^B \times z^{AB}) \cdot V(AB)$ vanishes; we can therefore restrict ourselves to the expression:

$$M'_{\text{exch}} = -\frac{e(f\mu)^2}{2\mu} \sum_{A<B} (\tau^A \times \tau^B)_3 \left\{ \frac{z^{AB} \cdot (z^{AB}, (\sigma^A \times \sigma^B))}{r_{AB}^2} \left(1 + \frac{1}{\mu\, r_{AB}}\right) - (\sigma^A \times \sigma^B) \right\} e^{-\mu r_{AB}}.$$

Our purpose is first to evaluate the matrix elements of M'_{exch} with respect to the variables S, M and T, N:

The vector products $(\sigma^A \times \sigma^B)$ satisfy the relations

$$[(\sigma^A \times \sigma^B)_i,\, S_k] = i \cdot (\sigma^A \times \sigma^B)_l$$

viz. the commutation rules of a vector; their diagonal elements with respect to S are consequently of the form:[13])

$$(S\alpha \mid (\sigma^A \times \sigma^B) \mid S\alpha') = (\alpha \mid A^{AB}(S) \mid \alpha') \cdot \vec{S}.$$

The matrix A is easily evaluated and becomes, for $S = 1/2$:

$$\begin{aligned} &(1 \mid A\,(1/2) \mid 1) = (2 \mid A\,(1/2) \mid 2) = 0, \\ &(1 \mid A\,(1/2) \mid 2) = -(2 \mid A\,(1/2) \mid 1) = 4i/\sqrt{3}, \end{aligned} \tag{29}$$

for each $AB = 12, 23, 31$.

Taking into account the complete symmetry in the description of the spin and the isotopic spin, we have at once

$$(T\mu \mid (\tau^A \times \tau^B)_3 \mid T\mu') = (\mu \mid A\,(T) \mid \mu') \cdot T_3.$$

The evaluation of $z^{AB} \cdot (z^{AB}, (\sigma^A \times \sigma^B))$ may be carried through in the following manner: We introduce the total orbital angular momentum $\vec{L}$ and the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. Then $(z^{AB}, (\sigma^A \times \sigma^B))$ is evaluated by means of the well known formulas for the scalar product[14]) and finally the diagonal element of the complete expression with respect to L, S, J is formed. We thus obtain for $L = 0$, $J = S = 1/2$

$$(JLS\alpha \mid z^{AB} \cdot (z^{AB}, (\sigma^A \times \sigma^B)) \mid JLS\alpha')_{\substack{L=0 \\ S=1/2}} = \frac{1}{3}\, r_{AB}^2\, (\alpha \mid A\,(1/2) \mid \alpha') \cdot \vec{J}$$

and therefore

$$(\alpha\nu \mid M' \mid \alpha'\nu') =$$
$$-\frac{e(f\mu)^2}{2\mu} (\alpha \mid A\,(1/2) \mid \alpha') \cdot (\nu \mid A\,(1/2) \mid \nu') \cdot T_3 \vec{J} \sum_{A<B} \frac{1}{3} \left(\frac{1}{\mu r_{AB}} - 2 \right) e^{-\mu r_{AB}}.$$

In terms of the new variables $\Gamma\varepsilon$, the matrix elements of M' are:

$$(\Gamma\varepsilon \mid M' \mid \Gamma'\varepsilon') = \sum_{\alpha\alpha'} \sum_{\nu\nu'} U^{\Gamma}_{\alpha\nu,\,\varepsilon}\, U^{\Gamma'}_{\alpha'\nu',\,\varepsilon'}\, (\alpha\nu \mid M' \mid \alpha'\nu').$$

We are interested in the expectation value of M' for the ground state of the three body system:

$$\left. \begin{aligned} \{M'\}_{\mathrm{av}} = -\frac{e\,(f\mu)^2}{2\mu} T_3 \vec{J} \sum_{\alpha\alpha'} \sum_{\nu\nu'} (\alpha \mid A\,(1/2) \mid \alpha') \cdot (\nu \mid A\,(1/2) \mid \nu') \\ U^{A}_{\alpha\nu}\, U^{A}_{\alpha'\nu'} \cdot \int dv \mid F^{\mathrm{I}}\,(123) \mid^2 \sum_{A<B} \frac{1}{3} \left(\frac{1}{\mu r_{AB}} - 2 \right) \cdot e^{-\mu r_{AB}}. \end{aligned} \right\} \tag{30}$$

*

From the $U^A_{\alpha\nu}$ as defined in (26) (and from (29)) it follows

$$\sum_{\alpha\alpha'}\sum_{\nu\nu'} U^A_{\alpha\nu}\, U^A_{\alpha'\nu'}\, (\alpha \mid A\,(1/2) \mid \alpha')\cdot(\nu \mid A\,(1/2) \mid \nu') = -\,16/3,$$

and therefore

$$\{M_{\text{exch}}\}_{\text{av}} = +\frac{8}{3}\, e\, \frac{(f\mu)^2}{\mu}\, N\, I \cdot \vec{J}$$

$$I = \int dv\, |F^{\mathrm{I}}\,(123)|^2 \left(\frac{1}{\mu r_{12}} - 2\right)\cdot e^{-\mu r_{12}}. \tag{31}$$

In order to have $\{M_{\text{exch}}\}_{\text{av}}$ in units of nuclear magnetons, let us introduce the ratio γ of the nucleon mass M to the meson mass μ. Then we have

$$e\,\frac{(f\mu)^2}{\mu} = 2\,\gamma\,(f\,\mu)^2\cdot\frac{e}{2\,M} = 2\,\gamma\,(f\,\mu)^2$$

nuclear magnetons. Considering still $J = S = 1/2$, we obtain for the exchange moment in nuclear magnetons

$$\underline{\mu_{\text{exch}} = +\,\frac{8}{3}\,\gamma\,(f\,\mu)^2\, N\, I} \tag{32}$$

N takes the values $+\,1/2$ (He_3) and $-\,1/2$ (H_3); I is given in (31).

Numerical evaluation

The space function F is supposed to be symmetrical in the arguments $x_1\, x_2\, x_3$ and to represent an S-State. We are only interested in the order of magnitude and the sign of the exchange effect; the choice of the simple trial function

$$F\,(123) = \text{const.}\ \exp\left(-\frac{\alpha}{2}\,(r_{12}^2 + r_{13}^2 + r_{23}^2)\right)$$

will be quite sufficient for this purpose. Since nothing precise is known about the best value for α, we calculate the integral I for different values of (μ^2/α). By elementary methods we find

$$I\,(\mu^2/\alpha) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{6\,\alpha}{\mu^2}} + \frac{4}{\sqrt{\pi}}\sqrt{\frac{\mu^2}{6\,\alpha}} - \left(3 + \frac{2\,\mu^2}{3\,\alpha}\right)\cdot e^{+\mu^2/6\,\alpha}\,(1 - \Phi\,(\mu/\sqrt{6\,\alpha}));$$

Φ is the Gauss error integral. The numerical values of I are:

μ^2/α:	0.5	1.0	2.0	3.0	4.0	5.0	7.0	10.0
I:	+0.132	−0.142	−0.227	−0.224	−0.205	−0.183	−0.156	−0.131

As to the values of γ and $(f\,\mu)^2$, the question arises whether the meson mass should be taken from cosmic ray data, which favour a value near 200 m_{el}[15]), or the value obtained from proton-

proton scattering[16]), which give $\mu = 327\ m_{el}$ and $(f\mu)^2 = 1/4$. This larger value μ has again received some interest since the connection between cosmic ray mesons and the mesons responsible for the nuclear forces has become somewhat questionable[17]). For a meson mass of 200 m $(f\mu)^2$ would be somewhat smaller, say 1/10. $\gamma \cdot (f\mu)^2$ varies thus from ~ 0.9 for the smallest up to ~ 1.4 for the largest mass considered. Large meson masses, however, involve larger values of μ^2/α, because the "radius" of the H_3-nucleus will be deter-

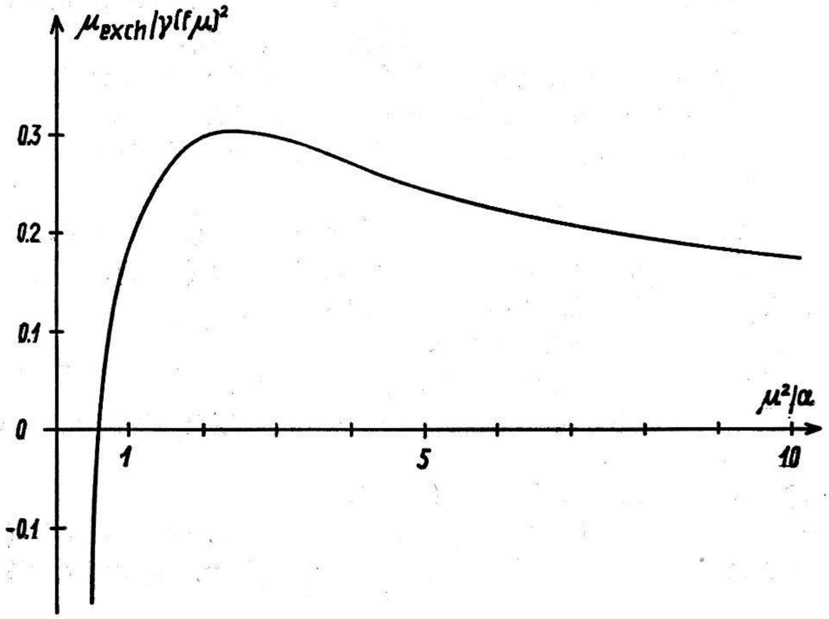

Fig. 1.

mined rather by the binding energy than by the extension of the potential well, and μ^2/α may thus vary say from 2 up to 5. A glance at fig. 1 shows then that the variation in the value of $\gamma(f\mu)^2$ is nearly compensated by a corresponding diminution of $I(\mu^2/\alpha)$. According to (32), we finally obtain for H_3 $(N = -1/2)$:

$$\mu_{exch} \cong +0.3 \text{ nuclear magnetons,}$$

a value which just fills up the gap between Bloch's value of 2.975 n. m. and the value calculated by Sachs and Schwinger[2]) without charge exchange and for a D-state probability of 4%: $\mu =$ 2.71 n. m.

The magnetic moment of He_3 is expected to have the value -1.86 n. m. $- |\mu_{exch}|$, viz. -2.1 n. m. (compare[2])).

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