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Higher Approximations in the external field for the Problem of Vacuum Polarization

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Summary. The convergence and gauge invariance of the integrals appearing in the higher approximations of vacuum polarization are here discussed for the cases of spinor and scalar fields. Only the comparatively simple case of an external electromagnetic field is treated. The more complicated problem of a quantizated field is not discussed.

As a direct result of the charge symmetry the vacuum expectation values of the current commutators vanish if an odd number of current operators are commuted. Hence the terms proportional to e^{2n+1} are identically zero, and in this form the statement is true for both the electrons and the bosons. Of the remaining terms it is shown that only the e^2 and e^4 approximations diverge, but that the still higher terms are both convergent and gauge invariant. It further appears that, apart from a numerical factor, which is the same in both approximations (and equal to $-\frac{1}{2}$), the strongest divergences are the same for the spinor and the scalar fields. The e^2 approximation has previously been treated by Jost and Rayski, who have shown that the non-gauge invariant (and divergent) terms compensate each other if one uses a suitable mixture of spinor and scalar fields. In this approximation, however, the logaritmically divergent charge renormalisation remains. The conditions of Jost and Rayski are

$$N = 2n \tag{I}$$

$$\sum_{i=1}^{N} M_i^2 = 2 \sum_{i=1}^{n} m_i^2 \tag{II}$$

where N and n are the number of scalar and spinor fields respectively and M_i and m_i are the corresponding masses. In the e^4 approximation the condition (I) alone is sufficient to secure a convergent and gauge invariante result.

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Introduction.

The e^2 approximation of the vacuum polarization has already been treated by several authors[1]) with the aid of the explicitely relativistically invariant quantum dynamics developed by SCHWINGER[2]) and others. It has been shown by WENTZEL[1]) that it is possible to obtain different results for the photon self energy (which can be considered as a special case of the more general problem of the vacuum polarization) if one uses different methods of computing the integrals which appear in the formulae for the vacuum polarization. This question of uniqueness has been further discussed by PAULI and VILLARS[1]). In their paper the latter authors have given a method of invariant regularization of the different Δ-functions with the help of auxiliary masses. These masses, however, are only regarded as a mathematical aid for the computation and in the final result they are allowed to tend to infinity. It has been observed independently by RAYSKI and UMEZAWA[3]) that it is also possible to regard these masses as observable if one assumes that the corresponding particles obey Bose-statistics. Detailed calculations by JOST and RAYSKI[1]) in the e^2 approximation have given as a result that the necessary assumptions in these realistic theories show a remarkable analogy with the more formalistical conditions of PAULI and VILLARS. We wish to extend the work of JOST and RAYSKI to the higher approximations in the fine-structure-constant and have as general equations

$$i \frac{\delta \psi[\sigma]}{\delta \sigma(x)} = H(x)\, \psi[\sigma] \tag{1}$$

$$H(x) = H_F(x) + H_B(x) \tag{1a}$$

$$H_F(x) = -\sum j_{\mu_F}(x)\, A_\mu(x) \tag{2}$$

$$j_{\mu_F}(x) = \frac{i\,e}{2}\left[\bar{\psi}(x)\,\gamma_\mu\,\psi(x) - \psi(x)\,\gamma_\mu^T\,\bar{\psi}(x)\right] \tag{2a}$$

$$H_B(x) = -\sum t_\mu(x)\, A_\mu(x) \tag{3}$$

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[1]) E. g. J. SCHWINGER, Phys. Rev. **75**, 651 (1949); G. WENTZEL, Phys. Rev. **74**, 1070 (1948); W. PAULI, F. VILLARS, Rev. Mod. Phys. **21**, 434 (1949); R. JOST, J. RAYSKI, Helv. Phys. Acta **22**, 457 (1949). For earlier work without use of invariant formalism a summary has been given by V. WEISSKOPF, Det. Kgl. Danske Vid. Selskab, XIV 6 (1936).

[2]) TOMONAGA, Progr. Theor. Phys. **1**, 27 (1946); J. SCHWINGER, Phys. Rev. **74**, 1439 (1948); ibid. **75**, 651 (1949); ibid. **76**, 790 (1949); F. J. DYSON: Phys. Rev. **75**, 486 (1949).

[3]) J. RAYSKI, Acta Physica Polonica **9**, 129 (1948); H. UMEZAWA, J. YUKAWA, E. YAMADA, Progr. Theor. Phys. **3**, 317 (1948).

$$t_\mu(x) = i\,e\,s_\mu(x) + \xi_{\mu\nu} A_\nu(x) \cdot e^2 \cdot \varphi^*(x)\,\varphi(x) \tag{3a}$$

$$s_\mu(x) = \frac{\partial \varphi^*(x)}{\partial x_\mu}\varphi(x) - \frac{\partial \varphi(x)}{\partial x_\mu}\varphi^*(x) \tag{3b}$$

$$\xi_{\mu\nu} = \delta_{4\mu} \cdot \delta_{4\nu} - \delta_{\mu\nu} \tag{3c}$$

Here[4]) $\psi(x)$ and $\overline{\psi}(x) = \psi^*(x)\,\gamma_4$ are the spinor fields and $\varphi(x)$ and $\varphi^*(x)$ the scalar fields[5]). The symbols $t_\mu(x)$, $s_\mu(x)$ and $\xi_{\mu\nu}$ are defined by equations (3a) (3b) and (3c). The current operator for the scalar field is given by[6])

$$j_{\mu_B}(x) = i\,e\,s_\mu(x) + 2\,\xi_{\mu\nu} A_\nu(x) \cdot e^2 \varphi^*(x)\,\varphi(x) \tag{4}$$

The total current is

$$j_\mu(x) = \sum j_{\mu_F}(x) + \sum j_{\mu_B}(x) \tag{4a}$$

It may be observed that none of the expressions (3a), (3b) or (3c) are tensors and that only the current $j_\mu(x)$ is a vector. The $A_\mu(x)$ are the four-dimensional vector potentials for the external electromagnetic field. They are here considered as given functions of space and time and not as operators. Hence we neglect the modification of the electromagnetic field due to polarization phenomena. The summations in equations (2) (3) and (4a) are to be extended over the spinor and scalar fields present.

The operators $\psi(x)$ and $\varphi(x)$ satisfy the following relations[4])

$$\{\psi_\alpha(x);\ \overline{\psi}_\beta(x')\} = -i\,S_{\alpha\beta}(x-x') \tag{5}$$

$$\{\psi_\alpha(x);\ \psi_\beta(x')\} = \{\overline{\psi}_\alpha(x);\ \overline{\psi}_\beta(x')\} = 0 \tag{5a}$$

$$[\varphi^*(x);\ \varphi(x')] = i\,\Delta(x-x') \tag{6}$$

$$[\varphi^*(x);\ \varphi^*(x')] = [\varphi(x);\ \varphi(x')] = 0 \tag{6a}$$

Different kinds of fields always commute.

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[4]) The notations are essentially the same as those used by SCHWINGER but with natural units ($\hbar = c = 1$).

[5]) Computations by FELDMANN have shown that it is not possible to compensate also the divergent charge renormalisation term by including fields with spin 1 (Unpublished letter to professor PAULI). The same result has also been gotten by H. UMEZAWA, R. KAWABE, Prog. Theor. Phys. (in press).

[6]) G. WENTZEL: Quantentheorie der Wellenfelder.

We want to study the modified current operator, the expectation value of which may be written as[2])

$$\langle j_\mu(x) \rangle = \sum_{n=0}^{\infty} (i)^n \int_{-\infty}^{\sigma} d\,x' \int_{-\infty}^{\sigma'} d\,x'' \cdots \int_{-\infty}^{\sigma^{(n-1)}} d\,x^n \times$$
$$\times \langle \big[H(x^n) \left[\cdots \left[H(x');\; j_\mu(x) \right] \cdots \right] \big] \rangle_0 \tag{7}$$

If the expressions (1a)—(4a) are substituted in equation (7) one gets a sum of commutators, some of which contains only one kind of fields (one of the $\psi(x)$ or one of the $\varphi(x)$ fields). The other terms contain at least one commutator between two different fields and are hence zero. This fact makes it possible to carry through the calculation for each field separately and in the end simply add the results together.

The Spinor fields.

We consider now only one of the spinor fields and write the corresponding parts of the current operator and of the hamiltonian as

$$j_\mu(x) = \frac{i\,e}{2} [\bar{\psi}(x);\; \gamma_\mu \psi(x)] \tag{8}$$

$$H(x) = -j_\mu(x)\, A_\mu(x) \tag{9}$$

In this case equation (7) gives

$$\langle j_\mu(x) \rangle = \sum_{n=0}^{\infty} i^{-n} \int_{-\infty}^{\sigma} d\,x' \cdots \int_{-\infty}^{\sigma^{(n-1)}} d\,x^n A_{\nu_1}(x') \cdots A_{\nu_n}(x^n) \times$$
$$\times \langle \big[j_{\nu_n}(x^n) \left[\cdots \left[j_{\nu_1}(x');\; j_\mu(x) \right] \cdots \right] \big] \rangle_0 \tag{10}$$

and our first task is to evaluate the iterated current commutators in this expression. If n is an even integer we have a commutator of an odd number of current operators, that is an expression of the following form

$$\left(\frac{i\,e}{2}\right)^{2n+1} \langle \big[[\bar{\psi}(x^{2n});\; \gamma_{\nu_{2n}} \psi(x^{2n})] \big[\cdots [[\bar{\psi}(x');\; \gamma_{\nu_1} \psi(x')];$$
$$[\bar{\psi}(x);\; \gamma_\mu \psi(x)]] \cdots \big] \big] \rangle_0 \tag{11}$$

If we use the charge conjugate spinor $\psi'(x)$ equation (8) might as well be written as

$$j_\mu(x) = -\frac{i\,e}{2} [\bar{\psi}'(x);\; \gamma_\mu \psi'(x)] \tag{12}$$

and equation (11) as

$$-\left(\frac{ie}{2}\right)^{2n+1}\langle\Big[[\bar{\psi}'(x^{2n});\ \gamma_{\nu_{2n}}\psi'(x^{2n})]\Big[\cdots[[\bar{\psi}'(x');\ \gamma_{\nu_1}\psi'(x')];\ [\bar{\psi}'(x);\ \gamma_\mu\psi'(x)]]\cdots\Big]\Big]\rangle_0 \qquad (13)$$

As the two vaccum expectation values in (11) and (13) are equal we conclude that they are both zero and that equation (10) may be simplified to

$$\langle j_\mu(x)\rangle = \sum_{n=0}^{\infty} i^{2n-1}\int_{-\infty}^{\sigma} dx'\cdots\int_{-\infty}^{\sigma^{2n}} dx^{2n+1} A_{\nu_1}(x')\cdots A_{\nu_{2n+1}}(x^{2n+1})\times$$
$$\times\langle\Big[j_{\nu_{2n+1}}(x^{2n+1})\big[\cdots[j_{\nu_1}(x');\ j_\mu(x)]\cdots\big]\Big]\rangle_0 \qquad (14)$$

If Ω is an arbitrary operator we have

$$[[\bar{\psi}(x);\ \Omega\,\psi(x')]\bar{\psi}_\alpha(x'')] = (\bar{\psi}(x)\,\Omega)_\beta\{\psi_\beta(x');\ \bar{\psi}_\alpha(x'')\} +$$
$$+\{\bar{\psi}_\alpha(x'');\ \psi_\beta(x')\}(\Omega^T\,\bar{\psi}(x))_\beta = -2i\,(\bar{\psi}(x)\,\Omega S(x'-x''))_\alpha \qquad (15)$$

In a similar way we get

$$[[\bar{\psi}(x);\ \Omega\,\psi(x')]\,\psi_\alpha(x'')] = 2i\,(S(x''-x)\,\Omega\,\psi(x'))_\alpha \qquad (16)$$

and hence

$$[[\bar{\psi}(x);\ \Omega\,\psi(x')]\,j_\mu(x'')] = \frac{ie}{2}\Big([[\bar{\psi}(x);\ \Omega\,\psi(x')]\,\bar{\psi}_\alpha(x'')]\,(\gamma_\mu\,\psi(x''))_\alpha +$$
$$+(\bar{\psi}(x'')\,\gamma_\mu)_\alpha[[\bar{\psi}(x);\ \Omega\,\psi(x')]\,\psi_\alpha(x'')] - [[\bar{\psi}(x);\ \Omega\,\psi(x')]\,\psi_\alpha(x'')]\times$$
$$\times(\gamma_\mu^T\bar{\psi}(x''))_\alpha - (\psi(x'')\,\gamma_\mu^T)_\alpha\,[[\bar{\psi}(x);\ \Omega\,\psi(x')]\,\bar{\psi}_\alpha(x'')\Big) =$$
$$= e\,\Big([\bar{\psi}(x);\ \Omega S(x'-x'')\,\gamma_\mu\,\psi(x'')] -$$
$$-[\bar{\psi}(x'');\ \gamma_\mu S(x''-x)\,\Omega\,\psi(x')]\Big). \qquad (17)$$

From equation (17) it is imidiately seen that the commutator

$$\Big[j_{\nu_n}(x^n)\,[\cdots[j_{\nu_1}(x')\,j_\mu(x)]\cdots]\Big]$$

consists of 2^{n+1} terms of the form

$$\pm\frac{i\,e^{n+1}}{2}[\bar{\psi}(x^i);\ \gamma_{\nu_i}S(x^i-x^j)\cdots\gamma_{\nu_r}S(x^r-x^s)\,\gamma_{\nu_s}\,\psi(x^s)] \qquad (18)$$

where i; ;... r; s is some permutation of the numbers 0; 1;... n. With the aid of the formula

$$\langle[\bar{\psi}_\alpha(x);\ \psi_\beta(x')]\rangle_0 = S^{(1)}_{\beta\alpha}(x'-x)$$

the vaccum expectation value of (18) is given by

$$\pm \frac{i\, e^{n+1}}{2} Sp[S^{(1)}(x^s - x^i)\gamma_{\nu_i} S(x^i - x^j)\cdots \gamma_{\nu_s} S(x^r - x^s)\gamma_{\nu_s}] \tag{19}$$

Hence each term in the series (14) may be written as a sum of 2^{2n+2} terms of the form

$$(-1)^n \left(\pm \frac{e^{2n+2}}{2}\right) \int\limits_{-\infty}^{\sigma} d\,x' \cdots \int\limits_{-\infty}^{\sigma^{2n}} d\,x^{2n+1}\, Sp[S^{(1)}(x^s - x^i)\gamma A(x^i) S(x^i - x^j) \cdots \gamma A(x^r) S(x^r - x^s)\gamma A(x^s)] \tag{20}$$

If we suppose that the external field does not allow any real pairs to be created we can transform the expression (20) into

$$(-1)^n \left(\pm \left(\frac{e}{2}\right)^{2n+2}\right) \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} d\,x' \cdots d\,x^{2n+1}\; \varepsilon(01)\,\varepsilon(12)\cdots \cdots \varepsilon(2n; 2n+1)\; Sp[S^{(1)}(s\,i)\gamma A(i)\cdots S(r\,s)\gamma A(s)]. \tag{21}$$

After changing the notations and rearrenging the terms in the traces it is always possible to write each term in equation (14) as

$$(-1)^n \left(\frac{e}{2}\right)^{2n+2} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} d\,x' \cdots d\,x^{2n+1} \big(Sp[\gamma_\mu S^{(1)}(01)\gamma A(1) S(12)\cdots \cdots S(2n+1,0)] E_1^{(2n+1)} + Sp[\gamma_\mu S(01)\gamma A(1) S^{(1)}(12)\cdots \cdots S(2n+1,0)] E_2^{(2n+1)} + \cdots + Sp[\gamma_\mu S(01)\gamma A(1) S(12)\cdots \cdots S^{(1)}(2n+1,0)] E_{2n+2}^{(2n+1)}\big) \tag{22}$$

where $E_1^{(2n+1)}$; $E_2^{(2n+1)}$; $\cdots$ $E_{2n+2}^{(2n+1)}$ are some functions of the ε (i, j). It will be proven in the appendix that the expression (22) may be written as

$$(-1)^n \frac{e^{2n+2}}{2} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} d\,x' \cdots dx^{2n+1} \big((Sp[\gamma_\mu S^{(1)}(01)\gamma A(1)\overline{S}(12)\cdots \cdots \overline{S}(2n+1,0)] + Sp[\gamma_\mu \overline{S}(01)\gamma A(1) S^{(1)}(12)\cdots \overline{S}(2n+1,0)]\cdots + \cdots + Sp[\gamma_\mu \overline{S}(01)\gamma A(1)\overline{S}(12)\cdots S^{(1)}(2n+1,0)]\big) \tag{23}$$

and hence

$$\langle j_\mu(x)\rangle = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n e^{2n+2} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} d\,x' \cdots d\,x^{2n+1} \big(Sp[\gamma_\mu S^{(1)}(01) \times \times \gamma A(1)\cdots \overline{S}(2n+1,0)] + \cdots + Sp[\gamma_\mu \overline{S}(01)\gamma A(1)\cdots S^{(1)}(2n+1,0)] \tag{24}$$

In momentum space this formula reads

$$\langle j_\mu(x) \rangle = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n e^{2n+2} \left(\frac{1}{2\pi}\right)^{8n+7} \int \cdots \int d p\, d p' \cdots d p^{2n+1} \times$$
$$\times e^{ix(p - p^{2n+1})} Sp[\gamma_\mu (i\gamma p - m) \gamma A (p' - p) (i \gamma p' - m) \cdots$$
$$\cdots \gamma A (p^{2n+1} - p^{2n}) (i\gamma p^{2n+1} - m)] \left(\frac{\delta(p^2 + m^2)}{(p'^2 + m^2) \cdots (p^{2n+1^2} + m^2)} + \right.$$
$$\left. + \frac{\delta(p'^2 + m^2)}{(p^2 + m^2)(p''^2 + m^2) \cdots (p^{2n+1^2} + m^2)} + \cdots + \frac{\delta(p^{2n+1^2} + m^2)}{(p^2 + m^2) \cdots (p^{2n^2} + m^2)} \right) \tag{25}$$

where

$$A\nu(p) = \int A\nu(x)\, e^{ipx} d x \tag{26}$$

The equations (24) and (25) give a formal expression for the expectation value of the current operator, but the questions of convergence and gauge invariance are still open.

We first want to give a formal proof of the gauge invariance in momentum space. It is true that this proof is only valid if the integrals converge but we will leave this question unsettled for the moment.

For this calculation we need the following formula

$$Sp[\Omega(i\gamma a - m)\gamma q(i\gamma(a+q) - m)] = i[(a+q)^2 + m^2] Sp[\Omega(i\gamma a - m)] -$$
$$- i[a^2 + m^2] Sp[\Omega(i\gamma(a+q) - m)] \tag{27}$$

In equation (27) a and q are two arbitrary fourdimensional vectors, m a number and Ω an arbitrary operator. The formula (27) may be proven by an explicite calculation e. g. in the following way

$$Sp[\Omega(i\gamma a - m)\gamma q(i\gamma(a+q) - m)] = - Sp[\Omega \cdot \gamma a \cdot \gamma q \cdot \gamma(a+q)] -$$
$$- i m (Sp[\Omega \cdot \gamma q \cdot \gamma (a+q)] + Sp[\Omega \cdot \gamma a \cdot \gamma q]) + m^2 Sp[\Omega] \tag{28}$$

Using the equation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\, \delta_{\mu\nu}$$

we get from (28)

$$- 2 q a Sp[\Omega \cdot \gamma a] + a^2 Sp[\Omega \cdot \gamma q] - q^2 Sp[\Omega \cdot \gamma a] -$$
$$- m i (q^2 + 2 a q) Sp[\Omega] + m^2 Sp[\Omega] = i[(a+q)^2 + m^2] \times$$
$$\times Sp[\Omega(i\gamma a - m)] - i(a^2 + m^2) Sp[\Omega(i\gamma(a+q) - m)]$$

which is the right hand side of (27).

One typical term in (25) may be written as

$$\int \cdots \int d p \, d p' \cdots d p^n \, e^{i x (p - p^n)} \, Sp \left[\gamma_\mu (i \gamma p - m) \gamma A (p' - p) \cdots \right.$$
$$\left. \cdots (i \gamma p^n - m) \right] \frac{\delta (p^{i^2} + m^2)}{(p^2 + m^2) \cdots (p^{i-1^2} + m^2)\,(p^{i+1^2} + m^2) \cdots (p^{n^2} + m^2)} \quad (29)$$

(In (29) n is supposed to be odd but for our present purpose we do not need this fact.)

If we make an infinitesimal gauge transformation

$$\left.\begin{aligned} A\nu(x) &\to A\nu(x) + \frac{\partial \Lambda(x)}{\partial x_\nu} \\ \varepsilon(p) &= \int e^{i x p} \Lambda(x) \, d x \end{aligned}\right\} \quad (30)$$

the integral (29) will be changed by the following amount

$$\frac{1}{i} \int \cdots \int d p \cdots d p^n e^{i x (p - p^n)} \frac{\delta (p^{i^2} + m^2)}{(p^2 + m^2) \cdots (p^{i-1^2} + m^2)\,(p^{i+1^2} + m^2) \cdots (p^{n^2} + m^2)} \times$$
$$\times \left(Sp \left[\gamma_\mu (i \gamma p - m) \gamma (p' - p) (i \gamma p' - m) \gamma A (p'' - p') \cdots \right] \times \right.$$
$$\times \varepsilon (p' - p) + Sp \left[\gamma_\mu (i \gamma p - m) \gamma A (p' - p) (i \gamma p' - m) \times \right.$$
$$\left. \times \gamma (p'' - p') \cdots \right] \varepsilon (p'' - p') + \cdots \Big) \quad (31)$$

Incerting equation (27) in formula (31) we get

$$\int \cdots \int d p \cdots d p^n e^{i x (p - p^n)} \frac{\delta (p^{i2} + m^2)}{(p^2 + m^2) \cdots (p^{i-1^2} + m^2)\,(p^{i+1^2} + m^2) \cdots (p^{n^2} + m^2)} \times$$
$$\times \left[(p'^2 + m^2) Sp \left[\gamma_\mu (i \gamma p - m) \gamma A (p'' - p') \cdots \right] \varepsilon (p' - p) - \right.$$
$$- (p^2 + m^2) Sp \left[\gamma_\mu (i \gamma p' - m) \gamma A (p' - p) \cdots \right] \varepsilon (p' - p) +$$
$$+ (p''^2 + m^2) Sp \left[\gamma_\mu (i \gamma p - m) \gamma A (p' - p) \cdots \right] \times \varepsilon (p'' - p') -$$
$$\left. - (p'^2 + m^2) Sp \left[\gamma_\mu (i \gamma p - m) \gamma A (p' - p) \cdots \right] \varepsilon (p'' - p') + \cdots \right] \quad (32)$$

With the notation

$$\int d p' A_\nu (p' - p) \, \varepsilon (p' - p'') = \int d q \, \varepsilon (q) \, A_\nu (p'' - p - q) = E_\nu (p'' - p) \quad (33)$$

and the formula

$$\int d p \, \varepsilon (p' - p) \, e^{i x p} = \left(\frac{1}{2 \pi} \right)^4 \Lambda (x) \, e^{i x p'} \quad (34)$$

the expression (32) may be written (the terms containing $p^{i2} + m^2$ give zero due to the factor $\delta\,(p^{i2} + m^2)$).

$$\int \cdots \int d p' \cdots d p^n e^{i x (p'-p'')} \frac{\delta\,(p^{i2}+m^2)}{(p'^2+m^2)\cdots(p^{i-1\,2}+m^2)\,(p^{i+1\,2}+m^2)\cdots(p^{n2}+m^2)} \times$$
$$\times \left[Sp\left[\gamma_\mu(i\gamma p'-m)\gamma E(p''-p')\,(i\gamma p''-m)\cdots\right]-\left(\frac{1}{2\pi}\right)^4 A\,(x) \times\right.$$
$$\times Sp[\gamma_\mu\,(i\gamma p'-m)\,\gamma\,A\,(p''-p')\,(i\gamma p''-m)\cdots]+$$
$$+ Sp[\gamma_\mu\,(i\gamma p'-m)\gamma\,A\,(p''-p')\,(i\gamma p''-m)\,\gamma E\,(p'''-p'')\cdots]-$$
$$- Sp[\gamma_\mu(i\gamma p'-m)\gamma E\,(p''-p')\,(i\gamma p''-m)\cdots]+\cdots$$
$$\left.\cdots+\left(\frac{1}{2\pi}\right)^4 A\,(x)\,Sp[\gamma_\mu\,(i\gamma p'-m)\gamma\,A\,(p''-p')\cdots]\right]=0\,. \qquad (35)$$

From (35) we conclude that each term in (25) if convergent is also gauge invariant.

To discuss the convergence properties of (25) we write the bracket as

$$\frac{\delta\,(p^2+m^2)}{(p'^2-p^2)\cdots(p^{n2}-p^2)}+\frac{\delta\,(p'^2+m^2)}{(p^2-p'^2)\,(p''^2-p'^2)\cdots(p^{n2}-p'^2)}+\cdots$$
$$\cdots+\frac{\delta\,(p^{n2}+m^2)}{(p^2-p^{n2})\cdots(p^{n-1\,2}-p^{n2})} \qquad (36)$$

We now use the formula

$$\frac{1}{(p'^2-p^2)\cdots(p^{n2}-p^2)}+\frac{1}{(p^2-p'^2)\cdots(p^{n2}-p'^2)}+\cdots$$
$$\cdots+\frac{1}{(p^2-p^{n2})\cdots(p^{n-1\,2}-p^{n2})}=0 \qquad (37)$$

to write (36) as

$$\frac{\delta\,(p'^2+m^2)-\delta\,(p^2+m^2)}{(p^2-p'^2)\,(p''^2-p'^2)\cdots(p^{n2}-p'^2)}+\cdots+\frac{\delta\,(p^{n2}+m^2)-\delta\,(p^2+m^2)}{(p^2-p^{n2})\cdots(p^{n-1\,2}-p^{n2})} \qquad (38)$$

Following Schwinger this can also be written

$$-\frac{1}{2}\int\limits_{-1}^{+1} d u \left[\frac{\delta'\left(\frac{1}{2}\,(p^2+p'^2)+m^2+\frac{1}{2}\,u\,(p^2-p'^2)\right)}{(p''^2-p'^2)\cdots(p^{n2}-p'^2)}+\cdots\right.$$
$$\left.\cdots+\frac{\delta'\left(\frac{1}{2}\,(p^2+p^{n2})+m^2+\frac{1}{2}\,u\,(p^2-p^{n2})\right)}{(p'^2-p^{n2})\cdots(p^{n-1\,2}-p^{n2})}\right] \qquad (39)$$

By repeating this, process it is obviously possible to write (36) in the following form

$$\left(-\frac{1}{2}\right)^{\frac{n(n+1)}{2}} \int_{-1}^{+1} d u_1 (1-u_1)^{n-1} \int_{-1}^{+1} d u_2 (1-u_2)^{n-2} \cdots \int_{-1}^{+1} d u_n \, \delta^{(n)} \Big(m^2 + \\ + \frac{p^2}{2} + \frac{p'^2}{4} \cdots + \frac{p^{n-1\,2}}{2^n} + \frac{p^{n\,2}}{2^n} - u_1 \Big(-\frac{p^2}{2} + \frac{p'^2}{4} + \frac{p''^2}{8} + \cdots + \\ + \frac{p^{n-1\,2}}{2^n} + \frac{p^{n\,2}}{2^n} \Big) - u_2 (1-u_1) \times \Big(-\frac{p'^2}{4} + \frac{p''^2}{8} + \cdots + \frac{p^{n-1\,2}}{2^n} + \frac{p^{n\,2}}{2^n} \Big) - \\ - \cdots - u_n (1-u_{n-1}) \cdots (1-u_1) \Big(-\frac{p^{n-1\,2}}{2^n} + \frac{p^{n\,2}}{2^n} \Big) \Big) \tag{40}$$

It is now convenient to make a translation of the origins and write (25) in the following way

$$\langle j_\mu (x) \rangle = \frac{1}{2} \sum_{n=0}^{\infty} e^{2n+2} (-1)^n \left(\frac{1}{2\pi}\right)^{8n+7} \int \cdots \int d q' \cdots d q^{2n+1} \times \\ \times e^{-ix(q' + \cdots + q^{2n+1})} K^{(2n+1)}_{\mu \nu_1 \cdots \nu_{2n+1}} (q' \cdots q^{2n+1}) A_{\nu_1}(q') \cdots \\ \cdots A_{\nu_{2n+1}}(q^{2n+1}) \tag{41}$$

where the kernel $K^{(n)}_{\mu \cdots \nu_n}$ according to (40) is given by

$$K^{(n)}_{\mu \nu_1 \cdots \nu_n} (q' \cdots q^n) = \left(-\frac{1}{2}\right)^{\frac{n(n+1)}{2}} \int d p \int_{-1}^{+1} d u_1 (1-u_1)^{n-1} \int_{-1}^{+1} d u_2 \times \\ \times (1-u_2)^{n-2} \cdots \int_{-1}^{+1} d u_n \, \delta^{(n)} (m^2 + p^2 + 2 p Q + Q^2 + \varphi) \, Sp \, [\gamma_\mu \times \\ \times (i \gamma p - m) \gamma_{\nu_1} (i \gamma (p + q') - m) \gamma_{\nu_2} \times (i \gamma (p + q' + q'') - m) \cdots \\ \cdots \gamma_{\nu_n} (i \gamma (p + q' + \cdots + q^n) - m)] \tag{42}$$

Here

$$Q = (1-u_1) \frac{q'}{2} + (1-u_1)(1-u_2) \frac{q''}{4} + \cdots + (1-u_1) \cdots (1-u_n) \frac{q^n}{2^n} \tag{43}$$

and φ is a bilinear expression in $q' \ldots q^n$.

Another translation of the origin transforms (42) into

$$K^{(n)}_{\mu \nu_1 \cdots \nu_n} (q' \cdots q^n) = \left(-\frac{1}{2}\right)^{\frac{n(n+1)}{2}} \int_{-1}^{+1} d u_1 (1-u_1)^{n-1} \cdots \int_{-1}^{+1} d u_n \int d q \times \\ \times \delta^{(n)} (q^2 + m^2 + \varphi) \, Sp [\gamma_\mu (i \gamma (q + a_1) - m) \gamma_{\nu_1} (i \gamma (q + a_2) - m) \cdots \\ \cdots \gamma_{\nu_n} (i \gamma (q + a_{n+1}) - m)] \tag{44}$$

where $a_1 \ldots a_{n+1}$ are linear combinations of $q' \ldots q^n$ with coefficients which depend on $u_1 \ldots u_n$.

We now suppose that the potentials $A_\nu(q)$ vanish for large momenta so that the integrations over $q' \ldots q^{2n+1}$ in (41) converge and are thus left with only the q integration in (44). The strongest divergence in this integral is of the form

$$\int dq\, \delta^{(n)}(q^2+\alpha^2)\, q_\mu q_{\nu 1} \cdots q_{\nu_n} \tag{45}$$

With the well-known representation of the δ-function (45) may be written

$$\frac{1}{2\pi} \cdot i^n \int_{-\infty}^{+\infty} x^n\, dx \int dq\, e^{ix(q^2+\alpha^2)}\, q_\mu q_{\nu 1} \cdots q_{\nu_n} \tag{46}$$

The q-space integration in (46) can be performed without difficulty and gives some constant factor λ multiplied by $|x| \cdot x^{-\frac{n+1}{2}-3}$. The integral (46) can therefore be written as

$$\frac{\lambda}{2\pi} \cdot i^n \int_{-\infty}^{+\infty} |x| \cdot x^{\frac{n-7}{2}}\, e^{i\alpha^2 x}\, dx \tag{47}$$

The expression (47) and the kernel (44) converge for $n \geqq 4$ but diverge for $n \leqq 3$. As we are only interested in odd integers n, the only divergent integrals are

$$(n=1) \int_{-\infty}^{+\infty} \frac{e^{i\alpha^2 x}}{x \cdot |x|}\, dx \tag{48a}$$

and

$$(n=3) \int_{-\infty}^{+\infty} \frac{e^{i\alpha^2 x}}{|x|}\, dx \tag{48b}$$

The first one appears in the e^2 approximation and as this case has already been treated by several authors we limit ourselves to the e^4 approximation, where the integral (48b) is the only divergent one. The kernel is here

$$K^{(3)}_{\mu\nu_1\nu_2\nu_3}(q' q'' q''') = \frac{1}{64} \int_{-1}^{+1} du_1 (1-u_1)^2 \int_{-1}^{+1} du_2 (1-u_2) \int_{-1}^{+1} du_3 \times$$
$$\times \int_{-\infty}^{+\infty} x^3\, dx \int dq\, e^{ix(q^2+m^2+\varphi)}\, Sp[\gamma_\mu (i\gamma(q+a_1)-m) \cdots$$
$$\cdots \gamma_{\nu_3}(i\gamma(q+a_4)-m)] \tag{49}$$

and the divergent term

$$\frac{1}{64}\int\limits_{-1}^{+1} du_1 (1-u_1)^2 \int\limits_{-1}^{+1} du_2 (1-u_2) \int\limits_{-1}^{+1} du_3 \int\limits_{-\infty}^{-\infty} x^3\, dx \int dq\, e^{ix(q^2+m^2+\varphi)} \times$$
$$\times Sp[\gamma_\mu \gamma q \gamma_{\nu_1} \gamma q \gamma_{\nu_2} \gamma q \gamma_{\nu_3} \gamma q] \tag{50}$$

The trace in (50) is easily evaluated and is equal to

$$4\{8 q_\mu q_{\nu_1} q_{\nu_2} q_{\nu_3} - 2 q^2 (q_\mu q_{\nu_1} \delta_{\nu_2 \nu_3} + q_\mu q_{\nu_3} \delta_{\nu_1 \nu_2} + q_{\nu_1} q_{\nu_3} \delta_{\mu \nu_3} +$$
$$+ q_{\nu_2} q_{\nu_2} \delta_{\mu \nu_1}) + q^4 (\delta_{\mu \nu_1} \delta_{\nu_2 \nu_3} - \delta_{\mu \nu_2} \delta_{\nu_1 \nu_3} + \delta_{\mu \nu_3} \delta_{\nu_1 \nu_2})\}. \tag{51}$$

The scalar fields.

In this case equation (7) reads

$$\langle j_\mu(x)\rangle = \sum_{n=0}^{\infty} i^{-n} \int\limits_{-\infty}^{\sigma} dx' \int\limits_{-\infty}^{\sigma'} dx'' \cdots \int\limits_{-\infty}^{\sigma^{(n-1)}} dx^n A_{\nu_1}(x') \cdots A_{\nu_n}(x^n) \times$$
$$\times \langle [t_{\nu_n}(x^n) [\cdots [t_{\nu_1}(x') j_\mu(x)] \cdots]]\rangle_0 . \tag{52}$$

As both $t_\nu(x)$ and $j_\mu(x)$ here contain two different powers of e, formula (52) is not an expansion in powers of the charge. It is, however, always possible to rearrenge the terms so as to get such an expansion. For this purpose we write

$$\langle j_\mu(x)\rangle = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} e^{n+k+1} \langle j_\mu(x)\rangle^{(n)(k)} = \sum_{n=0}^{\infty} e^{n+1} \langle j_\mu(x)\rangle^{(n)} \tag{53}$$

where

$$\langle j_\mu(x)\rangle^{(n)} = \sum_{k=0}^{\frac{n+1}{2}} \langle j_\mu(x)\rangle^{(n-k)(k)} \tag{53a}$$

for odd n, and

$$\langle j_\mu(x)\rangle^{(n)} = \sum_{k=0}^{\frac{n}{2}} \langle j_\mu(x)\rangle^{(n-k)(k)} \tag{53b}$$

when n is even. From equations (52) and (53) we also get

$$\langle j_\mu(x)\rangle^{(n)(k)} = \left(\frac{1}{2}\right)^n i^{1-k} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} dx' \cdots dx^n\, \varepsilon(01)\, \varepsilon(12) \cdots \varepsilon(n-1, n) \times$$
$$\times A_{\nu_1}(x') \cdots A_{\nu_n}(x^n) \{\langle [\varphi^*(x^n)\, \varphi(x^n) [\cdots [\varphi^*(x^{n-k+1})\, \varphi(x^{n-k+1}) \times$$
$$\times [s_{\nu_{n-k}}(x^{n-k}) [\cdots [s_{\nu_1}(x')\, s_\mu(x)] \cdots]] \cdots]]\rangle_0\, \xi(n-k+1, n+1) \cdots$$
$$\cdots \xi(n, n+k)\, A_{\nu_{n+1}}(x^{n-k+1}) \cdots A_{\nu_{n+k}}(x^n) + \cdots\} \tag{54}$$

The bracket in (54) contains one term for each way in which it is possible to pick out $n-k+1$ indices $\nu_{i_1} \ldots \nu_{i_{n-k+1}}$ from the series $\mu\nu_1 \ldots \nu_n$. If all $\nu_i \neq \mu$ the corresponding term is multiplied by a factor 2.

From the charge symmetry we see here too that the series in (53) will contain only even powers of e and hence that n in equation (53a) is an odd integer.

By an argument analogous to that leading from equation (14) to equation (23) but with differential operators instead of γ-matrices it is seen that (for n odd)

$$\left(\frac{1}{2}\right)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx' \cdots dx^n \, \varepsilon(01) \cdots \varepsilon(n-1, n) \, A_{\nu_1}(x') \cdots A_{\nu_n}(x^n)$$
$$\langle [s_{\nu_n}(x^n) [\cdots [s_{\nu_1}(x') \, s_\mu(x)] \cdots]] \rangle_0 = \frac{i}{2} \left(\frac{1}{2\pi}\right)^{4n+3} \int \cdots$$
$$\cdots \int dp \, dp' \cdots dp^n \, e^{ix(p-p^n)} A_{\nu_1}(p'-p) \cdots A_{\nu_n}(p^n - p^{n-1}) \times$$
$$\times P^{(n)} \times \left\{ \frac{\delta(p^2+m^2)}{(p'^2-p^2)\cdots(p^{n^2}-p^2)} + \frac{\delta(p'^2+m^2)}{(p^2-p'^2)\cdots(p^{n^2}-p'^2)} + \cdots \right.$$
$$\left. \cdots + \frac{\delta(p^{n^2}+m^2)}{(p^2-p^{n^2})\cdots(p^{n-1^2}-p^{n^2})} \right\} \tag{55}$$

where

$$P^{(n)} = \pi^{(n)} - \sum_j \delta_{4\nu_i} \, \delta_{4\nu_{i-1}} (p^{i-1^2} + m^2) \left[\pi^{(n)}_{i,i-1} - \frac{1}{2} \sum_j \delta_{4\nu_j} \, \delta_{4\nu_{j-1}} \times \right.$$
$$\left. \times (p^{j-1^2} + m^2) [\pi^{(n)}_{i,i-1,j,j-1} - \cdots] \right] \tag{56}$$

and

$$\pi^{(n)} = (p'_{\nu_1} + p_{\nu_1})(p''_{\nu_2} + p'_{\nu_2}) \cdots (p^n_{\nu_n} + p^{n-1}_{\nu_n})(p_\mu + p^n_\mu) \tag{57}$$

$$\pi^{(n)}_{i,j,\ldots} = \frac{\pi^{(n)}}{(p^i_{\nu_i} + p^{i-1}_{\nu_i})(p^j_{\nu_j} + p^{j-1}_{\nu_j})\cdots} \tag{58}$$

$$\pi^{(n)}_{i,j,\ldots} = 0 \text{ if two indices are equal.}$$

We here have also used the following formula given by JOST and RAYSKI

$$\varepsilon(x) \frac{\partial^2 \Delta(x)}{\partial x_\mu \, \partial x_\nu} = \frac{\partial^2 (\varepsilon(x) \, \Delta(x))}{\partial x_\mu \, \partial x_\nu} - 2 \, \delta(x) \, \delta_{4\mu} \, \delta_{4\nu} \tag{59}$$

The more general commutators in (54) can be evaluated in the same way. Putting

$$A_{\nu_{n+i}}(n-k+i) \, A_{\nu_{n-k+i}}(n-k+i) =$$
$$= \int \delta(n-k+i, \, n+i) \, A_{\nu_{n-k+i}}(n-k+i) \, A_{\nu_{n+i}}(n+i) \, dx^{n+i} \tag{60}$$

we get a formula, which is similar to (56) but with $\xi\,(ij)$ substituted for $\delta_{4\,\nu_i}\,\delta_{4\,\nu_{i-1}}$ in some of the terms. The expression (53a) may now be written

$$\langle j_\mu(x)\rangle^{(2n+1)} = \frac{1}{2}\left(\frac{1}{2\pi}\right)^{8n+7}(-1)^n\int\cdots$$
$$\cdots\int d p\, d p'\cdots d p^{2n+1}\, e^{i x\,(p-p^{2n+1})}$$
$$\times\,\overline{P}^{(2n+1)}_{\mu\nu_1\cdots\nu_{2n+1}}\left\{\frac{\delta\,(p^2+m^2)}{(p'^2-p^2)\cdots(p^{n2}-p^2)}+\cdots+\frac{\delta\,(p^{n2}+m^2)}{(p^2-p^{n2})\cdots(p^{n-1\,2}-p^{n2})}\right\}\times$$
$$\times\, A_{\nu_1}(p'-p)\cdots A_{\nu_n}(p^n-p^{n-1}) \tag{61}$$

where

$$\overline{P}^{(n)}_{\mu\nu_1\cdots\nu_n} = \pi^{(n)} - \sum_i f^{(1)}_{\nu_i\nu_{i-1}}\,(p^{i-1\,2}+m^2)\,\pi^{(n)}_{i,\,i-1} + \frac{1}{2}\sum_{i,\,j} f^{(2)}_{\nu_i\nu_{i-1}\nu_j\nu_{j-1}}\times$$
$$\times\,(p^{i-1\,2}+m^2)\,(p^{j-1\,2}+m^2)\,\pi^{(n)}_{i,\,i-1,\,j,\,j-1} - \frac{1}{6}\sum\cdots \tag{62}$$

and $f^{(k)}_{\nu_i\nu_{i-1}\nu_j\nu_{j-1}}\cdots$ is some function of the $\delta_{4\,\nu_i}$ and $\delta_{\nu_i\,\nu_{i-1}}$. As will be shown in the appendix, f is actually a tensor and equal to

$$f^{(k)}_{\nu_i\nu_{i-1}\nu_j\nu_{j-1}\cdots} = \delta_{\nu_i\,\nu_{i-1}}\,\delta_{\nu_i\,\nu_{j-1}}\cdots \tag{63}$$

The formulae (62) and (63) express the lorentz invariance of equation (61).

To prove the gauge invariance (supposing the integrals in (61) to converge) we need an identity of the same type as equation (27). We have

$$\overline{P}^{(n)}_{\mu\nu_1\cdots\nu_n}(p p'\cdots p^n)\,(p'_{\nu_1}-p_{\nu_1}) = (p'^2-p^2)\times$$
$$\times\left[\pi^{(n)}_1 - \sum_i \delta_{\nu_i\,\nu_{i-1}}\,(p^{i-1\,2}+m^2)\,\pi^{(n)}_{1,\,i,\,i-1}+\cdots\right] - (p^2+m^2)\times$$
$$\times\,(p'_\mu-p_\mu)\left[\pi^{(n)}_{1,0} - \sum_i \delta_{\nu_i\,\nu_{i-1}}\,(p^{i-1\,2}+m^2)\,\pi^{(n)}_{1,\,0,\,i,\,i-1}+\cdots\right] -$$
$$-\,(p'^2+m^2)\,(p'_{\nu_2}-p_{\nu_2})\left[\pi^{(n)}_{1,2} - \sum_i \delta_{\nu_i\,\nu_{i-1}}\,(p^{i-1\,2}+m^2)\times\right.$$
$$\left.\times\,\pi^{(n)}_{1,\,2,\,i,\,i-1}+\cdots\right] = (p'^2+m^2)\,\overline{P}^{(n-1)}_{\mu\nu_2\cdots\nu_n}(p\,p''\cdots p^n) -$$
$$-\,(p^2+m^2)\,\overline{P}^{(n-1)}_{\mu\nu_2\cdots\nu_n}(p'\,p''\cdots p^n)\,. \tag{64}$$

We can now repeat the calculation from equation (29) to equation (35) but start from (61) instead of (25) and use (64) instaed of (27). The result is obviously that, from this formal point of view, (61) is gauge invariant.

Writing (61) as (n odd)

$$\langle j_\mu(x)^{(n)}\rangle = \frac{1}{2}(-1)^{\frac{n+1}{2}}\left(\frac{1}{2\pi}\right)^{4n+3}\int\cdots\int dq'\cdots dq^n\, e^{-ix(q'+\cdots+q^n)}\times$$
$$\times \overline{K}^{(n)}_{\mu\nu_1\cdots\nu_n}(q'\cdots q^n)\, A_{\nu_1}(q)\cdots A_{\nu_n}(q^n) \qquad (65)$$

we have

$$\overline{K}^{(n)}_{\mu\nu_1\cdots\nu_n}(q'\cdots q^n) = \left(-\frac{1}{2}\right)^{\frac{n(n+1)}{2}}\int_{-1}^{+1} du_1(1-u_1)^{n-1}\cdots\int_{-1}^{+1} du_n\times$$
$$\times\int dq\,\delta^{(n)}(q^2+m^2+\varphi)\,\overline{P}^{(n)}_{\mu\nu_1\cdots\nu_n}(q+a_1, q+a_2,\cdots) \qquad (66)$$

(compare equation (44)).

Formula (66) converges or diverges as

$$\int dq\,\delta^{(n)}(q^2+\alpha^2)\, q_\mu q_{\nu_1}\cdots q_{\nu_n} \qquad (67)$$

which is the same equation as (45). We thus get exactly the same convergent cases as for the spinor field.

The divergent term of (66) in the case $n = 3$ may according to (62) and (46) be written

$$\frac{1}{64}\int_{-1}^{+1} du_1(1-u_1)^2\int_{-1}^{+1} du_2(1-u_2)\int_{-1}^{+1} du_3\int_{-\infty}^{+\infty} x^3\, dx\int dq\, e^{ix(q^2+m^2+\varphi)}\times$$
$$\times\{16\, q_\mu q_{\nu_1} q_{\nu_2} q_{\nu_3} - 4q^2(\delta_{\mu\nu_3} q_{\nu_1} q_{\nu_2} + \delta_{\nu_1\mu} q_{\nu_2} q_{\nu_3} + \delta_{\nu_2\nu_1} q_{\nu_3} q_\mu +$$
$$+\delta_{\nu_3\nu_2} q_{\nu_1} q_\mu) + q^4(\delta_{\mu\nu_3}\delta_{\nu_1\nu_2} + \delta_{\nu_3\nu_2}\delta_{\mu\nu_1})\}. \qquad (68)$$

Let us now consider a mixture of N scalar fields with masses M_i and n spinor fields with masses m_i. From equations (50), (51) and (68) we get the divergent term in the kernel of the e^4 approximation (after a suitable symmetrization of (51))

$$\frac{1}{64}\int_{-1}^{+1} du_1(1-u_1)^2\int_{-1}^{+1} du_2(1-u_2)\int_{-1}^{+1} du_3\int_{-\infty}^{+\infty} x^3\, dx\left[\left\{2\sum_{i=1}^{n} e^{ixm_i^2} - \sum_{i=1}^{N} e^{ixM_i^2}\right\}\times\right.$$
$$\times\int dq\, e^{ixq^2}\{16\, q_\mu q_{\nu_1} q_{\nu_2} q_{\nu_3} - 4q^2(\delta_{\mu\nu_3} q_{\nu_1} q_{\nu_2} + \delta_{\nu_1\mu} q_{\nu_2} q_{\nu_3} +$$
$$+\delta_{\nu_2\nu_1} q_\mu q_{\nu_3} + \delta_{\nu_3\nu_2} q_{\nu_1} q_\mu) + q^4(\delta_{\mu\nu_3}\delta_{\nu_1\nu_2} + \delta_{\nu_3\nu_2}\delta_{\mu\nu_1})\} +$$
$$+(e^{ix\varphi}-1)\times\left\{\sum_{i=1}^{n} e^{ixm_i^2}\, Sp[\gamma_\mu\gamma q\gamma_{\nu_1}\gamma q\gamma_{\nu_2}\gamma q\gamma_{\nu_3}\gamma q] -\right.$$
$$-\sum_{i=1}^{N} e^{ixM_i^2}\Big(\pi^{(3)}(qqqq) - q^2\sum_i \delta_{\nu_i\nu_{i-1}}\times$$
$$\left.\left.\times\,\pi^{(3)}_{i,i-1}(qq) + \frac{1}{2}\sum_{i\neq j}\delta_{\nu_i\nu_{i-1}}\delta_{\nu_j\nu_{j-1}}\Big).\right\}\right] \qquad (69)$$

The last term in (69) is convergent as $e^{i\varphi x}-1$ vanishes linear at the origin. The first term can be made convergent too by the aid of the assumption that

$$\lim_{x\to 0}\frac{1}{x}\left[2\sum_{i=1}^{n}e^{ixM_i^2}-\sum_{i=1}^{N}e^{ixM_i^2}\right] \tag{70}$$

is finite. This is the case if

$$2\,n = N \tag{71}$$

which is one of the conditions of JOST and RAYSKI. Their other condition

$$2\sum_{i=1}^{n} m_i^2 = \sum_{i=1}^{N} M_i^2 \tag{72}$$

makes (70) vanish, but this assumption is not needed to make (69) converge.

The symmetrisation of (51) that is necessary to get (69) is certainly allowed. In fact, the expressions given by (7) can always be made symmetric in the variables and the whole calculation can be carried through in this way. We have purpously destroyed this symmetry (equations (22) and (55)) to get formulae, which are more easy to handle.

All the integrals appearing in the kernel of the e^4 approximation are now convergent and hence our formal proof of the gauge-invariance may be applied in this case too. The only remaining divergent (but actually gauge invariant) expression in the phenomena is the charge renormalisation term in the e^2 approximation.

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Appendix.

Proof of equation (23).

From the formulae (17), (20) and (21) the symbols $E_k^{(n)}$ in (22) may be computed. The result is (for n an odd number)

$$E_{(n+1-k)}^{(n)} = (-1)^k [\varepsilon(0n)\,\varepsilon(n, n-1) \cdots \varepsilon(n-k, 1)\,\varepsilon(12)\,\varepsilon(23) \cdots \cdots \varepsilon(n-k-2, n-k-1) + \varepsilon(0n)\,\varepsilon(n, n-1) \cdots \cdots \varepsilon(n-k+1, 1)\,\varepsilon(1, n-k)\,\varepsilon(n-k, 2)\,\varepsilon(23) \cdots \cdots \varepsilon(n-k-2, n-k-1) + \cdots] \tag{73}$$

Denoting

$$(-1)^k [E_{(n+1-k)}^{(n)} + \varepsilon(01)\,\varepsilon(12) \cdots \varepsilon(n-k-1, n-k)\,\varepsilon(n-k+1, n-k+2) \cdots \cdots \varepsilon(n-1, n)\;\varepsilon(n0)]$$

with

$$S^{(n)\,(k)}(01 \cdots n-k-1;\quad n-k \cdots n)$$

we get from formula (73) the following recursion formulae

$$(k \neq 0)\; S^{(n)\,(k)}(01 \cdots n-k-1;\; n-k \cdots n) = \varepsilon(0n)\,S^{(n-1)\,(k-1)}(n\,12 \cdots \cdots n-k-1;\; n-k \cdots n-1) + \varepsilon(01)\,\varepsilon(1n)\,S^{(n-2)\,(k-1)}(n\,23 \cdots \cdots n-k-1;\; n-k \cdots n-1) + \cdots + \varepsilon(01)\,\varepsilon(12) \cdots \cdots \varepsilon(n-k-2, n)\,S^{(k+1)\,(k-1)}(n\, n-k-1;\; n-k \cdots n-1) + + (-1)^k\,\varepsilon(n-k, n-k+1) \cdots \varepsilon(n-1, n) \times \times S^{(n-k)\,(0)}(012 \cdots n-k-1;\, n) \tag{74}$$

and

$$S^{(n)\,(0)}(01 \cdots n-1, n) = \varepsilon(01)\,S^{(n-1)\,(0)}(12 \cdots n-1;\, n) + \varepsilon(12) \cdots \cdots \varepsilon(n-2, n-1)\,S^{(n-k)\,(0)}(012 \cdots n-k-1;\; n) \tag{75}$$

From the identity

$$S^{(2)\,(0)}(01;\, 2) = \varepsilon(02)\,\varepsilon(21) + \varepsilon(01)\,\varepsilon(12) + \varepsilon(01)\,\varepsilon(20) = -1 \tag{76}$$

and equation (75) it is imidiately seen, that $S^{(n)\,(0)}$ can be expressed as a sum of terms, each of which do not contain more than n-2 factors ε. From (74) the same statement is seen to be true also for the general expression $S^{(n)\,(k)}$. Using the property that no real pairs can be created, we now get (23) from (22).

Proof of equation (63).

The symbol $\overline{P}^{(n)}_{\mu\nu_1\cdots\nu_n}$ of equation (62) can be computed from equations (54)—(60). The result is (for n odd)

$$\begin{aligned}\overline{P}^{(n)}_{\mu\nu_1\cdots\nu_n} = \pi^{(n)} - \sum_i \delta_{4\nu_i}\,\delta_{4\nu_{i-1}}\,(p^{i-1^2}+m^2)\Big[\pi^{(n)}_{i,i-1} - \\ -\frac{1}{2}\sum_j \delta_{4\nu_j}\,\delta_{4\nu_{j-1}}\,(p^{j-1^2}+m^2)\left[\pi^{(n)}_{i,i-1,j,j-1} - \frac{1}{3}\sum[\cdots]\right]\Big] + \\ +\sum_i \xi(i,i-1)\,(p^{i-1^2}+m^2)\Big[\pi^{(n)}_{i,i-1} - \sum_j \delta_{4\nu_j}\,\delta_{4\nu_{j-1}}\,(p^{j-1^2}+m^2)\times \\ \times\left[\pi^{(n)}_{i,i-1,j,j-1} - \frac{1}{2}\sum[\cdots]\right]\Big] + \frac{1}{2!}\sum_{i,j}\xi(i,i-1)\,\xi(j,j-1)\times \\ \times(p^{i-1^2}+m^2)\,(p^{j-1^2}+m^2)\left[\pi^{(n)}_{i,i-1,j,j-1} - \sum[\cdots]\right] + \\ +\frac{1}{3!}\sum[\cdots]+\cdots \qquad (77)\end{aligned}$$

We here use the definition (3c) of $\xi(ij)$ and write down the coefficient for a term consisting of $2\,l$ factors $\delta_{4\nu_i}\,\delta_{4\nu_{i-1}}$ and λ factors $\delta_{\nu_i\nu_{i-1}}$. This coefficient is

$$\frac{(-1)^\lambda}{\lambda!}\sum_{s=0}^{l}\frac{(-1)^s}{s!\,(l-s)!} = \begin{cases}\dfrac{(-1)^\lambda}{\lambda!}\,\dfrac{(1-1)^l}{l!} = 0 & l \neq 0 \\[2ex] \dfrac{(-1)^\lambda}{\lambda!} & l = 0\end{cases} \qquad (78)$$

The only non-vanishing terms in (77) thus consist only of factors $\delta_{\nu_i\nu_{i-1}}$ with coefficients given by (78). This is formula (63).