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Charge Renormalization and the Identity of Ward

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Summary. The importance of the identity of Ward for the consistency of the charge renormalization is pointed out. A proof of the identity is given without aid of perturbation theory.

Introduction.

Charge renormalization was introduced in quantum electrodynamics in a paper by SCHWINGER[1]), where it was defined in the e^2-approximation and for the problem of vacuum polarization in an external, electromagnetic field. It was, however, soon remarked by DYSON[2]) that, actually, one has to deal with two different kinds of charge renormalization in quantum electrodynamics, in his paper called "external" and "internal" renormalization. DYSON introduced two different (infinite) constants Z_1 and Z_2 to take care of these renormalizations, but conjectured that they were possibly equal. Later on, it was proved by WARD[3]) that Z_1 is actually equal to Z_2, and consequently GUPTA[4]) introduced only one constant in his treatment of the charge renormalization. All these authors use, explicitly or implicitly, perturbation theory, and each renormalization constant is defined with the aid of a power series in e^2, where every coefficient is infinite[5]). In the treatment of the renormalization technique without aid of perturbation theory introduced by the author[6]), the GUPTA formalism for the charge renormalization was followed, and only one constant L was introduced to handle this problem, in other words, the identity of WARD was implicitly assumed to hold. It is the aim of the following note to make a clear

1) J. SCHWINGER, Phys. Rev. **75**, 651 (1949). Cf. also V. F. WEISSKOPF, Dan. Mat. Fys. Medd. **14**, no. 6 (1936).
2) F. J. DYSON, Phys. Rev. **75**, 486, 1736 (1949).
3) J. C. WARD, Phys. Rev. **78**, 182 (1950); Proc. Phys. Soc. **A 64**, 54 (1951).
4) S. GUPTA, Proc. Phys. Soc. **A 64**, 426 (1951).
5) Cf. also G. TAKEDA, Prog. Theor. Phys. **7**, 359 (1952), where an argument partly independent of perturbation theory is given.
6) G. KÄLLÉN, Helv. Phys. Acta **25**, 417 (1952), here quoted as I; Dan. Mat. Fys. Medd. **27**, no. 12 (1953), here quoted as II.

distinction between the two kinds of charge renormalization and to give a proof of their equivalence without aid of perturbation theory. The proof, which uses charge conservation for the total system over finite time intervals, may also be of some methodological interest itself. In the discussion below, the content of papers I and II is assumed to be known. The notation of these papers is often used without further explanations.

The External Charge Renormalization.

The definition of the charge renormalization in quantum electrodynamics was given in I as

$$\langle 0 \,|\, A_\mu(x) \,|\, k \rangle = \left[\delta_{\mu\nu} + M \frac{\partial^2}{\partial x_\mu \, \partial x_\nu}\right] \langle 0 \,|\, A_\nu^{(0)}(x) \,|\, k \rangle . \tag{1}$$

This was shown to lead to the formula

$$\frac{1}{1-L} = 1 + \overline{\Pi}\,(0) \tag{2}$$

which can be understood as an implicit formula for L. The physical meaning of this somewhat abstract formalism is perhaps better understood after a discussion of the vacuum expectation value of the current operator in a system with a very weak external field. It was shown in the appendix of I that this quantity can be written

$$\langle 0 \,|\, j_\mu(x) \,|\, 0 \rangle = \frac{1}{(2\,\pi)^4} \int dp \; e^{ipx} \, j_\mu^{ext}(p) \; \varepsilon\,(p^2) \tag{3}$$

where

$$j_\mu^{ext}(x) = \frac{1}{(2\,\pi)^4} \int dp \; e^{ipx} \, j_\mu^{ext}(p) = \text{the external current} \tag{4}$$

and

$$\begin{aligned} \varepsilon\,(p^2) &= 1 - \overline{\Pi}(p^2) + \overline{\Pi}(0) - i\,\pi\,\varepsilon\,(p)\,\Pi\,(p^2) = \\ &= \text{the “dielectric constant of the vacuum”.} \end{aligned} \tag{5}$$

The term $\overline{\Pi}(0)$ in (5) is a direct consequence of equation (2) and of the introduction of the renormalization term

$$-L\left(\Box A_\mu(x) - \frac{\partial^2 A_\nu(x)}{\partial x_\mu \, \partial x_\nu}\right) \tag{6}$$

in the definition of the current operator. Since the function $\Pi(p^2)$ is equal to zero for $-p^2 \leqq 0$, it follows that the dielectric constant of the vacuum is normalized to one for a light wave as a consequence of the charge renormalization. This shows the connection with SCHWINGER's way of defining the charge renormalization and corresponds to what DYSON calls “external” charge renormalization.

The Internal Charge Renormalization.

A quite different way of defining the charge renormalization in quantum electrodynamics would be to normalize the expectation value of the charge operator Q

$$Q = -i \int d^3x \, j_4(x) \tag{7}$$

to e for an one-electron state[7]). This corresponds to the "internal" charge renormalization of DYSON[2]). In order to discuss this point in more detail we note the following formulae in II (equations (23), (24) and (53)):

$$\begin{aligned}
\langle 0 | j_\mu | q, q' \rangle = {} & \langle 0 | j_\mu^{(0)} | q, q' \rangle \left[\varepsilon\left((q+q')^2\right) + 2 \frac{N-1}{1-L} \right] + \\
& + i\, e \langle 0 | \bar{\psi}^{(0)} | q' \rangle \, \Lambda_\mu(-q'; q) \, \langle 0 | \psi^{(0)} | q \rangle = \\
& = \langle 0 | j_\mu^{(0)} | q, q' \rangle \left[\varepsilon\left((q+q')^2\right) + 2 \frac{N-1}{1-L} + R\left((q+q')^2\right) \right] + \\
& + \frac{e}{2m} S\left((q+q')^2\right) (q_\mu - q'_\mu) \langle 0 | \bar{\psi}^{(0)} | q' \rangle \langle 0 | \psi^{(0)} | q \rangle \, ,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\frac{i\, e}{(2\pi)^8} \int\!\!\int dp\, dp'\, e^{ip'(3x) + ip(x4)} \Lambda_\mu(p'; p) = {} & N^2 \theta(x3)\, \theta(x4) \langle 0 | \{ f(3), \\
& [j_\mu(x), \bar{f}(4)] \} | 0 \rangle - N^2 \theta(x3)\, \theta(34) \langle 0 | [j_\mu(x), \{ f(3), \bar{f}(4) \}] | 0 \rangle - \\
& - 2\, i\, e (N-1) \frac{L}{1-L} \delta_{\mu 4} \gamma_4 \delta(x3)\, \delta(34) \, ,
\end{aligned} \tag{9}$$

$\varepsilon(p^2)$ is the function defined in equation (5), and where the functions $R(p^2)$ and $S(p^2)$ are defined with the aid of equation (8) above (or with the aid of equation (53) in II). From equation (22) in II we also conclude

$$\begin{aligned}
\langle q | j_\mu | q' \rangle = {} & \langle q | j_\mu^{(0)} | q' \rangle \left[\varepsilon\left((q'-q)^2\right) + 2 \frac{N-1}{1-L} + R\left((q'-q)^2\right) \right] + \\
& + \frac{e}{2m} S\left((q'-q)^2\right) (q'_\mu + q_\mu) \langle q | \bar{\psi}^{(0)} | 0 \rangle \langle 0 | \psi^{(0)} | q' \rangle = \\
& = \langle q | j_\mu^{(0)} | q' \rangle \left[\varepsilon\left((q'-q)^2\right) + 2 \frac{N-1}{1-L} + R\left((q'-q)^2\right) + \right. \\
& \left. + S\left((q'-q)^2\right) \right] + \frac{e}{4m} S\left((q'-q)^2\right) (q'_\nu - q_\nu) \langle q | \bar{\psi}^{(0)} | 0 \rangle \times \\
& \times (\gamma_\nu \gamma_\mu - \gamma_\mu \gamma_\nu) \langle 0 | \psi^{(0)} | q' \rangle \, .
\end{aligned} \tag{10}$$

In equation (10) both $|q\rangle$ and $|q'\rangle$ are one-electron states. The state $|q'\rangle$ in equation (8) is a one-positron state. If we put $\mu = 4$,

[7]) For the definition of particle numbers cf. I and G. KÄLLÉN, Physica **19** (1953).

$|q\rangle = |q'\rangle$ and integrate over three-dimensional space, we get from equation (10)

$$\langle q | Q | q \rangle = -i \int d^3 x \langle q | j_4^{(0)} | q \rangle \left[\varepsilon(0) + 2 \frac{N-1}{1-L} + R(0) + S(0) \right] = \\ = e \left[1 + 2 \frac{N-1}{1-L} + R(0) + S(0) \right] . \tag{11}$$

It now appears that the condition $\varepsilon(0) = 1$, which followed as a consequence of the charge renormalization, is not in itself sufficient to ensure the value e for the one-electron expectation value of the charge operator. To achieve this we must further have

$$R(0) + S(0) = -2 \frac{N-1}{1-L} . \tag{12}$$

Equation (12) corresponds in our notations to the identity of WARD.

One might be inclined to think that equation (12) could be understood as a consequence of a conservation of the charge of the state $|q\rangle$ from $-\infty$ to the time x_0. However, such an argument is not quite convincing. If the limit $t \to -\infty$ is performed in the formalism, certain formal prescriptions, usually described as an adiabatic switching off of the charge, have to be used to give the limit a well-defined meaning[7]). It appears at least doubtful to use a charge conservation for infinite time intervals under such circumstances. It has further been shown by SCHWINGER[8]) with the aid of an explicit calculation that one easily gets contradictions after a careless use of charge conservation over infinite time intervals. On the other hand, the argument given in the next section will prove equation (12) as a consequence of charge conservation over finite time intervals. The result of our analysis can then be interpreted as a conservation of the quantity $\langle q | Q | q \rangle / e$ also during the switching on process.

Proof of Equation (12).

We start by putting $\mu = 4$ and integrating over three-dimensional space in equation (9)

$$\frac{i e}{(2\pi)^5} \int\int dp\, dp' e^{-i p_0' (x_0^{\mathrm{III}} - x_0) - i p_0 (x_0 - x_0^{\mathrm{IV}})} \delta(\bar{p} - \bar{p}') e^{i \bar{p} (\bar{x}^{\mathrm{III}} - \bar{x}^{\mathrm{IV}})} \Lambda_4(p'; p) = \\ = i N^2 \theta(x3) \theta(x4) \langle 0 | \{ f(3), [Q, \bar{f}(4)] \} | 0 \rangle - \\ - i N^2 \theta(x3) \theta(34) \langle 0 | [Q, \{ f(3), \bar{f}(4) \}] | 0 \rangle - \\ - 2 ie (N-1) \frac{L}{1-L} \gamma_4 \delta(34) \delta(x_0 - x_0^{\mathrm{III}}) . \tag{13}$$

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[8]) J. SCHWINGER, Phys. Rev. **76**, 790 (1949), appendix.

As the operator Q is a constant of motion we can write

$$[Q, f(3)] = -i \int\limits_{x_0^{\rm III} = x_0} d^3x \, [j_4(x), f(3)] = -e\, f(3) \tag{14a}$$

and

$$[Q, \bar{f}(4)] = e\, \bar{f}(4) \,. \tag{14b}$$

Note that only the time independence of the operator Q over the finite time intervals $x_0 - x_0^{\rm III}$ and $x_0 - x_0^{\rm IV}$ is used in the argument. With the aid of equations (13), (14a) and (14b) we obtain

$$\begin{aligned} \frac{i\,e}{(2\,\pi)^5} \int\!\!\int dp\, dp'\, e^{ip(34) + i(p_0' - p_0)(x_0 - x_0^{\rm III})}\, \delta(\bar{p} - \bar{p}')\, \Lambda_4(p'; p) = \\ = i\, e\, N^2\, \theta(x3)\, \theta(x4) \langle 0 | \{ f(3), \bar{f}(4) \} | 0 \rangle - \\ - 2\, ie\, (N-1) \frac{L}{1-L} \gamma_4\, \delta(34)\, \delta(x_0 - x_0^{\rm III}) \,. \end{aligned} \tag{15}$$

Multiplying with $e^{iq_0 x_0}$ and integrating over x_0, we get

$$\begin{aligned} \frac{1}{(2\,\pi)^4} \int dp\, e^{ip(34) + iq_0 x_0^{\rm III}}\, \Lambda_4(\bar{p}, p_0 - q_0; p) = N^2 \int dx_0 e^{iq_0 x_0}\, \theta(x3)\, \theta(x4) \times \\ \times \langle 0 | \{ f(3), \bar{f}(4) \} | 0 \rangle - 2(N-1) \frac{L}{1-L} \gamma_4\, \delta(34)\, e^{iq_0 x_0^{\rm III}} \,. \end{aligned} \tag{16}$$

From the formula

$$\begin{aligned} \int dx_0\, e^{iq_0 x_0}\, \theta(x3)\, \theta(x4) = \Big\{ \pi\, \delta(q_0) + i [\theta(34) + \\ + \theta(43)\, e^{iq_0 (x_0^{\rm IV} - x_0^{\rm III})}]\, P \frac{1}{q_0} \Big\}\, e^{iq_0 x_0^{\rm III}} \end{aligned} \tag{17}$$

we obtain

$$\begin{aligned} \frac{1}{(2\,\pi)^4} \int \Lambda_4(\bar{p}, p_0 - q_0; p)\, dp\, e^{ip(34)} = \pi\, N^2\, \delta(q_0) \frac{-1}{(2\,\pi)^3} \int dp\, e^{ip(34)} \times \\ \times\, \varepsilon(p) \{ \Sigma_1(p^2) + (i\gamma p + m)\, \Sigma_2(p^2) \} - \\ - \frac{N^2}{(2\,\pi)^4} \int dp\, e^{ip(34)} \Big\{ \bar{\Sigma}_1(p^2) + (i\gamma p + m)\, \bar{\Sigma}_2(p^2) + \\ + i\pi\varepsilon(p) \big(\Sigma_1(p^2) + (i\gamma p + m)\, \Sigma_2(p^2) \big) - \\ - \bar{\Sigma}_1\big(\bar{p}^2 - (p_0 - q_0)^2\big) - \big[i\big(\gamma_k p_k + i\gamma_4 (p_0 - q_0)\big) + m \big] \times \\ \times\, \bar{\Sigma}_2\big(\bar{p}^2 - (p_0 - q_0)^2\big) + i\pi\, \varepsilon(p - q) \Big(\Sigma_1\big(\bar{p}^2 - (p_0 - q_0)^2\big) + \\ + \big[i\big(\gamma_k p_k + i\gamma_4 (p_0 - q_0)\big) + m \big]\, \Sigma_2\big(\bar{p}^2 - (p_0 - q_0)^2\big) \Big) \Big\} \times \\ \times\, P \frac{1}{q_0} - 2(N-1) \frac{L}{1-L} \gamma_4 \frac{1}{(2\,\pi)^4} \int dp\, e^{ip(34)} \,. \end{aligned} \tag{18}$$

Equation (18) can be solved directly for $\Lambda_4(\overline{p}, p_0 - q_0; p)$. The result is

$$\Lambda_4(\overline{p}, p_0 - q_0; p) = -N^2 \left\{ \frac{\bar{\Sigma}_1(p^2) - \bar{\Sigma}_1(p^2 + 2\,p_0 q_0 - q_0^2)}{q_0} + (i\gamma p + m) \times \right.$$
$$\left. \times \frac{\bar{\Sigma}_2(p^2) - \bar{\Sigma}_2(p^2 + 2\,p_0 q_0 - q_0^2)}{q_0} - \gamma_4 \bar{\Sigma}_2(p^2 + 2\,p_0 q_0 - q_0^2) \right\} -$$
$$- 2(N-1)\frac{L}{1-L}\gamma_4 . \tag{19}$$

Equation (19) holds in the domain

$$-p^2 \leqq (m+\mu)^2; \; -p^2 - 2\,p_0 q_0 + q_0^2 \leqq (m+\mu)^2 .$$

From our point of view, the interesting quantity is

$$ie \langle q \,|\, \bar{\psi}^{(0)} \,|\, 0 \rangle\, \Lambda_4(q; q)\, \langle 0 \,|\, \psi^{(0)} \,|\, q \rangle = \big(R(0) + S(0)\big) \langle q \,|\, j_4^{(0)} \,|\, q \rangle . \tag{20}$$

From equation (19) follows

$$ie \langle q \,|\, \bar{\psi}^{(0)} \,|\, 0 \rangle\, \Lambda_4(q; q)\, \langle 0 \,|\, \psi^{(0)} \,|\, q \rangle = \Big\{ N^2 \big[2m\, \bar{\Sigma}_1'(-m^2) +$$
$$+ \bar{\Sigma}_2(-m^2)\big] - 2\frac{N-1}{1-L} L \Big\} \langle q \,|\, j_4^{(0)} \,|\, q \rangle . \tag{21}$$

(Note that

$$\langle q \,|\, \bar{\psi}^{(0)} \,|\, 0 \rangle\, q_0 \,\langle 0 \,|\, \psi^{(0)} \,|\, q \rangle = m \langle q \,|\, \bar{\psi}^{(0)} \,|\, 0 \rangle\, \gamma_4 \langle 0 \,|\, \psi^{(0)} \,|\, q \rangle) .$$

Using the definition of $R(p^2)$ and $S(p^2)$ we conclude from I equation (75)

$$R(0) + S(0) = N^2 \big[2m\, \bar{\Sigma}_1'(-m^2) + \bar{\Sigma}_2(-m^2)\big] - 2(N-1)\frac{L}{1-L} =$$
$$= -2\frac{N-1}{1-L} . \tag{22}$$

Formula (22) is identical with (12) and we have now proved the equivalence of the "external" and the "internal" way of defining the charge renormalization.