On the Magnetic Moments of the Neutron and Proton

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by **J. M. Luttinger**[1]), ETH., Zürich.

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Zusammenfassung: In dieser Arbeit wird das magnetische Moment eines Nukleons berechnet für die neutrale, geladene und symmetrische pseudoskalare Mesonentheorie. Die Nukleonen werden durch die Löchertheorie beschrieben, so dass die Rechnung vollständig relativistisch ist. Die üblichen Divergenzen werden durch eine vom Verfasser angegebene Methode[2]) vermieden und — wie beim Electron — werden alle Resultate endlich. Die magnetischen Momente der Nukleonen lassen sich, in der g^2-Näherung, durch geschlossene Formeln darstellen.

I. Introduction.

Is is the purpose of this paper to show that a method previously developed[2]) for calculating the radiative correction to the g-factor of the electron may be extended to calculation of the g-factors of the neutron and proton in meson theory. For concreteness we have choosen the case of pseudoscalar mesons. Within the framework of this theory there are still two types of coupling possible[3]): that in which a pseudovector is constructed out of the meson field, and that in which a pseudoscalar is constructed. It may be shown, however, by simple generalization and correction of a proof due to Dyson[4]) that the two theories will yield identical results for all the magnetic moments calculated in this paper. We shall therefore only give the calculation for the pseudoscalar type of

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[1]) National Research Fellow.

[2]) J. M. Luttinger, Phys. Rev., **74**, 893 (1948). Henceforth cited as I.

[3]) N. Kemmer, Proc. Roy. Soc., A **166**, 127 (1938).

[4]) F. J. Dyson, Phys. Rev., **73**, 929 (1948). Dyson drops the terms quadratic in the coupling constant on the grounds that they are without physical significance. Actually these terms play an essential role in making the two types of coupling equivalent. The same proof shows that the nuclear forces are identical in the g^2 approximation. Direct calculation verifies this and shows that the interaction terms quadratic in the coupling constant remove the $1/r^3$ singularity in the forces, and thus possibly allow a ground state for the deuteron. These forces, however, are not at all static in nature and cannot be applied to calculate the ground state of the deuteron. Similar conclusions have heen reached independently by Dr. L. van Hove (unpublished). I am grateful to Dr. van Hove for a valuable correspondence on this point.

coupling, as it is considerably simpler in structure. As a check on the algebraic work we have also carried out the calculations for the pseudovector coupling, and of course obtained the same results. It may be mentioned that it is the pseudovector type of coupling which has usually been discussed in connection with the meson theory of nuclear forces.

In handling the heavy particles we have used hole theory throughout, as it represents the only consistent relativistic theory for particles of spin one half. That is, we have taken the neutron and the proton to be quantized fields which obey Fermi-Dirac commutation relations and the Dirac equation without the presence of "Pauli terms" – it is just the hope of meson theory that it will account for the existence of such terms.

Three types of pseudoscalar meson field have been considered: neutral, charged and symmetric. From the point of view of comparison with the experimental facts perhaps only the latter is of interest, the nuclear forces generally being assumed to be charge independent[1]). However, for completeness the other two cases are included.

II. Neutral Theory.

For this case the meson field will only yield an additional magnetic moment for the proton – the magnetic moment of the neutron remaining zero. The Lagrangian density is given by[2]).

$$L = L_p + L_M + L' + L'' \tag{1}$$

$$L_p = -\psi_p^*\left(\frac{1}{i}\frac{\partial}{\partial t} + \frac{1}{i}\alpha\cdot(\nabla - ie A_0) + M\beta\right)\psi_p$$

$$L_M = \frac{1}{2}\left(\left(\frac{\partial\Phi}{\partial t}\right)^2 - (\nabla\Phi)^2 - \mu^2\Phi^2\right)$$

$$L' = -\sqrt{4\pi}\, ig\,(\psi_p^*\,\beta\,\gamma_5\,\Phi\psi_p)$$

$$L'' = -2\pi g^2\lambda\,[\psi_p^*\sigma\,\psi_p)\cdot(\psi_p^*\sigma\,\psi_p) - (\psi_p^*\gamma_5\,\psi_p)\,(\psi_p^*\gamma_5\,\psi_p)]$$

In these equations A_0 is the vector potential of an external homogeneous magnetic field, ψ_p the (quantized) proton wave function, α, β the well known Dirac matrices[3]), σ the (four rowed) spin ma-

[1]) This charge independence has recently been questioned. See J. M. Blatt, Phys. Rev., **74**, 92 (1948). Blatt's results may, however, be dependent on his special assumptions concerning the potential.

[2]) We have used natural units, $\hbar = c = 1$. M is the nucleon mass, μ the meson mass.

[3]) The specific representation which we use, however, has the usual representations of α_x and α_y interchanged. Cf. I, appendix.

trices, $\gamma_5 = i\alpha_x \alpha_y \alpha_z$ and g the coupling constant. The number λ which appears in L'' is an arbitrary constant. The presence of this term corresponds to the fact that one may get different versions of the pseudoscalar theory by adding an invariant term quadratic in the coupling constant (see Kemmer, op. cit.). The value of λ cannot be fixed a priori, but we shall see that L'' gives no contribution to the magnetic moment and therefore this arbitrariness is in no way disturbing for our calculations.

Using this Lagrangian density it is an easy matter to find the Hamiltonian density.

$$H = H_p + H_M + H' + H'' \tag{2}$$

Where

$$H_p = \psi_p^* \left(\frac{1}{i} \alpha \cdot (\nabla - i e A_0) + M\beta\right) \psi_p$$

$$H_M = \frac{1}{2} (\pi^2 + (\nabla \Phi)^2 + \mu^2 \Phi^2)$$

$$H' = \sqrt{4\pi}\, i g\, (\psi_p\, \beta\, \gamma_5\, \Phi \psi_p)$$

$$H'' = 2\pi g^2 \lambda [(\psi_p^* \sigma \psi_p) \cdot (\psi_p^* \sigma \psi_p) - (\psi_p^* \gamma_5 \psi_p)(\psi_p^* \gamma_5 \psi_p)].$$

To carry out the quantization[1]) we write[2]):

$$\psi_p = \sum_n a_n \psi_n$$

$$\Phi = \sum_p \frac{\alpha_p + \alpha_{-p}^*}{\sqrt{2\,\omega_p}} e^{i p \cdot x}$$

$$\pi = i \sum_p \sqrt{\frac{\omega_p}{2}} (\alpha_{-p}^* - \alpha_p)\, e^{i p \cdot x}.$$

The ψ_n are the solutions corresponding to the energy E_n of the Dirac equation in the presence of an homogeneous external field (cf. I, appendix). The quantities a_n and α_p satisfy the commutation rules of Fermi-Dirac and Einstein-Bose particles respectively, i. e. all

$$a_n^* a_{n'} + a_{n'} a_n^* = \delta_{n n'}$$

$$\alpha_p \alpha_{p'}^* - \alpha_{p'}^* \alpha_p = \delta_{p p'}$$

other commutators being zero. ω_p represents the energy of a free meson, $\omega_p = \underset{+}{\sqrt{\mu^2 + p^2}}$.

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1) As a general reference for the methods of quantization used here, see G. Wentzel, Einf. in die Quant. Theorie der Wellenfelder, Franz Deuticke, Wien (1943).

2) Our solutions are normalized in a large box of volume V. However, since in all end results the quantity V drops out, we simply put it equal to 1.

Substituting in the Hamiltonian we obtain:

$$H_p = \sum_n a_n^* a_n E_n$$

$$H_M = \sum_p \alpha_p^* \alpha_p \omega_p$$

$$H' = \sqrt{4\pi}\, i g \sum_{n,n'} \sum_p a_n^* a_{n'} \alpha_p^* \frac{Q_{nn'}^{(p)}}{\sqrt{2\,\omega_p}} + C.\,C.$$

$$H'' = 2\pi g^2 \lambda \sum_{\substack{n,n'\\ m,m'}} a_n^* a_{n'} a_m^* a_{m'} R_{nn'\,mm'}$$

where

$$Q_{nn'}^{(p)} \equiv \int d^3x\, (\psi_n^* \beta \gamma_5 \psi_{n'})\, e^{-ip\cdot x}$$

$$R_{nn'mm'} \equiv \int d^3x\, [(\psi_n \sigma \psi_{n'})\cdot(\psi_m^* \sigma \psi_{m'}) - (\psi_n^* \gamma_5 \psi_{n'})(\psi_m^* \gamma_5 \psi_{m'})].$$

Proceeding exactly as in the case of the electron, we calculate the energy of one proton in a state "m" (and no mesons present), to terms proportional to the square of the coupling constant. This means calculating the mean value of H'' for this state, and performing second order perturbation theory with respect to H'. The method is straightforward and proceeds exactly as in I. The results are:

$$E = E' + E'' \tag{3}$$

$$E' = -2\pi g^2 \left[\sum_p \frac{1}{\omega_p} \left\{ \sum_{E_n>0} \frac{|Q_{nm}^{(p)}|^2}{\omega_p + E_n - E_m} - \sum_{E_n<0} \frac{|Q_{mn}^{(p)}|}{\omega_p + |E_n| + E_m} \right\} + \sum_p \frac{Q_{nn}^{(p)*} Q_{mm}^{(p)} + Q_{mm}^{(p)*} Q_{nn}^{(p)}}{\omega_p^2} \right]$$

$$E'' = 2\pi g^2 \lambda \left(\sum_{E_n>0} - \sum_{E_n<0} \right) R_{mnnm} + 2\pi g^2 \lambda \sum_{E_n<0} R_{mmnn}.$$

The characteristic minus signs which arise in the summations over negative energies come from the use of hole theory, and the definition of the true energy as the energy of the particle plus vacuum, minus the energy of the vacuum.

To make use of the method of I, it is now necessary to specialize the state "m". As was there shown, if we take "m" to be tha solution of the Dirac equation in which the energy is simply M, then any infinities in the mass or charge of the electron will not effect the magnetic moment. This means that simply by developing the quantity E in a power series in H_0 (external field) the linear term in H_0 will converge and will give us directly the change in the magnetic moment of the nucleon. This expectation is borne out by the detailed calculation, from which one may conclude that in the pseudoscalar theory[1]) all the infinities are simply mass and charge infinities, just as in electrodynamics.

With this choice of state the evaluation of the matrix elements simple, and we shall only quote the results[2]):

$$E' = -\pi g^2 \sum_p \frac{1}{\omega_p} \left\{ \sum_{n=0}^{\infty} \left[\frac{\left(\frac{2 e H_0}{E_n (E_n + M)}\right)}{E_n + \omega_p - M} + \frac{\left(\frac{p_3^2}{E_n (E_n + M)}\right)}{E_n + \omega_p - M} \right] \frac{\xi^n}{n!} e^{-\xi} \right\} - (M \to -M) \tag{4}$$

where

$$E_n \equiv \sqrt[+]{M^2 + p_3^2 + 2 e H_0 n}$$

$$\xi = \frac{p_1^2 + p_2^2}{2 e H_0}$$

and $(M \to -M)$ means that the same function of $-M$ is to be constructed.

$$E'' = \pi g^2 \lambda \sum_{p_2, p_3} \sum_{n=0}^{\infty} \left(\frac{2 (E_{n+1} + M)}{E_{n+1}} + \frac{E_n + M}{E_n} - \frac{E_n - M}{E_n} \right. - M \to -M) \tag{5}$$

Simplification gives

$$E' = -\pi g^2 \sum_p \frac{1}{\omega_p} \left\{ \sum_{n=0}^{\infty} \frac{E_n - M}{\omega_p + E_n - M} \frac{1}{E_n} \frac{\zeta^n}{n!} e^{-\xi} \right\} - (M \to -M) \tag{6}$$

$$E'' = (2 M)(e H_0) \lambda g^2 \sum_{p_3} \sum_{n=0}^{\infty} \left(\frac{1}{E_{n+1}} + \frac{1}{E_n} \right). \tag{7}$$

In (7) we have replaced the summation over p_2 by $\frac{e H_0}{2 \pi}$, which corresponds to keeping the particles present within the normali-

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[1]) VILLARS (in unpublished calculations) has shown that even with this special state the magnetic moment of the nucleons diverges for the vector theory—with tensor coupling—which seems to indicate that in this version of the vector meson theory there are infinities in addition to the mass and charge terms. This result, incidently, prevents the calculation of the magnetic moments on the basis of a MØLLER-ROSENFELD mixture.

The other version of vector meson theory (vector coupling, similar to coupling of electron to the electromagnetic field) has been investigated and found to give convergent magnetic moments.

[2]) The calculation involves an integration over Hermite polynomials which has been carried through in I.

zation volume[1]). Let us first calculate E''. To terms linear in H_0 we have

$$E'' = 2\,M\,e\,H_0\,\lambda\,g^2 \sum_{p_3} \sum_{n=0}^{\infty} \left(\frac{2}{E_n} - \frac{e\,H_0}{E_n^3}\right).$$

The Euler sum formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x)\,d\,x + \frac{f(0)}{2} + \cdots$$

provides an expansion in powers of H_0. Keeping only the linear terms one obtains

$$E'' = 2\,M\,(e\,H_0)\,\lambda\,g^2 \sum_{p_3} \left\{ \int_0^{\infty} d\,n \left(\frac{2}{E_n} - \frac{e\,H_0}{E_n^3}\right) + \left[\frac{1}{E_n}\right]_{n=0} \right\}.$$

The first term is independent of H_0, and represents an infinite self energy. Dropping this, we get for the terms linear in H_0:

$$E'' = 2\,M\,(e\,H_0)\,\lambda\,g^2 \sum_{p_3} \left(\left[\frac{1}{E_n}\right]_{n=0} - e\,H_0 \int_0^{\infty} \frac{1}{E_n^3}\,d\,n \right).$$

However,

$$\int_0^{\infty} d n\,\frac{e\,H_0}{E_n^3} = \left[\frac{1}{E_n}\right]_{n=0}$$

and therefore

$$E'' = 0. \tag{8}$$

Returning now to the discussion of E', we make use of the identity

$$\frac{1}{\omega_p} \cdot \frac{E_n - M}{\omega_p + E_n - M} = \left(\frac{1}{\omega_p} - \frac{1}{\omega_p + E_n - M}\right)$$

which transforms (6) into

$$E' = \pi\,g^2 \sum_p \sum_{n=0}^{\infty} \frac{1}{E_n}\,\frac{1}{(\omega_p + E_n - M)}\,\frac{\xi^n}{n!}\,e^{-\xi} - (M \to -M). \tag{9}$$

Defining a quantity (Cf. I)

$$f(n) = \frac{1}{E_n}\,\frac{1}{\omega_p + E_n - M}$$
$$= f(\xi) + (n - \xi)\,f'(\xi) + \frac{(n-\xi)^2}{2}\,f''(\xi) + \cdots$$

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[1]) Bethe-Sommerfeld, Hand. der Phys., XXIV, 2, page 478, Julius Springer, Berlin (1933).

we get

$$E' = \pi g^2 \sum_p \sum_{n=0}^{\infty} \left(f(\xi) + (n-\xi) f'(\xi) + \frac{(n-\xi)^2}{2} f''(\xi_1) + \cdots \right) \frac{\xi^n}{n!} e^{-\xi} - (M \to -M)$$

$$= \pi g^2 \sum_p \left(f(\xi) + \frac{\xi}{2} f''(\xi) + \cdots \right) - (M \to M) .$$

The first term of this expression is independent of H_0 and represents a self energy of the proton. We shall therefore drop it along with the remainder terms in the expansion, the latter representing higher powers of H_0 than the first. Therefore, for the purpose of calculating the magnetic moment we have

$$E' = \pi g^2 \sum_p \frac{\xi}{2} f''(\xi) - (M \to -M) . \tag{10}$$

Carrying through the indicated differentiations one obtains

$$E' = \frac{\pi g^2 (e H_0)}{4} \sum_p (p_1^2 + p_2^2) \left[\frac{3}{\Omega^2(\Omega+\omega-M)} + \frac{3}{\Omega(\Omega+\omega-M)^2} + \frac{2}{(\Omega+\omega-M^3)} \right] \frac{1}{\Omega^3} - (M \to -M) . \tag{11}$$

Here

$$\Omega \equiv \underset{+}{\sqrt{M^2+p^2}} , \quad \omega \equiv \underset{+}{\sqrt{\mu^2+p^2}} .$$

Noting the fact that the factor multiplying $p_1^2 + p_2^2$ in (11) is a function of p^2 only, so that we may replace $p_1^2 + p_2^2$ by $\left(\frac{2}{3}\right) p^2$, we get

$$E' = \frac{\pi g^2 (e H_0)}{3} (M) \sum_p p^2 \frac{1}{\Omega^3}$$

$$\left[\frac{3}{\Omega^2((\omega+\Omega)^2 - M^2)} + \frac{6(\omega+\Omega)}{\Omega((\omega+\Omega)^2-M^2)^2} + \frac{2(3(\omega+\Omega)^2+M^2)}{((\omega+\Omega)^2-M^2)^3} \right] .$$

Finally, we replace the sum by an integral

$$\left(\sum_p \to \left(\frac{1}{2\pi}\right)^3 \int d^3 p \right) ,$$

and carry out the angle integration:

$$E' = \frac{g^2 (e H_0) M}{6\pi} \int_0^{\infty} dp \frac{p^2}{\Omega^3}$$

$$\left[\frac{3}{\Omega^2((\omega+\Omega)^2 - M^2)} + \frac{6(\omega+\Omega)}{\Omega((\omega+\Omega)^2-M^2)^2} + \frac{2(3(\omega+\Omega)^3+M^2)}{((\omega+\Omega)^2-M^2)^3} \right] . \tag{12}$$

The integral in (12) is elementary, as may be seen by introducing the new variable z, defined by $\Omega + \omega = \sqrt{z}$. The result is

$$E_p' = \frac{g^2 (e H_0)}{8 \pi M} \left(1 + 2 \delta^2 - \frac{2 \delta^3 (3 - \delta^2)}{\sqrt{4 - \delta^2}} \cos^{-1} \frac{\delta}{2} - 2 (1 - \delta^2) \delta^2 \log \frac{1}{\delta}\right)$$

where δ is the ration of meson to proton (or neutron) mass.

$$\mu_p = -\frac{\partial E_p'}{\partial H_0} = -\frac{g^2}{\pi} \left(\frac{1}{4} + \frac{\delta^2}{2} - \frac{\delta^3 (3 - \delta^2) \cos^{-1} \delta/2}{2 \sqrt{4 - \delta^2}} - \frac{(1 - \delta^2) \delta^2}{2} \log \frac{1}{\delta}\right) \quad (13)$$

in units of the nuclear magneton.

III. Charged Theory.

The Lagrangian density is given by

$$L_c = L_p + L_N + L_{MC} + L_c' + L_c'' \quad (14)$$

Here L_p is defined as above, while

$$L_N = -\psi_N^* \left(-\frac{1}{i} \frac{\partial}{\partial t} + \frac{1}{i} \alpha \cdot \nabla + M \beta\right) \psi_N$$

$$L_{MC} = -\left(\frac{\partial \Phi^*}{\partial t}\right)\left(\frac{\partial \Phi}{\partial t}\right) - (\nabla \Phi^* + i e A_0 \Phi^*) \cdot (\nabla \Phi - i e A_0 \Phi) - \mu^2 \Phi^* \Phi$$

$$L_c' = -\sqrt{8 \pi}\, i g [(\psi_p^* \beta \gamma_5 \Phi \psi_N) + (\psi_N^* \beta \gamma_5 \Phi^* \psi_p)]$$

$$L_c'' = 4 \pi g^2 \lambda [(\psi_p^* \sigma \psi_N) \cdot (\psi_N^* \sigma \psi_p) - (\psi_p^* \gamma_5 \psi_N) (\psi_N^* \gamma_5 \psi_p) + (\psi_N^* \sigma \psi_p) \cdot (\psi_p^* \sigma \psi_N) - (\psi_N^* \gamma_5 \psi_p) (\psi_p^* \gamma_5 \psi_N)]$$

ψ_N is the (quantized) wave function of the neutron, Φ the charged meson field. The term L_c'' gives a contribution ($-L_p''$) in the Hamiltonian density. Just as in the case of neutral mesons this term will give no contribution to the magnetic moment, and in what follows we shall drop it entirely. We then obtain for the Hamiltonian density

$$H_c = H_p + H_N + H_{MC} + H_c'$$

$$H_N = \psi_N^* (1/i\, \alpha \cdot \nabla + M \beta) \psi_N \quad (15)$$

$$H_{MC} = \pi^* \pi + (\nabla \Phi^* + i e A_0 \Phi^*) \cdot (\nabla \Phi - i e A_0 \Phi) + \mu^2 \Phi^* \Phi$$

$$H_c' = -L_c'.$$

We now carry through the quantization by means of the relationships

$$\begin{aligned} \psi_N &= \sum_n b_n \tilde{\psi}_n \\ \psi_p &= \sum_n a_n \psi_n \\ \Phi &= \sum_l \frac{\alpha_l + \beta_l^*}{\sqrt{2\varepsilon_l}} \Phi_l \\ \pi^* &= -i \sum_l \sqrt{\frac{\varepsilon_l}{2}} (\alpha_l - \beta_l^*) \Phi_l . \end{aligned} \tag{16}$$

ψ_n are solutions of the DIRAC equation in the presence of an homogeneous magnetic field, $\tilde{\psi}_n$ are solutions of the field free DIRAC equation (corresponding to the energy $\tilde{E}_n$), and the Φ_l are solutions of the KLEIN-GORDON equation in the magnetic field. The quantities ε_l are the corresponding (positive) frequencies, ie

$$\left((\nabla - i e A_0)^2 - \mu^2\right) \Phi_l = \varepsilon_l^2 \Phi_l \tag{17}$$

The solutions of (17) are well known[1]). One finds that l is given by an integer $N \geqq 0$ and two momenta k_2, k_3:

$$\Phi_l = e^{i(k_2 y + k_3 z)} e^{-\eta^2/2} H_N(\eta) \frac{(e H_0)^{1/4}}{\pi^{1/4} 2^{N/2} \sqrt{N!}}, \qquad \eta \equiv \sqrt{e H_0}\left(x - \frac{k_2}{e H_0}\right)$$

$$\varepsilon_l = \sqrt{\mu^2 + k_3^2 + 2 e H_0 (N + 1/2)} .$$

The quantities H_N are the ordinary Hermite polynomials.

The quantities a_n, b_n satisfy FERMI-DIRAC commutation rules, the quantities α_l, β_l EINSTEIN-BOSE ones. Using (16) we obtain

$$\begin{aligned} H_p &= \sum_n a_n^* a_n E_n \\ H_N &= \sum_n b_n^* b_n \tilde{E}_n \\ H_{MC} &= \sum_l (\alpha_l^* \alpha_l + \beta_l^* \beta_l) \varepsilon_l \\ H_c' &= \sqrt{4\pi} i g \sum_{\substack{n,m \\ l}} \frac{1}{\sqrt{\varepsilon_l}} a_n^* b_m T_{nm}^{(l)} (\alpha_l + \beta_l^*) + c.c. \\ T_{nm}^{(l)} &\equiv \int d^3 x \, (\psi_n^* \beta \gamma_5 \Phi_l \tilde{\psi}_m) . \end{aligned} \tag{18}$$

It is now necessary to calculate the energy of a neutron or proton in the state "m", where "m" is again the special state which avoids the divergencies in the mass and charge. This choice is necessary

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[1]) BETHE-SOMMERFELD, Hand. der Phys., XXIV, 2, page 478, Julius Springer, Berlin (1933).

here only in the case of the proton: for the neutron (since its energy is always independent of H_0) any state would do. We calculate with the state "m", however, because it is mathematically the simplest state to handle. Straightforward second order perturbation theory then leads to the result (E_N^c, E_P^c being the corrections to the energy of the neutron and proton respectively):

$$E_N^c = -4\pi g^2 \sum_l \frac{1}{\varepsilon_l}\left\{\sum_{e_m>0} \frac{|T_{nm}^{(l)}|^2}{E_n+\varepsilon_l-M} - \sum_{\tilde{E}_n<0} \frac{|T_{nm}^{(l)}|^2}{|E_n|+\varepsilon_l+M}\right\}. \tag{19}$$

$$E_P^c = -4\pi g^2 \sum_l \frac{1}{\varepsilon_l}\left\{\sum_{\tilde{E}_n>0} \frac{|T_{mn}^{(l)}|^2}{\tilde{E}_n+\varepsilon_l-M} - \sum_{E_n<0} \frac{|T_{nm}^{(l)}|^2}{|\tilde{E}_n|+\varepsilon_l+M}\right\}. \tag{20}$$

Calculation of the matrix elements gives

$$E_N^c = -2\pi g^2 \sum_{p_2,p_3} \sum_{n=0}^{\infty}$$

$$\left\{\frac{2eH_0(n+1)}{\varepsilon_{n+3/2}E_{n+1}(E_{n+1}+M)(E_{n+1}+\varepsilon_{n+3/2}-M)} + \frac{p_3{}^2}{\varepsilon_{n+1/2}E_n(E_n+M)(E_n+\varepsilon_{n+1/2}-M)}\right\}$$

$$-(M\to -M). \tag{21}$$

$$E_P^c = -2\pi g^2 \sum_p \sum_{N=0}^{\infty}$$

$$\left\{\frac{p_1{}^2+p_2{}^2}{(\Omega+\varepsilon_{N+1/2}-M)} + \frac{p_3{}^2}{(\Omega+\varepsilon_{N+1/2}-M)}\right\}\frac{e^{-\xi}}{\Omega(\Omega+M)}\frac{\xi^N}{N!} - (M\to -M). \tag{22}$$

$$E_n \equiv \underset{+}{\sqrt{M^2+p_3{}^2+2eH_0 n}}$$

$$\varepsilon_n \equiv \underset{+}{\sqrt{\mu^2+p_3{}^2+2eH_0 n}}$$

$$\Omega = \underset{+}{\sqrt{M^2+p^2}}$$

$$\xi = \frac{p_1{}^2+p_2{}^2}{2eH_0}.$$

Let us first calculate E_N^c.

$$E_N^c = -2\pi g^2 \frac{eH_0}{2\pi}\sum_{p_3}\sum_{n=0}^{\infty}\left\{\frac{p_3{}^2+2eH_0 n}{\varepsilon_{n+1/2}E_n(E_n+M)(E_n+\varepsilon_{n+1/2}-M)}\right\} - (M\to -M).$$

$$= -g^2(eH_0)\sum_{p_3}\sum_{n=0}^{\infty}\left\{\frac{E_n-M}{\varepsilon_{n+1/2}E_n(E_n+\varepsilon_{n+1/2}-M)}\right\} - (M\to M).$$

$$= g^2(eH_0)\sum_{P_3}\sum_{n=0}^{\infty}\frac{1}{E_n(E_n+\varepsilon_{n+1/2}-M)} - (M\to -M).$$

$$= g^2(eH_0)\sum_{P_3}\sum_{n=0}^{\infty}\left\{\frac{1}{E_n(E_n+\varepsilon_n-M)} - \frac{1}{E_n(E_n+\varepsilon_n-M)^2}\frac{eH_0}{2\varepsilon_n}\right\}$$

$$+O(H_0^2) - (M\to -M_0^2).$$

Using once more the EULER sum formula used in the evaluation of (7), and dropping self energy terms and those which are proportional to the square of the field strength, we get

$$E_N^c = g^2 (e H_0) \sum_{p_3} \left\{ \frac{1}{2} \left[\frac{1}{E_n (E_n + \varepsilon_n - M)} \right]_{n=0} - \frac{e H_0}{2} \int_0^\infty dn \frac{1}{E_n \varepsilon_n (E_n + \varepsilon_n - M)^2} \right\} - (M \to -M).$$

$$= g^2 \frac{e H_0}{2\pi} \int_{-\infty}^{+\infty} dp_3 \left\{ \frac{1}{2} \left[\frac{1}{E_n (E_n + \varepsilon_n - M)} \right]_{n=0} - \frac{e H_0}{2} \int_0^\infty dn \frac{1}{E_n \varepsilon_n (E_n + \varepsilon_n - M)^2} \right\} - (M \to -M).$$

The second integral may be considerably simplified. Let $r^2 = 2\, eH \cdot n$ $\omega = \sqrt{\mu^2 + p^2}$, $\Omega = \sqrt{\mu^2 + p^2}$, $p^2 = p_3{}^2 + r^2$, then

$$\int_{-\infty}^{\infty} dp_3 \int_0^\infty dn \frac{e H_0}{E_n \varepsilon_n (E_n + \varepsilon_n - M)^2} = \int_{-\infty}^{\infty} dp_3 \int_0^\infty r\, dr \frac{1}{\Omega \omega (\omega + \Omega - M)^2}$$

$$= \frac{1}{2\pi} \int d^3 p \frac{1}{\omega \Omega (\omega + \Omega - M)^2} = 2 \int_0^\infty dp \frac{p^2}{\omega \Omega (\omega + \Omega - M)^2}$$

$$= \int_{-\infty}^{\infty} dp \frac{p^2}{\omega \Omega (\omega + \Omega - M)^2}.$$

Therefore

$$E_N^c = \frac{(e H_0) g^2}{4\pi} \int_{-\infty}^{\infty} dp \left\{ \frac{1}{\Omega (\omega + \Omega - M)} - \frac{p^2}{\Omega \omega (\omega + \Omega - M)^2} \right\} - (M \to -M)$$

$$= \frac{(e H_0) g^2}{\pi} \int_0^\infty dp \left\{ \frac{1}{\Omega ((\omega + \Omega)^2 - M^2)} - \frac{p^2}{\omega \Omega} \frac{\omega + \Omega}{((\omega + \Omega)^2 - M^2)^2} \right\}. \qquad (23)$$

The integral in (23) is elementary, and may be reduced to known form by the same substition that was used in (12). The result is

$$E_N^c = \frac{(e H_0) g^2}{2\pi} \frac{1}{M} \left(1 - \frac{\delta (2 - \delta^2)}{\sqrt{4 - \delta^2}} \cos^{-1} \frac{\delta}{2} + \delta^2 \log \frac{1}{\delta} \right)$$

$$\mu_N^c = -\frac{g^2}{\pi} \left(1 - \frac{\delta (2 - \delta^2)}{\sqrt{4 - \delta^2}} \cos^{-1} \frac{\delta}{2} + \delta^2 \log \frac{1}{\delta} \right) \qquad (24)$$

*

Proceeding to E_P^c, we may write (22) as

$$E_P^c = +2\pi g^2 \sum_p \sum_{N=0}^{\infty} \left\{ \frac{1}{\Omega(\Omega+\varepsilon_{N+\frac{1}{2}}-M)} \right\} \frac{\xi^N}{N!} e^{-\xi} - (M \to -M)$$

$$= 2\pi g^2 \sum_p \sum_{N=0}^{\infty} \left\{ \frac{1}{\Omega(\Omega+\varepsilon_N-M)} - \frac{eH_0}{2\varepsilon_N} \frac{1}{(\Omega+\varepsilon_N-M)^2} \right\} \frac{\xi^N}{N!} e^{-\xi}$$

$$- (M \to -M) + O(H_0^2).$$

Define a function

$$g(N) = \frac{1}{\varepsilon_N + \Omega - M} = g(\xi) + (N-\xi) g'(\xi) + \frac{(N-\xi)^2}{2} g''(\xi) + \cdots$$

Then, after dropping the self energy terms and terms in H_0^2, we get

$$E_P^c = 2\pi g^2 \sum_p \frac{1}{\Omega} \left\{ \frac{\xi}{2} g''(\xi) - \frac{eH_0}{2\omega} \frac{1}{(\omega+\Omega-M)^2} \right\} - (M \to -M).$$

Carrying out the x differentiation yields:

$$E_P^c = \pi g^2 (eH_0) \sum_p \frac{1}{\Omega} \left\{ \frac{p_1{}^2 + p_2{}^2}{2(\omega+\Omega-M)^2 \omega^2} \left(\frac{1}{\omega} + \frac{2}{\omega+\Omega-M} \right) - \frac{1}{\omega} \frac{1}{(\omega+\Omega-M)^2} \right\}$$
$$- (M \to -M)$$

$$= \pi g^2 (eH_0) \sum_p \frac{1}{\Omega} \left\{ \frac{p^2}{3(\omega+\Omega-M)^2 \omega^2} \left(\frac{1}{\omega} + \frac{2}{\omega+\Omega-M} \right) - \frac{1}{\omega} \frac{1}{(\omega+\Omega-M)^2} \right\}$$
$$- (M \to -M)$$

$$= \frac{g^2(eH_0)}{2\pi} \int_0^{\infty} dp \frac{p^2}{\Omega\omega} \left[\frac{2}{3} \frac{p^2}{\omega(\omega+\Omega-M)^3} + \frac{1}{(\omega+\Omega-M)^2} \left(\frac{1}{3} \frac{p^2}{\omega^2} - 1 \right) \right]$$
$$- (M \to -M)$$

$$= \frac{2g^2(eH_0)}{\pi} \int_0^{\infty} dp \frac{p^2}{\omega\Omega} \left[\frac{1}{3} \frac{p^2[3(\omega+\Omega)^2+M^2]}{\omega((\omega+\Omega)^2-M^2)^3} - \frac{(\Omega+\omega)}{((\omega+\Omega)^2-M^2)^2} \left(1 - \frac{1}{3} \frac{\omega^2}{p^2} \right) \right] \tag{25}$$

Just as in (12) and (23) the integral of (25) is elementary. One obtains

$$E_P^c = -\frac{g^2(eH_0)}{2M\pi} \left(\frac{1}{2} - \delta^2 - \frac{\delta(2-4\delta^2+\delta^4)}{\sqrt{4-\delta^2}} \cos^{-1}\frac{\delta}{2} + \delta^2(2-\delta^2)\log\frac{1}{\delta} \right)$$

$$\mu_P^c = \frac{g^2}{\pi} \left(\frac{1}{2} - \delta^2 - \frac{\delta(2-4\delta^2+\delta^4)}{\sqrt{4-\delta^2}} \cos^{-1}\frac{\delta}{2} + \delta^2(2-\delta^2)\log\frac{1}{\delta} \right). \tag{26}$$

IV. The Pseudovector Coupling.

With this coupling the interactions L' and L'_c are replaced by

$$L' \text{ (pseudovector coupling)} = -\sqrt{4\pi}\, f \left(\psi_p^* \left(\sigma \cdot \nabla \Phi - \gamma_5 \frac{\partial \Phi}{\partial t}\right)\psi_p\right)$$

$$L'_c \text{(pseudovector coupling)} = -\sqrt{8\pi}\, f \Big[\left(\psi_p^* \left[\sigma \cdot (\nabla - i e A_0)\, \Phi - \gamma_5 \frac{\partial \Phi}{\partial t}\right] \psi_N\right) + \left(\psi_N^* \left[\sigma \cdot (\nabla + (i e A_0)\, \Phi^* - \gamma_5 \frac{\partial \Phi^*}{\partial t}\right] \psi_p\right)\Big]$$

When the coupling constants are related by $g^2 = (2\,Mf)^2$, then the two types of coupling give identical results. The values usually given in the literature[1]) are $(fM)^2 \sim 9$, for the meson mass equal to either 200 or 325 electron masses. These values are extremely tentative, however, and possibly should be subject to suspicion even in so far as order of magnitude goes.

V. Summary of Results[2]).

Neutral Mesons.

$$\mu_N = 0$$

$$\mu_p = -\frac{g^2}{\pi}\left(\frac{1}{4} + \frac{\delta^2}{2} - \frac{\delta^3(3-\delta^2)}{2\sqrt{4-\delta^2}} \cos^{-1}\frac{\delta}{2} - \frac{\delta^2}{2}(1-\delta^2)\log\frac{1}{\delta}\right).$$

Charged Mesons.

$$\mu_N^c = -\frac{g^2}{\pi}\left(1 - \frac{\delta(2-\delta^2)}{\sqrt{4-\delta^2}} \cos^{-1}\frac{\delta}{2} + \delta^2 \log\frac{1}{\delta}\right).$$

$$\mu_P^c = \frac{g^2}{\pi}\left(\frac{1}{2} - \delta^2 - \frac{\delta(2-4\,\delta^2+\delta^2)}{\sqrt{4-\delta^2}} \cos^{-1}\frac{\delta}{2} + \delta^2(2-\delta^2)\log\frac{1}{\delta}\right).$$

Symmetrical Theory.

$$\mu_N^s = \mu_N^c$$

$$\mu_P^s = \mu_P^c + \mu_P\,.$$

The units are those of the nuclear magneton, and δ is the ratio of meson to proton mass. When the above numerical values for the coupling constant and meson mass are used, one obtains:

A) $\mu \sim 325\; m$ *(electron)*.

$$\mu_N = 0 \qquad \mu_N^c \sim -8 \qquad \mu_N^s \sim -8$$

$$\mu_P \sim -2.8 \qquad \mu_P^c \sim 2.5 \qquad \mu_P^s \sim -.3\,.$$

[1]) F. Villars, Helv. Phys. Acta, XX, 476 (1947).

[2]) These results satisfy the relationship $\mu_N^c + \mu_P^c = 2\,\mu_N^N$, which has a general validity in the g^2-approximation. (The factor 2 appears because we have choosen the coupling constant differently for charged and neutral theory.)

B) $\mu \sim$ *200 m (electron).*

$$\mu_N = 0 \qquad \mu^c_N \sim -9.3 \qquad \mu^s_N \sim -9.3$$
$$\mu_P \sim -2.8 \qquad \mu^c_P \sim 3.6 \qquad \mu^s_P \sim .8\,.$$

$$\mu_N \text{ (experimental)} = -1.9103$$
$$\mu_P \text{ (experimental)} = 1.7896$$

It is clear that none of the theories give agreement with experiment. More disturbing than the incorrectness of the absolute values is that of the ratio of the two moments. This is independent of the value of the coupling constant, and depends only on the ratio δ. It seems impossible to fit the data with any value of δ. Whether the error lies in the model (pseudoscalar mesons) or in the use of perturbation theory remains an open question.

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