Relativistic Heisenberg picture

Autor(en): **Zumino, Bruno**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **23 (1950)**

Heft [3]: **Supplementum 3. Internationaler Kongress über Kernphysik und Quantenelektrodynamik**

PDF erstellt am: **11.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-422278

Relativistic Heisenberg picture

by **Bruno Zumino.**
Centro di Studio per la Fisica Nucleare del Consiglio Nazionale delle Ricerche, Roma.

Greek indices μ, ν, λ run from 0 to 3, small italic indices r, s from 1 to 3; $h = c = 1$; the metric tensor is $g_{\mu\nu} = 0$ for $\mu \neq \nu$, $g_{00} = 1$ $g_{11} = g_{22} = g_{33} = -1$.

We start from the simpler case of a single electron in a given electromagnetic field, the wave equation of which is Dirac's equation. The corresponding Heisenberg picture, as usually expressed, is not in relativistic form owing to the fact that the time is treated in a different way then the space coordinates. This inconvenience can be avoided by the introduction of an independent space-like surface S, from which all dynamical variables, including the time, depend (see also DIRAC[3])).

In the following x_μ, x'_μ, x''_μ will denote coordinates of a free world point, and z'_μ, z''_μ of a world point lying on S. To write the equations of the theory we define for each $\zeta[S]$ functional of S, the operation

$$D_\mu \xi = \int_S \frac{\delta \zeta[S]}{\delta S(z')} d S'_\mu; \text{ here } \frac{\delta \zeta[S]}{\delta S(z')} = \lim_{\delta\omega \to 0} \frac{\zeta[S_1] - \zeta[S]}{\delta \omega}$$

is the functional derivative with respect to S in the point z' of S (see SCHWINGER[4]); S_1 is a space-like surface different from S only in the neighborhood of the point z' and $\delta\omega$ the four-dimensional volume enclosed between the two surfaces; dS'_μ are the components of a vector of length dS, normal to S at the point z'; the formula

$$\frac{\delta}{\delta S(z'')} \int_S f(z')\, d S'_\mu = \frac{\partial f(z'')}{\partial z''^\mu} \tag{1}$$

and Schwinger's lemma, holding for closed systems,

$$\int_S \frac{\partial}{\partial z'^\mu} f(z')\, d S'_\nu = \int_S \frac{\partial}{\partial z'^\nu} f(z')\, d S'_\mu \tag{2}$$

are often useful; it can also be demonstrated that $D_\mu D_\nu \zeta = D_\nu D_\mu \zeta$).

Dirac's equation can be written in the form

$$\left\{\gamma_\mu\left[\frac{\partial}{\partial x_\mu}+i\,e\,A^\mu(x)\right]+m\right\}\langle x\,|=0; \tag{3}$$

the basic vector (see DIRAC[2]). $\langle x|$ has four components and γ_μ are four-rowed matrices; γ_0 is anti-Hermitian, γ_r Hermitian and they satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=-2\,g_{\mu\nu}$.

Defining $|\,x\rangle^+=-i\,|\,x\rangle\,\gamma_0$, the adjoint equation is

$$|\,x\rangle^+\left\{\gamma_\mu\left[\frac{\overleftarrow{\partial}}{\partial x_\mu}-i\,e\,A^\mu(x)\right]-m\right\}=0;$$

it follows $\frac{\partial}{\partial x_\mu}\{-i\,|\,x\rangle^+\gamma_\mu\langle x\,|\,\}=0$ so that

$$1=-i\int_S|\,z'\rangle^+\gamma_\lambda\langle z'\,|\,d\,S'^\lambda \tag{4}$$

is independent from S.

In the case in which an electromagnetic field is present, no simple expression can be given for $\langle x'\,|\,x''\rangle^+$; but we can always assert that

$$f(z'')=-i\int_S f(z')\,\gamma_\mu\langle z'\,|\,z''\rangle^+d\,S'^\mu=-i\int_S\langle z''\,|\,z'\rangle^+\gamma_\mu f(z')\,d\,S'^\mu \tag{5}$$

provided z' and z'' are both on S; here $f(x)$ needs not to satisfy Dirac's equation.

As a relativistic generalization of the coordinates and momenta of the electron at a given time, we define

$$z_\mu[S]=-i\int_S|\,z'\rangle^+\gamma_\lambda z'_\mu\langle z'\,|\,d\,S'^\lambda$$

and

$$p_\mu[S]=\int_S|\,z'\rangle^+\gamma_\lambda\frac{\partial}{\partial z'^\mu}\langle z'\,|\,d\,S'^\lambda=-\int_S\left(\frac{\partial}{\partial z'^\mu}|\,z'\rangle^+\right)\gamma_\lambda\langle z'\,|\,d\,S'^\lambda;$$

in force of (5) they satisfy

$$\langle z'\,|\,z_\mu[S]=z'_\mu\langle z'\,|,\; z_\mu[S]\,|\,z'\rangle^+=|\,z'\rangle^+z'_\mu$$

and

$$\langle z'\,|\,p_\mu[S]=i\frac{\partial}{\partial z'^\mu}\langle z'\,|,\; p_\mu[S]\,|\,z'\rangle^+=-i\frac{\partial}{\partial z'^\mu}|\,z'\rangle^+$$

(z' always on S). In a similar way

$$\gamma_\mu[S]=-i\int_S|\,z'\rangle^+\gamma_\lambda\gamma_\mu\langle z'\,|\,d\,S'^\lambda,\;\langle z'\,|\,\gamma_\mu[S]=\gamma_\mu\langle z'\,|,$$

whereas $\gamma_\mu[S]\,|\,z'\rangle^+$

is not even independent from the particular surface S through z'; we have instead, for

$$\overline{\gamma}_\mu[S] = i\int_S |z'\rangle^+ \gamma_\mu \gamma_\lambda \langle z'| dS'^\lambda, \text{ the relation } \overline{\gamma}_\mu[S]|z'\rangle^+ = -|z'\rangle^+ \gamma_\mu .$$

Using the definitions of the dynamical variables and their properties just exposed, it is easy to find their Poisson brackets. For instance, remembering also (1) (2) and (4),

$$[p_\mu, z_\nu] = \frac{p_\mu z_\nu - z_\nu p_\mu}{i} = -\int_S \left(-i\frac{\partial}{\partial z'^\mu}|z'\rangle^+\right)\gamma_\lambda z'_\nu \langle z'| dS'^\lambda +$$

$$+\int_S |z'\rangle^+ z'_\nu \gamma_\lambda i \frac{\partial}{\partial z'^\mu}\langle z'| dS'^\lambda = i\int_S \frac{\partial}{\partial z'^\mu}\{|z'\rangle^+ \gamma_\lambda z'_\nu \langle z'|\} dS'^\lambda -$$

$$- i\int_S |z'\rangle^+ \gamma_\lambda \langle z'| \frac{\partial z'_\nu}{dz'^\mu} dS'^\lambda = i\int_S \frac{\partial}{\partial z'_\lambda}\{|z'\rangle^+ \gamma_\lambda z'_\nu \langle z'|\} dS'_\mu +$$

$$+ g_{\mu\nu} = -D_\mu z_\nu + g_{\mu\nu} .$$

The complete scheme is

$$[z_\mu, z_\nu] = 0, \quad [z_\mu, \gamma_\nu] = 0, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2g_{\mu\nu}, \tag{6}$$

$$[p_\mu, z_\nu] = g_{\mu\nu} - D_\mu z_\nu, \quad [p_\mu, \gamma_\nu] = -D_\mu \gamma_\nu, \quad [p_\mu, p_\nu] = D_\nu p_\mu - D_\mu p_\nu . \tag{7}$$

Attention must be paid to formulas such as (7). The first of it, for instance, specializing S to have the equation $x_0 = z'_0$, becomes

$$[p_r, z_s] = -\delta_{rs}, \quad [p_0, z_s] = -\frac{dz_s}{dz'_0}, \quad [p_0, z_0] = 0,$$

so that it can be considered a relativistic expression for both the Poisson bracket relations between coordinates and momenta and the Hamiltonian law of dependence of the coordinates from the time.

(6) and (7) can be taken as the starting point of the theory and it becomes of interest to check their consistency. This can be done showing that they have as a consequence the relations

$$D_\lambda[z_\mu, z_\nu] = 0 \qquad D_\lambda[z_\mu, \gamma_\nu] = 0 \qquad D_\lambda(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = 0$$

$$D_\lambda[p_\mu, z_\nu] = D_\mu[p_\lambda, z_\nu] \qquad D_\lambda[p_\mu, \gamma_\nu] = D_\mu[p_\lambda, \gamma_\nu]$$

$$D_\lambda[p_\mu, p_\nu] + D_\mu[p_\nu, p_\lambda] + D_\nu[p_\lambda, p_\mu] = 0$$

which correspond to properties of the second members of (6) and (7) themselves (remember $D_\mu D_\nu = D_\nu D_\mu$). For instance

$$D_\lambda [p_\mu, z_\nu] - D_\mu [p_\lambda, z_\nu] = [D_\lambda p_\mu - D_\mu p_\lambda, z_\nu] + [p_\mu, D_\lambda z_\nu] - [p_\lambda, D_\mu z_\nu] =$$

$$= \big[[p_\mu, p_\lambda], z_\nu\big] + \big[p_\mu, [z_\nu, p_\lambda]\big] - \big[p_\lambda [z_\nu, p_\mu]\big] = 0$$

in force of the identity of Poisson Jacoby.

The relations (6) and (7) make no reference to the particular kind of forces to which the particle is subjected; they are the relativistic generalization of the non relativistic Hamiltonian formulation of the one-particle theory if the form of the Hamiltonian function is not yet specified. We can specialize the problem adding the statement that

$$F = \gamma_\mu [p^\lambda - e A^\lambda (z)] + i m = 0 \tag{8}$$

be valid for every space-like surface S. (This follows from (3) by multiplication on the left with $| x\rangle^+ \gamma_\varrho d S^\varrho$ and integration.)

We have as a consequence, using (6), (7) and the properties of the Poisson brackets,

$$0 = [F, z_\mu] = \gamma_\mu - \gamma_\lambda D^\lambda z_\mu; \tag{9}$$

$$0 = [F, p_\mu] = D_\mu \gamma_\lambda (p^\lambda - e A^\lambda) + \gamma_\lambda (D_\mu p^\lambda - D^\lambda p_\mu) -$$

$$- e \gamma_\lambda \frac{\partial A^\lambda}{\partial z^\varrho} (D_\mu z^\varrho - \delta^\varrho_\mu)$$

but

$$0 = D_\mu F = D_\mu \gamma_\lambda (p^\lambda - e A^\lambda) + \gamma_\lambda D_\mu p^\lambda - e \gamma_\lambda \frac{\partial A^\lambda}{\partial z^\varrho} D_\mu z^\varrho$$

so that

$$\gamma_\lambda D^\lambda p_\mu = e \gamma_\lambda \frac{\partial A^\lambda}{\partial z^\mu}; \tag{10}$$

$$0 = [F, \gamma_\mu] = - i (\gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda)(p^\lambda - e A^\lambda) - \gamma_\lambda D^\lambda \gamma_\mu$$

but

$$- i (\gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda) = 2 i (\gamma_\mu \gamma_\lambda + g_{\mu\lambda})$$

so that, using (8) again,

$$\gamma_\lambda D^\lambda \gamma_\mu = 2 i \{ - i m \gamma_\mu + p_\mu - e A_\mu (z) \}. \tag{11}$$

(9), (10) and (11) are the relativistic generalization of the equation of motion for our dynamical variables.

The foregoing theory can be easily generalized to give the relativistic Heisenberg picture corresponding to Dirac's many times theory usually expressed correctly only by means of wave equations (see DIRAC, FOCK and PODOLSKY[1]).

References.

[1]) DIRAC, FOCK and PODOLSKY, Phys. ZS. d. Sowjetunion **2,** 468 (1932).
[2]) DIRAC, Proc. Camb. Philos. Soc. **35,** 416 (1939).
[3]) DIRAC, Phys. Rev. **73,** 1092 (1948).
[4]) SCHWINGER, Phys. Rev.,, **74,** 1439 (1948).