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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **25 (1952)**

Heft IV

PDF erstellt am: **08.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-112316

On the Definition of the Renormalization Constants in Quantum Electrodynamics

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(14. II. 1952.)

Summary. A formulation of quantum electrodynamics in terms of the renormalized Heisenberg operators and the experimental mass and charge of the electron is given. The renormalization constants are implicitly defined and expressed as integrals over finite functions in momentum space. No discussion of the convergence of these integrals or of the existence of rigorous solutions is given.

Introduction.

The renormalization method in quantum electrodynamics has been investigated by many authors, and it has been proved by DYSON[1]) that every term in a formal expansion in powers of the coupling constant of various expressions is a finite quantity. No serious attempt at a discussion of the convergence of the series has been published, and the definition of the renormalization constants is always given as a formal series where every coefficient is infinite. It is the aim of the present paper to give a formulation of quantum electrodynamics where only the renormalized operators (in the Heisenberg representation) will appear and where the renormalization constants are defined in terms of these operators and the experimental mass and charge of the electron. There thus exists a possibility of studying the renormalized quantities directly without the aid of a power series expansion and especially to decide if they are really finite and not only a divergent sum of finite terms. No discussion of this point, however, will be given in this paper, only the formulation of the theory.

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[1]) F. J. DYSON, Phys. Rev. **83**, 608; 1207 (1951) and earlier papers.

The starting point of our analysis is the following formal Lagrangian

$$\mathfrak{L} = -\frac{N^2}{4}\left[\bar{\psi}(x), \left(\gamma\frac{\partial}{\partial x}+m\right)\psi(x)\right] - \frac{N^2}{4}\left[-\frac{\partial\bar{\psi}(x)}{\partial x_\nu}\gamma_\nu + m\bar{\psi}(x), \psi(x)\right] +$$
$$+\frac{K}{2}\cdot N^2[\bar{\psi}(x), \psi(x)] - \frac{1}{2}\frac{\partial A_\nu(x)}{\partial x_\nu}\frac{\partial A_\mu(x)}{\partial x_\mu} -$$
$$-\frac{1-L}{4}\left(\frac{\partial A_\nu(x)}{\partial x_\mu}-\frac{\partial A_\mu(x)}{\partial x_\nu}\right)\left(\frac{\partial A_\nu(x)}{\partial x_\mu}-\frac{\partial A_\mu(x)}{\partial x_\nu}\right)+$$
$$+\frac{ie}{4}\cdot N^2\{A_\mu(x), [\bar{\psi}(x), \gamma_\mu\psi(x)]\}. \tag{1}$$

In this expression $A_\mu(x)$ is the renormalized vector-potential of the electromagnetic field, $\psi(x)$ the renormalized Dirac-operator of the electron-positron field, m and e the experimental mass and charge of the electron, and K, L and N three universal constants, the definition of which will be given later. The three quantities K, N^{-1} and $(1-L)^{-1}$ might be infinite but in spite of that, we will adopt the convention that the usual algebraic operations can be performed with them. K and L describe respectively the mass- and the charge-renormalization, and N is a normalization constant for the ψ-field. The other notations are nowadays standard symbols in quantum field theory.

From the above Lagrangian we obtain without difficulty the following equations of motion for the field operators in the Heisenberg representation

$$\left(\gamma\frac{\partial}{\partial x}+m\right)\psi(x) = \frac{ie}{2}\{A_\nu(x), \gamma_\nu\psi(x)\} + K\psi(x) \equiv f(x) \tag{2}$$

$$\Box A_\mu(x) = \frac{-ie}{2}N^2[\bar{\psi}(x), \gamma_\mu\psi(x)] + L\left(\Box A_\mu(x) - \frac{\partial^2 A_\nu(x)}{\partial x_\mu \partial x_\nu}\right) \equiv -j_\mu(x). \tag{3}$$

In Eq. (2) and (3) $f(x)$ and $j_\mu(x)$ are only to be considered as abbreviations for the right-hand sides.

From our Lagrangian we can also obtain the commutators for the electromagnetic operators and their time-derivatives in two points, the distance of which is space-like, and the corresponding anticommutators for the electron field. As the terms with K and L contain the time-derivatives of $A_\mu(x)$ but not of $\psi(x)$, the canonical commutators involving the electromagnetic potentials will be rather complicated and really meaningless if $L = 1$, but the anticommutators of the matter-field will have the simple form

$$\{\bar{\psi}_\alpha(x), \psi_\beta(x')\} = (\gamma_4)_{\beta\alpha}\cdot N^{-2}\cdot\delta(\bar{x}-\bar{x}') \quad \text{for } x_0 = x_0' \tag{4}$$

$$\delta(\bar{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,. \tag{4a}$$

For the electromagnetic potentials we get for $(x - x')^2 > 0$

$$[A_\mu(x), A_\nu(x')] = 0 \tag{5}$$

$$\left[\frac{\partial A_\mu(x)}{\partial t}, A_\nu(x')\right] = -i\,\delta(\overline{x} - \overline{x}')\left[\frac{\delta_{\mu\nu}}{1-L} - \frac{L}{1-L}\,\delta_{\mu 4}\,\delta_{\nu 4}\right] \tag{6}$$

$$\left[\frac{\partial A_\mu(x)}{\partial t}, \frac{\partial A_\nu(x')}{\partial t'}\right] = \frac{L}{1-L}\left(\delta_{\mu 4}\frac{\partial}{\partial x_\nu} + \delta_{\nu 4}\frac{\partial}{\partial x_\mu}\right)\delta(\overline{x} - \overline{x}')\,. \tag{7}$$

Besides, every component A_μ will commute with every component of ψ on a space-like surface.

General Properties of the Operators.

The two equations of motion are formally integrated with the help of the retarded singular functions and the operators for the free fields

$$\psi(x) = \psi^{(0)}(x)\cdot N^{-1} - \int S_R(x - x')\,f(x')\,dx' \tag{8}$$

$$A_\mu(x) = A_\mu^{(0)}(x) + \int D_R(x - x')\,j_\mu(x')\,dx' \tag{9}$$

$$S_R(x - x') = \overline{S}(x - x') - \frac{1}{2}\,S(x - x') \tag{10}$$

$$D_R(x - x') = \overline{D}(x - x') - \frac{1}{2}\,D(x - x')\,. \tag{11}$$

The integral equations (8) and (9) have the same solutions as the differential equations (2) and (3) but contain also the boundary conditions for $t = -\infty$. The operators $\psi^{(0)}(x)$ fulfill the following formulae

$$\{\overline{\psi}_\alpha^{(0)}(x), \psi_\beta^{(0)}(x')\} = -\,i\,S_{\beta\alpha}(x' - x) \tag{12}$$

$$\langle 0|[\overline{\psi}_\alpha^{(0)}(x), \psi_\beta^{(0)}(x')]|0\rangle = S_{\beta\alpha}^{(1)}(x' - x)\,. \tag{13}$$

The properties of the operators $A_\mu^{(0)}(x)$ are a little more delicate. In practical calculations, we want to use the formulae

$$[A_\mu^{(0)}(x), A_\nu^{(0)}(x')] = -\,i\,\delta_{\mu\nu}D(x' - x) \tag{14}$$

$$\langle 0|\{A_\mu^{(0)}(x), A_\nu^{(0)}(x')\}|0\rangle = \delta_{\mu\nu}D^{(1)}(x' - x) \tag{15}$$

but it is well-known that these formulae are inconsistent with the Lorentz-condition

$$\frac{\partial A_\mu(x)}{\partial x_\mu}|\psi\rangle = \frac{\partial A_\mu^{(0)}(x)}{\partial x_\mu}|\psi\rangle = 0\,. \tag{16}$$

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On the other hand, it can be shown that this inconsistency is of no importance if only gauge-invariant expressions are calculated[2]). However, in what follows we will not only be interested in gauge-invariant quantities and are thus forced to discuss Eq. (14) and (15) in more detail. For our purpose, the most convenient way to do this will be to adopt the indefinite metric of GUPTA[3]) and BLEULER[4]). In this formalism the Lorentz-condition (16) is replaced by the weaker condition

$$\frac{\partial A_\mu^{(0)(+)}(x)}{\partial x_\mu} \mid \psi \rangle = 0 \tag{17}$$

and as a consequence of this, equation (14) can be fulfilled. ($F^{(+)}(x)$ means the positive-frequency part of the operator $F(x)$.) Further, if the vacuum is suitably defined (no scalar, transversal or longitudinal photons present) Eq. (15) follows from the formalism, but it must be understood that this is a non-gauge-invariant convention. The special gauge chosen corresponds to

$$\langle 0 \mid A_\mu(x) \mid 0 \rangle = \langle 0 \mid A_\mu^{(0)}(x) \mid 0 \rangle = 0 . \tag{18}$$

In what follows we will, when necessary, use this gauge.

From our Lagrangian we can construct an energy-momentum tensor $T_{\mu\nu}$ and from this one a displacement operator P_μ fulfilling

$$[P_\mu, P_\nu] = 0 \tag{19}$$

$$[P_\mu, F(x)] = i \frac{\partial F(x)}{\partial x_\mu} . \tag{20}$$

In Eq. (20) $F(x)$ is an arbitrary operator depending on the point x. Eq. (19) thus expresses the fact that the P_μ's are constants of motion. As all the operators P_μ commute with each other, we can use a representation in the Hilbert space where every state vector is an eigenvector of all the P_μ's with the eigenvalues p_μ. In this representation Eq. (20) reads

$$\langle a \mid [P_\mu, F(x)] \mid b \rangle = (p_\mu^{(a)} - p_\mu^{(b)}) \langle a \mid F(x) \mid b \rangle = \\ = i \frac{\partial}{\partial x_\mu} \langle a \mid F(x) \mid b \rangle . \tag{21}$$

Hence

$$\langle a \mid F(x) \mid b \rangle = \langle a \mid F \mid b \rangle e^{i (p^{(b)} - p^{(a)}) x} . \tag{22}$$

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[2]) Cf. *e. g.* S. T. MA, Phys. Rev. **80**, 729 (1950) and other papers quoted by him.
[3]) S. N. GUPTA, Proc. Phys. Soc. London **63**, 681 (1950).
[4]) K. BLEULER, Helv. Phys. Acta **23**, 567 (1950).

Here $p_\mu^{(a)}$ and $p_\mu^{(b)}$ are the eigenvalues of P_μ in the states $| a \rangle$ and $| b \rangle$. In this representation, the x-dependence of an arbitrary operator is thus given by Eq. (22). The detailed form of the operators P_μ is of very little interest for the present investigation, and we will not write them down explicitly but make the following assumptions concerning the eigenvalues p_μ:

a) Every vector p_μ is time-like.

b) There exists a state with a smallest eigenvalue of the time-component p_0. This state by definition will be called the vacuum and, with a suitable renormalization of the energy, this eigenvalue of p_0 can be put equal to zero.

It is supposed that the above definition of the vacuum is not in contradiction with Eq. (18).

Definition of the Constant L.

We are now able to turn to the main problem of this paper, *i. e.* the definition of the universal constants K, L and N in the Lagrangian (1). We begin with the definition of L that describes the charge-renormalization. This is conveniently stated in terms of the matrix elements of the operators $A_\mu(x)$ between the vacuum state and a state where only one photon is present. (As we are working in the Heisenberg representation some care is necessary when we are speaking of a state with a given number of particles present. If, however, it is understood that we hereby always specify the system for $t = -\infty$, no ambiguities will arise. The occupation-number operators are then constructed from the special operators $A_\mu^{(0)}$ and $\psi^{(0)}$ introduced in Eq. (8) and (9).) At the first moment it would seem natural to introduce the following condition for the matrix elements

$$\langle 0 | A_\mu(x) | k \rangle = \langle 0 | A_\mu^{(0)}(x) | k \rangle \tag{23}$$

where $| k \rangle$ decribes a state with only one photon with energy-momentum vector k, but as the calculation below shows, this can only be fulfilled for the transversal photons. If also the longitudinal and scalar photons are considered, the correct condition for $\langle 0 | A_\mu(x) | k \rangle$ is

$$\langle 0 | A_\mu(x) | k \rangle = \left(\delta_{\mu\nu} + M \frac{\partial^2}{\partial x_\mu \partial x_\nu}\right) \langle 0 | A_\nu^{(0)}(x) | k \rangle \tag{24}$$

where M is another universal constant (*i. e.* independent of x and k). Eq. (24), together with the commutators (5), (6) and (7) and the equation of motion (9), determines the two constants L and M uniquely in terms of the matrix elements of the current operator in Eq. (3). As this contains the constant L explicitly and the other constants implicitly, the definitions of L and M are only implicit.

We now compute the vacuum expectation value of the commutator between the electromagnetic potentials and obtain

$$\begin{aligned}\langle 0 | [A_\mu(x), A_\nu(x')] | 0 \rangle &= \langle 0 | [A_\mu^{(0)}(x), A_\nu^{(0)}(x')] | 0 \rangle + \\ + \langle 0 | [A_\mu(x) - A_\mu^{(0)}(x), A_\nu^{(0)}(x')] | 0 \rangle &+ \langle 0 | [A_\mu^{(0)}(x), A_\nu(x') - A_\nu^{(0)}(x')] | 0 \rangle + \\ + \int\int dx''\, dx'''\, D_R(x-x'')\, D_R(x'-x''') &\langle 0 | [j_\mu(x''), j_\nu(x''')] | 0 \rangle . \end{aligned} \tag{25}$$

It is here convenient to introduce a special notation for the vacuum expectation value of the current commutator. Considering the definition of a matrix product and Eq. (22), we obtain

$$\begin{aligned}\langle 0 | [j_\mu(x), j_\nu(x')] | 0 \rangle &= \sum_z \langle 0 | j_\mu | z \rangle \langle z | j_\nu | 0 \rangle e^{-ip^{(z)}(x'-x)} - \\ - \sum_z \langle 0 | j_\nu | z \rangle \langle z | j_\mu | 0 \rangle e^{ip^{(z)}(x'-x)} &\to \frac{-1}{(2\pi)^3} \Big\{ \int_{p_0>0} dp\, e^{ip(x'-x)} \pi_{\mu\nu}^{(+)}(p) - \\ &- \int_{p_0<0} dp\, e^{ip(x'-x)} \pi_{\mu\nu}^{(-)}(p) \Big\} \end{aligned} \tag{26}$$

$$\pi_{\mu\nu}^{(+)}(p) \to V \sum_{p^{(z)}=p} \langle 0 | j_\nu | z \rangle \langle z | j_\mu | 0 \rangle \tag{26a}$$

$$\pi_{\mu\nu}^{(-)}(p) \to V \sum_{p^{(z)}=-p} \langle 0 | j_\mu | z \rangle \langle z | j_\nu | 0 \rangle . \tag{26b}$$

In Eq. (26a) and (26b) V is a large volume in which the fields are supposed to be enclosed, and summation over states and integration in p-space are freely interchanged. Due to the equation

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0 \tag{27}$$

(which is easily verified from Eq. (3)) we must have

$$p_\mu \pi_{\mu\nu}^{(+)} = \pi_{\mu\nu}^{(+)} p_\nu = p_\mu \pi_{\mu\nu}^{(-)} = \pi_{\mu\nu}^{(-)} p_\nu = 0 . \tag{28}$$

From reasons of invariance, on the other hand, $\pi_{\mu\nu}^{(\pm)}$ must have the form

$$\pi_{\mu\nu}^{(\pm)}(p) = \delta_{\mu\nu} A^{(\pm)}(p^2) + p_\mu p_\nu B^{(\pm)}(p^2) . \tag{29}$$

Combining (28) and (29) we get

$$\pi_{\mu\nu}^{(\pm)}(p) = (-p^2\,\delta_{\mu\nu} + p_\mu\, p_\nu)\,\pi^{(\pm)}(p^2) \tag{30}$$

with

$$\pi^{(+)}(p^2) = \frac{V}{-3\,p^2} \sum_{p^{(z)}=p} \langle 0\,|\, j_\nu\,|\, z\rangle \langle z\,|\, j_\nu|\, 0\rangle = \pi^{(-)}(p^2)\,. \tag{31}$$

We thus have

$$\langle 0\,|\,[j_\mu(x), j_\nu(x')]\,|\,0\rangle =$$

$$= \frac{-1}{(2\,\pi)^3} \int dp\, e^{ip(x'-x)}\,\varepsilon(p)\,(-p^2\,\delta_{\mu\nu} + p_\mu\, p_\nu)\,\pi(p^2) \tag{32}$$

$$\pi(p^2) \to \frac{V}{-3\,p^2} \sum_{p^{(z)}=p} \langle 0\,|\, j_\nu\,|\, z\rangle \langle z\,|\, j_\nu\,|\, 0\rangle\,. \tag{32a}$$

Here it can be observed that if we compute the vacuum expectation value of the anticommutator instead of the commutator we get

$$\langle 0\,|\,\{j_\mu(x), j_\nu(x')\}\,|\,0\rangle =$$

$$= \frac{1}{(2\,\pi)^3} \int dp\, e^{ip(x'-x)}\,(-p^2\,\delta_{\mu\nu} + p_\mu\, p_\nu)\,\pi(p^2) \tag{33}$$

with the same function $\pi\,(p^2)$ in (33) as in (32). This follows immediately from the analysis above. Noting that $\pi(p^2) = 0$ unless $p^2 < 0$ (this follows from Eq. (32a) and (24)) we can further write

$$\langle 0\,|\,[j_\mu(x), j_\nu(x')]\,|\,0\rangle =$$

$$= \frac{-1}{(2\,\pi)^3} \int dp\, e^{ip(x'-x)}\,\varepsilon(p) \int_0^\infty da\,\delta(p^2 + a)\,(-p^2\,\delta_{\mu\nu} + p_\mu\, p_\nu)\,\pi(-a) =$$

$$= -i \int_0^\infty da \left(\Box\,\delta_{\mu\nu} - \frac{\partial^2}{\partial x_\mu\,\partial x_\nu}\right) \Delta\,(x' - x, a)\,\pi(-a)\,. \tag{34}$$

Here $\Delta(x' - x, a)$ is the usual singular function constructed with the "mass" $\sqrt{a}$. Thus we also have

$$-\frac{i}{2}\,\varepsilon\,(x' - x)\langle 0\,|\,[j_\mu(x), j_\nu(x')]\,|\,0\rangle =$$

$$= \int_0^\infty da \left(\Box\delta_{\mu\nu} - \frac{\partial^2}{\partial x_\mu\,\partial x_\nu}\right) \bar{\Delta}\,(x' - x, a)\,\pi(-a) =$$

$$= \frac{1}{(2\,\pi)^4} \int dp\, e^{ip(x'-x)}\,(-p^2\,\delta_{\mu\nu} + p_\mu\, p_\nu)\,\bar{\pi}(p^2) \tag{35}$$

$$\bar{\pi}(p^2) = P \int_0^\infty \frac{\pi(-a)}{p^2 + a}\, da\,. \tag{35a*}$$

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*) The letter P in Eq. (35a) indicates that the principal value of the integral has to be taken in the point $a = -p^2$.

Returning to Eq. (25) we get with the aid of (24) and (32)

$$\langle 0 | [A_\mu(x), A_\nu(x')] | 0 \rangle =$$
$$= \frac{-1}{(2\pi)^3} \int dp\, e^{ip(x'-x)} \varepsilon(p) \int_0^\infty da\, \delta(p^2+a) F_{\mu\nu} \tag{36}$$

$$F_{\mu\nu} = \delta(a)(\delta_{\mu\nu} - 2M p_\mu p_\nu) + \frac{\pi(-a)}{a}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{a}\right) =$$
$$= \delta_{\mu\nu}\left(\delta(a) + \frac{\pi(-a)}{a}\right) + p_\mu p_\nu \left(\frac{\pi(-a)}{a^2} - 2M\delta(a)\right). \tag{37}$$

Putting $x_0' = x_0$ in (36) it follows from (5) that we must have

$$M = \frac{1}{2} \int_0^\infty \frac{\pi(-a)}{a^2}\, da. \tag{38}$$

The integral in (38) may diverge both for $a = 0$ and for $a = \infty$. The first divergency is of a kind usually classified as "infrared". It can always be avoided if we introduce a small photon mass μ. The function π will then vanish for $a < 9\mu^2$, and the denominator is zero only for $a = \mu^2$. We will not investigate this point further. The convergence of the integral at infinity will be discussed later in this paragraph.

Performing a differentiation with respect to the time t in (36) and putting the two times equal afterwards, we obtain

$$\left\langle 0 \left| \left[\frac{\partial A_\mu(x)}{\partial t}, A_\nu(x')\right] \right| 0 \right\rangle_{x_0 = x_0'} = -i\,\delta_{\mu\nu}\,\delta(\bar{x} - \bar{x}')\left(1 + \bar{\pi}(0)\right) -$$
$$- \frac{i}{(2\pi)^3} \int dp\, e^{i\bar{p}(\bar{x}'-\bar{x})} \int_0^\infty da\, p_0 p_\mu p_\nu \varepsilon(p)\,\delta(p^2+a)\left[\frac{\pi(-a)}{a^2} - 2M\delta(a)\right]. \tag{39}$$

If μ and ν are both equal to 1, 2 or 3, the last integral in (39) is zero due to Eq. (38). If they are both equal to 4, we get

$$\frac{-i}{(2\pi)^3} \int d^3p\, e^{i\bar{p}(\bar{x}'-\bar{x})} \int_0^\infty da\,(-\bar{p}^2 - a)\left(\frac{\pi(-a)}{a^2} - 2M\delta(a)\right) =$$
$$= i\,\delta(\bar{x}' - \bar{x})\,\bar{\pi}(0).$$

If only one of the indices μ and ν is equal to 4, the integral will vanish due to reasons of symmetry. We thus have

$$\left\langle 0 \left| \left[\frac{\partial A_\mu(x)}{\partial t}, A_\nu(x')\right] \right| 0 \right\rangle_{x_0 = x_0'} =$$
$$= -i\,\delta(\bar{x}' - \bar{x})\left[\delta_{\mu\nu}\left(1 + \bar{\pi}(0)\right) - \delta_{\mu 4}\delta_{\nu 4}\bar{\pi}(0)\right]. \tag{40}$$

Equations (40) and (6) are identical if

$$1+\bar{\pi}(0)=\frac{1}{1-L} \tag{41}$$

which is our formula for L[5]. It now only remains to verify that Eq. (7) is consistent with the formulae (38) and (41). With the same method as used above we find from Eq. (36)

$$\left\langle 0\left|\left[\frac{\partial A_\mu(x)}{\partial t},\frac{\partial A_\nu(x')}{\partial t'}\right]\right|0\right\rangle=\bar{\pi}(0)\left[\delta_{\mu 4}\frac{\partial}{\partial x_\nu}+\delta_{\nu 4}\frac{\partial}{\partial x_\mu}\right]\delta(\bar{x}'-\bar{x}) \tag{42}$$

which is in fact identical with (7) due to (41).

If we consider the definition of the function $\pi(p^2)$ in Eq. (32a), we observe that it is defined as a sum over only a finite number of terms or rather as an integral over a finite domain in p-space. (The two surfaces $p^2=0$ and $p^2=-m^2$, where in fact an infinite number of states exist, are of no importance, as the first kind of matrix elements will vanish in view of Eq. (24) and the second kind will vanish as a consequence of the charge-invariance.) If a solution to our equations exists at all, $\pi(p^2)$ is thus a finite quantity for all values of p^2. The question if L, M and $\bar{\pi}(p^2)$ are finite or not is thus answered, if we know the behaviour of $\pi(p^2)$ for large values of $-p^2$. The assumption of the renormalization method is that although $\bar{\pi}(0)$ might be infinite (and hence $L=1$) the difference $\bar{\pi}(p^2)-\bar{\pi}(0)$ (and thus also M) is finite. This means that $\pi(p^2)$ is *not* allowed to increase as strongly as $-p^2$ for large values of $-p^2$.

Concerning the general behaviour of the function $\pi(p^2)$, we will here only mention that it must be positive. This follows *e. g.* from the formula

$$V\sum_{p^{(z)}=p}\langle 0|j_\nu|z\rangle\langle z|j_\mu|0\rangle=(-p^2\,\delta_{\mu\nu}+p_\mu p_\nu)\,\pi(p^2) \tag{43}$$

if we here put $\mu=\nu=1$. As the x-component of the current operator is self-adjoint (not Hermitian, in view of the indefinite metric!) we have

$$\langle z|j_x|0\rangle=(\langle 0|j_x|z\rangle)^*(-1)^{N_4^{(z)}}. \tag{44}$$

In Eq. (44) $N_4^{(z)}$ means the number of scalar photons in the state $|z\rangle$. To obtain (44) we have made use of the explicit form of the

[5]) A similar formula for the charge-renormalization but in terms of the unrenormalized current operator has been given by UMEZAWA and KAMEFUCHI, Prog. Theor. Phys. **6**, 543 (1951).

metric operator as given by GUPTA[3]) and BLEULER[4]). Eq. (43) and (44) now give

$$\pi(p^2) = \frac{V}{p_x^2 - p^2} \sum_{p^{(z)}=p} |\langle 0 | j_x | z \rangle|^2 (-1)^{N_4^{(z)}}. \tag{45}$$

The negative contribution to the sum in (45) from a state with a scalar photon will exactly cancel with the contribution from a similar state with a longitudinal photon, if we observe that the current is a gauge-invariant operator, and hence that we can write

$$\langle 0 | j_\mu | z, k \rangle = F_{\mu\lambda} \langle 0 | A_\lambda^{(0)} | k \rangle \tag{46}$$

with

$$F_{\mu\lambda} k_\lambda = 0. \tag{47}$$

In (46) $| z, k \rangle$ means a state with one photon with energy-momentum k and other particles present, some of which may also be photons. The quantity $F_{\mu\lambda}$ then depends on the vector k (but not on the polarization vector of the photon) and on the annihilation operators of the other particles. The only surviving terms in (45) will then be the contributions from the transversal photons and these terms are all positive. It thus follows

$$\pi(p^2) \geqq 0 \tag{48}$$

$$\bar{\pi}(0) = \int_0^\infty \frac{\pi(-x)}{x} dx > 0 \tag{49}$$

$$0 < \frac{\bar{\pi}(0)}{1 + \bar{\pi}(0)} = L \leqq 1. \tag{50}$$

This property of the charge-renormalization has earlier been proved by SCHWINGER (unpublished) in a somewhat different way. I am indebted to professor PAULI for information concerning SCHWINGER's proof.

Definition of the Constant K.

The definition of K (the mass-renormalization) can be carried out in a way similar to that of the definition of L. Here we state the renormalization in terms of the matrix elements of the operators ψ between the vacuum state and a state $| q \rangle$ with only one electron present

$$\langle 0 | \psi(x) | q \rangle = \langle 0 | \psi^{(0)}(x) | q \rangle \tag{51}$$

or

$$\langle 0 | f(x) | q \rangle = 0. \tag{52}$$

To investigate the last condition we compute the vacuum expectation value of the anticommutator between $f(x)$ and $\bar{\psi}^{(0)}(x')$.

$$\langle 0|\{\bar{\psi}^{(0)}_\alpha(x'),\, f_\beta(x)\}|0\rangle = \sum_q (\langle 0|\bar{\psi}^{(0)}_\alpha(x')|q\rangle\langle q|f_\beta(x)|0\rangle + \\ + \langle 0|f_\beta(x)|q\rangle\langle q|\bar{\psi}^{(0)}_\alpha(x')|0\rangle). \tag{53}$$

With the aid of the formula

$$N^{-1}\cdot\bar{\psi}^{(0)}(x') = \int_{-\infty}^{x_0''=x_0} \bar{f}(x'')\, S(x''-x')\, dx'' - i\int_{x_0''=x_0} \bar{\psi}(x'')\,\gamma_4 S(x''-x')\, d^3x'' \tag{54}$$

(Eq. (54) follows from the equation of motion (8) — or rather its adjoint — and some well-known properties of the S-function) we can write

$$N^{-1}\cdot\langle 0|\{\bar{\psi}^{(0)}(x'), f(x)\}|0\rangle = \int_{-\infty}^{x} \langle 0|\{\bar{f}(x''), f(x)\}|0\rangle\, S(x''-x')\, dx'' - \\ - i\int_{x_0''=x_0} \langle 0|\{\bar{\psi}(x''), f(x)\}|0\rangle\,\gamma_4 S(x''-x')\, d^3x''. \tag{55}$$

The three-dimensional integral in (55) contains the anticommutator of operators for equal times and can thus be calculated with the aid of (4). In view of the definition of $f(x)$ in Eq.(2) we get

$$\{\bar{\psi}(x''), f(x)\}_{x_0=x_0''} = (ie\gamma A(x) + K)\,\gamma_4\,\delta(\bar{x}-\bar{x}'')\cdot N^{-2} \tag{56}$$

$$\langle 0|\{\bar{\psi}(x''), f(x)\}|0\rangle_{x_0=x_0''} = K'\gamma_4\,\delta(\bar{x}-\bar{x}'') \tag{57}$$

(in view of Eq. (18))

$$N^{-1}\langle 0|\{\bar{\psi}^{(0)}(x'), f(x)\}|0\rangle = \int_{-\infty}^{x} \langle 0|\{\bar{f}(x''), f(x)\}|0\rangle\, S(x''-x')\, dx'' - \\ - iK'S(x-x')\,;\ \ (K' = K\cdot N^{-2}). \tag{58}$$

Here again it is convenient to introduce a special notation for the anticommutator in (58). In analogy with (26) we write

$$\langle 0|\{\bar{f}_\alpha(x''), f_\beta(x)\}|0\rangle = \\ = \frac{-1}{(2\pi)^3}\Bigl(\int_{p_0>0} dp\, e^{ip(x-x'')}\{\textstyle\sum_1^{(+)}(p^2) + (i\gamma p+m)\sum_2^{(+)}(p^2)\}_{\beta\alpha} + \\ + \int_{p_0<0} dp\, e^{ip(x-x'')}\{\textstyle\sum_1^{(-)}(p^2) + (i\gamma p+m)\sum_2^{(-)}(p^2)\}_{\beta\alpha}\Bigr) \tag{59}$$

$$\Big(\sum_1^{(+)}(p^2)+(i\gamma p+m)\sum_2^{(+)}(p^2)\Big)_{\beta\alpha}=-V\sum_{p^{(z)}=p}\langle 0\,|\,f_\beta\,|\,z\rangle\,\langle z\,|\,\bar{f}_\alpha\,|\,0\rangle \quad (59\text{a})$$

$$\Big(\sum_1^{(-)}(p^2)+(i\gamma p+m)\sum_2^{(-)}(p^2)\Big)_{\beta\alpha}=-V\sum_{p^{(z)}=-p}\langle 0\,|\,\bar{f}_\alpha\,|\,z\rangle\,\langle z\,|\,f_\beta\,|\,0\rangle\,. \quad (59\text{b})$$

In view of the charge-invariance of the theory, we must have

$$-V\sum_{p^{(z)}=-p}\langle 0\,|\,(C^{-1}f)_\alpha\,|\,z\rangle\,\langle z\,|\,(C\bar{f})_\beta|\,0\rangle =$$

$$=\sum_1^{(-)}(p^2)\,\delta_{\beta\alpha}+(i\gamma p+m)_{\beta\alpha}\sum_2^{(-)}(p^2) \quad (60)$$

where C is the charge-conjugation matrix of SCHWINGER[6]), which has the following properties

$$C^T=-C \quad (61)$$

$$-C^{-1}\gamma_\mu\,C=\gamma_\mu^T\,. \quad (62)$$

If we compare (60) and (59a) we get, considering (61) and (62)

$$\sum_1^{(-)}(p^2)=-\sum_1^{(+)}(p^2) \quad (63)$$

$$\sum_2^{(-)}(p^2)=-\sum_2^{(+)}(p^2)\,. \quad (63\text{a})$$

We thus have

$$\langle 0\,|\{\bar{f}_\alpha(x''),\,f_\beta(x)\}|\,0\rangle =$$

$$=\frac{-1}{(2\pi)^3}\int dp\,e^{ip(x-x'')}\varepsilon(p)\{\sum_1(p^2)+(i\gamma p+m)\sum_2(p^2)\}_{\beta\alpha}\,. \quad (64)$$

As on page 423 we also have

$$\langle 0\,|\,[\bar{f}_\alpha(x''),\,f_\beta(x)]\,|\,0\rangle =$$

$$=\frac{1}{(2\pi)^3}\int dp\,e^{ip(x-x'')}\{\sum_1(p^2)+(i\gamma p+m)\sum_2(p^2)\}_{\beta\alpha} \quad (65)$$

and

$$-\frac{i}{2}\,\varepsilon(x-x'')\,\langle 0|\,\{\bar{f}_\alpha(x''),\,f_\beta(x)\}\,|\,0\rangle =$$

$$=\frac{1}{(2\pi)^4}\int dp\,e^{ip(x-x'')}\{\bar{\sum}_1(p^2)+(i\gamma p+m)\,\bar{\sum}_2(p^2)\}_{\beta\alpha} \quad (66)$$

$$\bar{\sum}_i(p^2)=P\int_{m^2}^{\infty}\frac{\Sigma_i(-a)}{p^2+a}\,da\,;\quad (i=1{,}2)\,. \quad (67)$$

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[6]) J. SCHWINGER, Phys. Rev. **74**, 1439 (1948).

With these notations we can write Eq. (58) as

$$N^{-1}\langle 0|\{\bar\psi^{(0)}(x'), f(x)\}|0\rangle =$$
$$= \frac{-1}{(2\pi)^3}\int dx''\int dp\, e^{ip(x-x'')}\frac{\varepsilon(p)}{2}\{\Sigma_1(p^2)+(i\gamma p+m)\Sigma_2(p^2)\}S(x''-x')+$$
$$+\frac{i}{(2\pi)^4}\int dx''\int dp\, e^{ip(x-x'')}\{\bar\Sigma_1(p^2)+(i\gamma p+m)\bar\Sigma_2(p^2)\}S(x''-x')-$$
$$-iK'S(x-x'). \tag{68}$$

As

$$S(x)=\frac{-i}{(2\pi)^3}\int dp\, e^{ipx}\varepsilon(p)(i\gamma p-m)\delta(p^2+m^2)$$

Eq. (68) can also be written

$$N^{-1}\cdot\langle 0|\{\bar\psi^{(0)}(x'), f(x)\}|0\rangle =$$
$$= -i[K'-\bar\Sigma_1(-m^2)-i\pi\varepsilon(p)\Sigma_1(-m^2)]\,S(x-x'). \tag{69}$$

From Eq. (52), however, it follows that the right-hand side of (69) must vanish and, as $\Sigma_1(-m^2)=0$ in view of *e. g.* (59a), this means

$$K'=\bar\Sigma_1(-m^2)=\int_{m^2}^{\infty}\frac{\Sigma_1(-a)}{a-m^2}\,da=K\cdot N^{-2}. \tag{70}$$

Eq. (70) gives the formula for K. Returning now to Eq. (51), we can write the matrix element of ψ between the vacuum and an one-electron state

$$\langle 0|\psi(x)|q\rangle=\frac{1}{N}\langle 0|\psi^{(0)}(x)|q\rangle+\left(1-\frac{1}{N}\right)\langle 0|\psi^{(0)}(x)|q\rangle. \tag{71}$$

The normalization constant N can be determined from the anti-commutator of $\bar\psi$ and ψ for equal times. Computing the vacuum expectation value of this quantity, we get in analogy with Eq. (25)

$$\langle 0|\{\bar\psi(x),\psi(x')\}|0\rangle=\frac{-i}{N^2}S(x'-x)[1+2(N-1)]+$$
$$+\int\int dx''dx'''\,S_R(x'-x'')\langle 0|\{f(x''),\bar f(x''')\}|0\rangle S_A(x'''-x)=$$
$$=\frac{-1}{(2\pi)^3}\int dp\, e^{ip(x'-x)}\varepsilon(p)\Big[\delta(p^2+m^2)\frac{1+2(N-1)}{N^2}+$$
$$+\frac{i\gamma p-m}{(p^2+m^2)^2}\{\Sigma_1(p^2)+(i\gamma p+m)\Sigma_2(p^2)\}\Big](i\gamma p-m). \tag{72}$$

As

$$\frac{i\gamma p-m}{p^2+m^2}(\Sigma_1+(i\gamma p+m)\Sigma_2)\frac{i\gamma p-m}{p^2+m^2}=$$
$$=(i\gamma p-m)\left[\frac{-\Sigma_2}{p^2+m^2}-\frac{2m\Sigma_1}{(p^2+m^2)^2}\right]-\frac{\Sigma_1}{p^2+m^2} \tag{73}$$

we get for equal times with the aid of (4)

$$N^{-2}\gamma_4\,\delta(\bar{x}-\bar{x}') = \gamma_4\,\delta(\bar{x}-\bar{x}')\left[\frac{1+2(N-1)}{N^2} - \right.$$
$$\left. - \int_{m^2}^{\infty} d a \left\{\frac{-\Sigma_2(-a)}{a-m^2} + \frac{2\,m\,\Sigma_1(-a)}{(a-m^2)^2}\right\}\right] \tag{74}$$

and hence

$$\frac{N-1}{N^2} = \frac{-1}{2}\left(\overline{\Sigma}_2(-m^2) + 2\,m\,\overline{\Sigma}_1{}'(-m^2)\right) \tag{75}$$

$$\overline{\Sigma}_1{}'(-m^2) = -\int_{m^2}^{\infty} \frac{\Sigma_1(-a)\,d a}{(a-m^2)^2} = \frac{d\,\overline{\Sigma}_1(p^2)}{d\,p^2}\bigg|_{p^2=-m^2}. \tag{76}$$

As was the case with the function $\bar{\pi}(p^2)$, it is necessary if the renormalization method is consistent that the difference

$$\overline{\Sigma}_i(p^2) - \overline{\Sigma}_i(-m^2) \qquad (i = 1, 2)$$

is finite, or that the integrals

$$\int_{m^2}^{\infty} \frac{\Sigma_i(-a)\,d a}{(p^2+a)\,(a-m^2)}$$

will converge. The last term in (75) is thus a finite quantity (apart from an infrared divergency for $a = m^2$) but the first integral might be infinite. This is, however, not serious, as the normalization constant itself is not observable. As a matter of fact, it has been shown by WARD[7]) that, for an observable quantity, all infinities of this kind will disappear from the coefficients in an expansion in powers of the charge.

We will end this paragraph with the observation that if one considers Eq. (68) as an identity in x', one concludes *e. g.*

$$\langle 0\,|\,f(x)\,|\,q\rangle = \frac{N}{(2\,\pi)^4}\int d\,x''\int d\,p\,e^{i\,p(x-x'')}\times$$
$$\times\left[K' - \overline{\Sigma}_1(p^2) - i\,\pi\,\varepsilon(p)\,\Sigma_1(p^2) - \right.$$
$$\left. -(i\,\gamma\,p+m)\left(\overline{\Sigma}_2(p^2) + i\,\pi\,\varepsilon(p)\,\Sigma_2(p^2)\right)\right]\langle 0\,|\,\psi^{(0)}(x'')\,|\,q\rangle \tag{77}$$

where the equation of motion for $\psi^{(0)}$ has *not* been used. One could

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[7]) J. C. WARD, Phys. Rev. **78**, 182 (1950).

then try to compute the normalization constant N from (77) in the following way

$$\left(1-\frac{1}{N}\right)\langle 0\,|\,\psi^{(0)}(x)\,|\,q\rangle = -\int S_R(x-x')\,\langle 0\,|\,f(x')\,|\,q\rangle\,d\,x' =$$

$$= \frac{N}{(2\,\pi)^4}\int d\,x''\int d\,p\left[\frac{\overline{\Sigma}_1(p^2)-K'}{p^2+m^2}\,(i\gamma\,p-m)+i\,\pi\,\varepsilon(p)\,\frac{\Sigma_1(p^2)}{p^2+m^2}\,(i\gamma\,p-m)-\right.$$

$$\left.-\overline{\Sigma}_2(p^2)-i\,\pi\,\varepsilon(p)\,\Sigma_2(p^2)\right]\langle 0\,|\,\psi^{(0)}(x'')\,|\,q\rangle\,e^{i\,p\,(x-x'')} =$$

$$= N\left\{-\overline{\Sigma}_2(-m^2)-2\,m\lim_{p^2+m^2\to 0}\frac{\overline{\Sigma}_1(p^2)-\overline{\Sigma}_1(-m^2)}{p^2+m^2}\right\}\langle 0\,|\,\psi^{(0)}(x)\,|\,q\rangle. \quad (78)$$

In Eq. (78) the equation of motion for $\psi^{(0)}$ and the fact that $\Sigma_i(p^2)$ vanishes for $-p^2 < (m+\mu)^2$ has been used in the *last* step of the computation. (μ is the small photon mass introduced to avoid infrared divergencies.) The value obtained in this way for N is, however, not the correct value in Eq. (75). This error comes from the way in which we have ambiguously put

$$-\int S_R(x-x')\left(\gamma\,\frac{\partial}{\partial\,x'}+m\right)\psi^{(0)}(x')\,d\,x' = \psi^{(0)}(x'). \qquad (79)$$

The left-hand side of (79) is not a well-defined mathematical symbol, and the example above shows that a formula of the kind of Eq. (79) is not always to be trusted. Similar observations have been made in the past by many authors[8]).

I want to express my deep gratitude to professor W. Pauli for his kind interest and valuable criticism and to the *Swedish Atomic Committee* for financial support.

Appendix.

The physical meaning of the functions $\pi(p^2)$ and $\overline{\pi}(p^2)$ can be made clearer if we consider a system with an external electromagnetic field. The influence of such a field can be taken into account if we add the following two terms to the Lagrangian (1)

$$\frac{i\,e\,N^2}{2}\,A_\mu^{(e)}(x)\,[\overline{\psi},\gamma_\mu\,\psi]+L\,A_\mu^{(i)}\,j_\mu^{(e)}(x). \qquad (A.1)$$

[8]) Cf. *e.g.* R. Karplus-N. M. Kroll, Phys. Rev. **77**, 542 (1950); F. J. Dyson, Phys. Rev. **75**, 1736 (1949) and G. Källén, Ark. f. Fys. **2**, 371 (1950).

We then get the following equations of motion

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi(x) = \frac{i e}{2} \{A_\nu^{(i)}(x), \gamma_\nu \psi(x)\} + i e A_\nu^{(e)}(x) \gamma_\nu \psi(x) + \\ + K \psi(x) \qquad \text{(A. 2)}$$

$$\Box A_\mu^{(i)}(x) = -\frac{i e N^2}{2} [\bar{\psi}(x), \gamma_\mu \psi(x)] + L\Big(\Box A_\mu^{(i)}(x) - \\ - \frac{\partial^2 A_\nu^{(i)}(x)}{\partial x_\mu \partial x_\nu} - j_\mu^{(e)}(x)\Big). \qquad \text{(A. 3)}$$

In the formulae above, $A_\mu^{(e)}(x)$ is the external field and $A_\mu^{(i)}(x)$ the induced field. The former is a c-number and the latter an operator. The external current $j_\mu^{(e)}(x)$ is given by

$$j_\mu^{(e)}(x) = -\left(\Box A_\mu^{(e)}(x) - \frac{\partial^2 A_\nu^{(e)}(x)}{\partial x_\mu \partial x_\nu}\right). \qquad \text{(A. 4)}$$

If we suppose that we know the solution ($\boldsymbol{\psi}(x)$ and $\boldsymbol{A}_\mu(x)$) when the external field is zero, and that we have a situation where the external field is very weak, we can expand the operators above in a power series of the *external field*. It can be verified without difficulty that the first two terms in such an expansion are respectively

$$\psi(x) = \boldsymbol{\psi}(x) - i \int_{-\infty}^{x} [\boldsymbol{j}_\nu(x'), \boldsymbol{\psi}(x)] A_\nu^{(e)}(x') \, dx' \qquad \text{(A. 5)}$$

and

$$A_\mu^{(i)}(x) = \boldsymbol{A}_\mu(x) - i \int_{-\infty}^{x} [\boldsymbol{j}_\nu(x'), \boldsymbol{A}_\mu(x)] A_\nu^{(e)}(x') \, dx' + \\ + \frac{L}{1-L} (\delta_{\mu\nu} - \delta_{4\mu} \delta_{4\nu}) A_\nu^{(e)}(x). \qquad \text{(A. 6)}$$

Substituting *e. g.* Eq. (A. 6) into the left-hand side of Eq. (A. 3) we obtain

$$\Box A_\mu^{(i)}(x) = -\boldsymbol{j}_\mu(x) - \frac{e N^2}{2(1-L)} \int_{-\infty}^{x} [[\boldsymbol{j}_\nu(x'), \bar{\boldsymbol{\psi}}(x)], \gamma_\mu \boldsymbol{\psi}(x)] A_\nu^{(e)}(x') \, dx' - \\ - \frac{e N^2}{2(1-L)} \int_{-\infty}^{x} [\bar{\boldsymbol{\psi}}(x) \gamma_\mu, [\boldsymbol{j}_\nu(x'), \boldsymbol{\psi}(x)]] A_\nu^{(e)}(x') \, dx' + \\ + i \int_{x_0 = x_0'} d^3 x' \left([\boldsymbol{j}_\nu(x'), \boldsymbol{A}_\mu(x)] \frac{\partial A_\nu^{(e)}(x')}{\partial t'} + \left[\boldsymbol{j}_\nu(x'), \frac{\partial \boldsymbol{A}_\mu(x)}{\partial t}\right] A_\nu^{(e)}(x')\right) + \\ + \frac{i L}{1-L} \int_{-\infty}^{x} \left[\boldsymbol{j}_\nu(x'), \frac{\partial^2 \boldsymbol{A}_\lambda(x)}{\partial x_\mu \partial x_\lambda}\right] A_\nu^{(e)}(x') \, dx' + \frac{L}{1-L} (\delta_{\mu\nu} - \delta_{\mu 4} \delta_{\nu 4}) \Box A_\nu^{(e)}(x). \qquad \text{(A. 7)}$$

Using Eq. (A. 5), (A. 6) and the formulae

$$i \int\limits_{x_0 = x_0'} d^3 x' \left([\boldsymbol{j}_\nu(x'), \boldsymbol{A}_\mu(x)] \frac{\partial A_\nu^{(e)}(x')}{\partial t'} + \left[\boldsymbol{j}_\nu(x'), \frac{\partial \boldsymbol{A}_\mu(x)}{\partial t} \right] A_\nu^{(e)}(x') \right) =$$

$$= \frac{-L}{1-L} \frac{\partial^2 A_\nu^{(e)}(x)}{\partial x_\mu \partial x_\nu} + \frac{L}{1-L} \delta_{\mu 4} \Box A_4^{(e)}(x) \qquad \text{(A. 8)}$$

and

$$\int\limits_{-\infty}^{x} \left[\boldsymbol{j}_\nu(x'), \frac{\partial^2 \boldsymbol{A}_\lambda(x)}{\partial x_\mu \partial x_\lambda} \right] A_\nu^{(e)}(x')\, dx' = \frac{\partial^2}{\partial x_\mu \partial x_\lambda} \int\limits_{-\infty}^{x} [\boldsymbol{j}_\nu(x'), \boldsymbol{A}_\lambda(x)]\, A_\nu^{(e)}(x')\, dx' +$$

$$+ \frac{i L}{1-L} \frac{\partial}{\partial x_\mu} \sum_{k=1}^{3} \frac{\partial A_k^{(e)}(x)}{\partial x_k} \qquad \text{(A. 9)}$$

we can simplify the right-hand side of (A. 7) to

$$\frac{-i e N^2}{2(1-L)} \left([\bar{\psi}(x), \gamma_\mu \psi(x)] - [\bar{\psi}(x) - \bar{\boldsymbol{\psi}}(x), \gamma_\mu (\psi(x) - \boldsymbol{\psi}(x))] \right) -$$

$$- \frac{L}{1-L} \left(\frac{\partial^2 A_\nu^{(i)}(x)}{\partial x_\mu \partial x_\nu} + j_\mu^{(e)}(x) \right). \qquad \text{(A. 10)}$$

This expression differs from the correct current only in terms which are of second order in $A_\mu^{(e)}(x)$. The verification of (A. 5) can be performed along similar lines and will not be given explicitly.

If we now compute the vacuum-expectation value of the current operator, we obtain from (A. 6) with the aid of (A. 8)

$$\langle 0 | j_\mu^{(i)}(x) | 0 \rangle = \frac{-i}{2} \int\limits_{-\infty}^{+\infty} \left(1 + \varepsilon(x - x')\right) \langle 0 | [\boldsymbol{j}_\nu(x'), \boldsymbol{j}_\mu(x)] | 0 \rangle\, A_\nu^{(e)}(x')\, dx' +$$

$$+ \frac{L}{1-L} j_\mu^{(e)}(x) \qquad \text{(A. 11)}$$

or

$$\langle 0 | j_\mu^{(i)}(x) | 0 \rangle =$$

$$= \frac{1}{(2\pi)^4} \int dp\, e^{ipx} \left(-\bar{\pi}(p^2) + \bar{\pi}(0) - i\pi\, \varepsilon(p)\, \pi(p^2)\right) j_\mu^{(e)}(p) \qquad \text{(A. 12)}$$

where

$$j_\mu^{(e)}(x) = \frac{1}{(2\pi)^4} \int dp\, e^{ipx} j_\mu^{(e)}(p). \qquad \text{(A. 13)}$$

The vacuum thus behaves as a medium with the complex dielectricity-constant

$$\varepsilon(p^2) = 1 - \bar{\pi}(p^2) + \bar{\pi}(0) - i\pi\, \varepsilon(p)\, \pi(p^2). \qquad \text{(A. 14)}$$

The connection between the real and the imaginary part of $\varepsilon(p^2)$, which is expressed in Eq. (35a), is very similar to a formula that has been given by KRAMERS[9]) for a dielectricum.

The expectation value of the energy which is transferred per unit time from the external field to the system of particles is given by

$$\int d^3x \langle 0 | \frac{\partial L}{\partial t} | 0 \rangle = \int d^3x \Big(\frac{\partial A_\nu^{(e)}(x)}{\partial t} \langle 0 | j_\nu^{(i)}(x) | 0 \rangle +$$

$$+ L \Big[\langle 0 | A_\nu^{(i)}(x) | 0 \rangle \frac{\partial j_\nu^{(e)}(x)}{\partial t} - \langle 0 | j_\nu^{(i)}(x) | 0 \rangle \frac{\partial A_\nu^{(e)}(x)}{\partial t} - j_\nu^{(e)}(x) \frac{\partial A_\nu^{(e)}(x)}{\partial t} \Big] \Big). \quad \text{(A.15)}$$

The time average of the terms proportional to L is zero, and the average of the first term is equal to

$$\frac{1}{(2\pi)^4} \int dp \, \pi(p^2) \frac{|p_0|}{-p^2} \frac{1}{2} j_\nu^{(e)}(p) \, j_\nu^{(e)}(-p) \quad \text{(A. 16)}$$

which is thus the energy of the real particles (photons and electron pairs) which are created per unit time by the external field.

[9]) M. H. A. KRAMERS, Cong. Int. d. Fisici, Como, Settembre 1927.