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A Note Concerning the Quantization of Spinor Fields

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Abstract. It is shown that the anticommutation rules between ψ and $\overline{\psi}$ determine to a large extent those between ψ at different space-time points. The remaining freedom can be characterized by a real parameter ϱ, $0 \leqslant \varrho \leqslant 1$. The special case $\varrho = 0$ corresponds to the usual method of quantization. The cases $\varrho \neq 0$ can only be used for the description of particles with no electromagnetic interaction. The special case $\varrho = 1$ represents the Majorana field. There is no physical principle or experimental result known to rule out the cases $\varrho \neq 0$.

1. The commutation rules for a spinor field.

The quantization of a spinor field $\psi(x)$ can be obtained by essentially two different approaches. In the method of JORDAN and WIGNER[1]) and that of FOCK[2]) one stresses the equivalence of the quantized theory with the many particle problems in the configuration space of antisymmetrical wave functions. In order to establish this equivalence it is necessary to have a complete set of orthogonal wave functions for the one-particle problem out of which the complete set of antisymmetrical wave functions can be constructed. It is well-known that the relativistic one-electron problem includes in such a set the solutions with negative energy. Since negative energies are never observed we must consider the relativistic one-body problem as a mathematical fiction with no counterpart in reality. It is of course possible in the many body theory to render the negative energy solutions innocuous with the familiar methods of the hole theory.

This roundabout procedure is avoided in the second approach which was recently developed by SCHWINGER[3]). By introducing as

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1) P. JORDAN and E. WIGNER, Z. Phys. **47**, 631 (1928).
2) V. FOCK, Z. Phys. **75**, 622 (1932).
3) J. SCHWINGER, Phys. Rev. **82**, 914 (1951).

a basic postulate a generalized Action Principle and by combining it with the principle of invariance under time reversal he was able to conclude that a spinor field must be quantized in accordance with the well-known anticommutation rules

$$\{\psi(x), \overline{\psi}(x')\} = i(\gamma\,\partial - m)\, D(x - x'), \tag{1}$$

$$\{\psi(x), \tilde{\psi}(x')\} = 0. \tag{2}$$

The left-hand sides are here and in the following always written as matrix expressions. The spinor ψ is thus a one-column spinor matrix and $\tilde{\psi}$, its transposed, a one-row matrix. $\psi^+ = \tilde{\psi}^*$ and $\overline{\psi} = \psi^+ A$ where A is defined by

$$-\gamma_\mu^+ = A\,\gamma_\mu\, A^{-1}, \tag{3}$$

$$A^+ = A. \tag{4}$$

$\gamma\,\partial = \gamma^\mu\,\partial_\mu$ denotes the invariant scalar product, D is the homogeneous D-function with rest mass and the γ_μ are assumed to satisfy

$$\{\gamma_\mu\,\gamma_\nu\} = 2\, g_{\mu\nu}\,. \tag{5}$$

The generalized Action Principle seems to give a satisfactory solution of the quantization problem of a spinor field except for the fact that there exists a well-known example of a spinor field which satisfies all the basic postulates implied in the Action Principle but which violates Eq. (2). It is the spinor field quantized according to the method of MAJORANA[1]).

In order to write down most conveniently the commutation rules for the Majorana field which replaces Eq. (2) we introduce the associated spinor $\widehat{\psi}$ defined in the following manner:

$$\widehat{\psi} = \tilde{\psi}\, B\,\gamma_5, \quad \gamma_5 = \gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 \tag{6}$$

where the matrix B has the property

$$\tilde{\gamma_\mu} = B\,\gamma_\mu\, B^{-1}. \tag{7}$$

It follows from (7) that B is antisymmetrical[2])

$$\tilde{B} = -B. \tag{8}$$

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[1]) E. MAJORANA, Nuovo Cimento **14**, 171 (1937). – G. RACAH, ibid. **14**, 322 (1937). – W. H. FURRY, Phys. Rev. **54**, 56 (1938).

[2]) W. PAULI, Zeeman Verhandelingen, Haag, Martinus Nijhoff, 1935, p. 41.

The associated spinor ψ defined by (6) is related to the charge conjugate spinor ψ^c defined by

$$\psi^c = C^* \psi^* \tag{9}$$

where C is the matrix with the property

$$\gamma_\mu^* = C \gamma_\mu C^{-1}, \tag{10}$$

$$C^* C = 1. \tag{11}$$

The matrices A, B, C are related because of the fact that the three operations on the γ's so far considered are not independent, in fact

$$\gamma_\mu^+ = \tilde{\gamma}_\mu^*. \tag{12}$$

One can always choose the as yet undetermined multiplicative factor in B in such a way that this relation assumes the simple form

$$A = C^+ B \gamma_5. \tag{13}$$

Relation (13) determines B only up to a phase factor since C is only determined up to a phase factor and (13) only fixes the relative phase between the two.

With the help of (13) one obtains immediately the desired relationship in the form

$$\widehat{\psi}^c = \overline{\psi}. \tag{14}$$

It follows from this that the spinor ψ transforms under proper Lorentz transformations according to

$$\widehat{\psi}' = \psi S^{-1} \tag{15}$$

if under the same transformation ψ transforms according to

$$\psi' = S \psi. \tag{16}$$

A Majorana field φ is defined as a field which is equal to its own charge conjugate field

$$\varphi^c = \varphi. \tag{17}$$

Thus it follows from (14) and (17)

$$\widehat{\varphi} = \widehat{\varphi}^c = \overline{\varphi}. \tag{18}$$

The commutation rules which replace (2) are then obtained from (1) and (18) in the form:

$$\{\varphi(x), \widehat{\varphi}(x')\} = i(\gamma\,\partial - m)\,D(x - x')\,. \tag{2'}$$

This example of the Majorana field shows clearly enough that the equation (2) cannot follow from the Action Principle. In order to make sure that none of the basic principles are violated we must verify that (2′) is invariant under proper Lorentz transformations, space reflections and the transformation of time reversal. As regards the proper Lorentz transformations there is very little comment needed. The form of the equation (2′) together with the transformation property (15) guarantees this invariance.

The space reflections require a more careful examination because of the ambiguity in the sign of S^2. We define the transformed spinor φ' by the condition

$$\varphi'(x') = S\,\varphi(x) \tag{19}$$

where $x_k' = -\,x_k$, $x_0' = x_0$ and S has the properties

$$\gamma_\mu' = S^{-1}\gamma_\mu\,S\,, \tag{20}$$

$$S^2 = \pm 1\,. \tag{21}$$

From the definition of B [Eq. (7)] and the anticommuting property of γ^5 follows in either case

$$S^\sim\,B\,\gamma_5 = -\,B\,\gamma_5\,S\,. \tag{22}$$

Thus

$$\widehat{\varphi}' = -\,\widehat{\varphi}\,S \tag{23}$$

and

$$\begin{aligned}\{\varphi'(x), \widehat{\varphi}'(y)\} &= -\,i\,S(\gamma\,\partial' - m)\,S\,D(x' - y'),\\ &= -\,i(\gamma\,\partial - m)\,S^2\,D(x - y).\end{aligned} \tag{24}$$

The commutation rules are invariant under the space reflections only if[1])

$$S^2 = -\,1\,. \tag{25}$$

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[1]) It follows from (25) that if an argument could be found which requires S^2 to be $+1$ the Majorana theory and all the other cases to be discussed below could be ruled out on the ground of non-invariance under space reflections. Such an argument has recently been given by CAIANIELLO [Phys. Rev. **86**, 564 (1952)].

The transformation of time inversion in the relativistic theory is given by

$$\psi \rightarrow \psi^{\tau} \quad \psi^{\tau}(x') = D\, \psi^{x}(x) \tag{26}$$

where ψ^{x} denotes the complex conjugate (not hermitian conjugate!) spinor operator and D is the spinor matrix defined by[1])

$$\gamma_{\mu}'^{*} = D^{-1}\, \gamma_{\mu}\, D \tag{27}$$

$$D^{*}\, D = -1 \tag{28}$$

and

$$x_0' = -x_0\,, \quad x_k' = x_k \quad (k = 1, 2, 3)\,,$$

$$\gamma_0' = -\gamma_0\,, \quad \gamma_k' = \gamma_k\,. \tag{29}$$

The matrix D is related to C and γ_5 by

$$D = \gamma_0\, \gamma_5\, C^{-1} \tag{30}$$

which is consistent with (28).

For the associated spinor $\widehat{\varphi}$ we obtain

$$\widehat{\varphi}^{\tau}(x) = \tilde{\varphi}^{x}(x')\, D\, B\, \gamma_5\,. \tag{31}$$

With the help of (30) and (13) one verifies without difficulty

$$D^{\sim} B = B^{*}\, D^{-1} \tag{32}$$

and

$$D^{-1}\, \gamma_5 = -\gamma_5^{*}\, D^{-1}\,. \tag{33}$$

Thus (31) may also be written as

$$\widehat{\varphi}^{\tau}(x) = -\,\widehat{\varphi}^{x}(x')\, D^{-1}. \tag{34}$$

For the commutation rules between the time reversal fields we obtain finally

$$\{\varphi^{\tau}(x), \widehat{\varphi}^{\tau}(y)\} = -\, i\, D(\gamma^{*}\, \partial' - m)\, D^{-1}\, D(x' - y') \tag{35}$$

or using (27) and $D(z') = -\,D(z)$

$$\{\varphi^{\tau}(x), \widehat{\varphi}^{\tau}(y)\} = i(\gamma\, \partial - m)\, D(x - y)\,. \tag{36}$$

Thus we have also established the invariance of the commutation rules under time reversal.

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[1]) For a detailed discussion of the time inversion transformation in the relativistic case see S. WATANABE, Phys. Rev. **84**, 1008 (1951). Our definition corresponds to "standpoint II" in WATANABE's paper.

In view of this example of a theory which does not satisfy Eq. (2) one may well ask the question just how much one is able to derive from the generalized Action Principle[1]).

We shall prove the following theorem:

A spinor field quantized according to (1) satisfies instead of (2) in general commutation rules of the form

$$\{\psi(x), \widehat{\psi}(y)\} = i\,\varrho(\gamma\,\partial - m)\,D(x-y) \tag{37}$$

and the phase of ψ can be chosen in such a way that ϱ is real. ϱ is restricted to

$$0 \leqslant \varrho \leqslant 1\,. \tag{38}$$

2. Proof of the theorem.

In order to make the problem mathematically well-defined we must specify that we are only interested in irreducible representations of the operators ψ. The operators $\overline{\varphi}$ are then also irreducible. It follows from this that an operator which commutes with all the operators ψ and $\overline{\varphi}$ is a multiplum of the unit operator, that is, a c-number.

We begin the proof by defining $C_{\varrho\sigma}(x, y)$

$$\{\psi_\varrho(x), \widehat{\psi}_\sigma(y)\} = i\,C_{\varrho\sigma}(x, y) \tag{39}$$

and show first that $C_{\varrho\sigma}(x, y)$ must be a c-number. According to the foregoing remark this is established if we show for instance that

$$[C_{\varrho\sigma}(x, y), \overline{\psi}_\tau(z)] = 0\,. \tag{40}$$

The left hand side of (40) when written out is given by

$$\begin{aligned}[C_{\varrho\sigma}(x, y), \overline{\psi}_\tau(z)] = {} & \psi_\varrho(x)\,\{\widehat{\psi}_\sigma(y), \overline{\psi}_\tau(z)\} - \{\psi_\varrho(x), \overline{\psi}_\tau(z)\}\,\psi_\sigma(y) \\ & + \widehat{\psi}_\sigma(y)\,\{\psi_\varrho(x), \overline{\psi}_\tau(z)\} - \{\widehat{\psi}_\sigma(y), \overline{\psi}_\tau(z)\}\,\psi_\varrho(x)\,. \end{aligned} \tag{41}$$

All that matters is that the anticommutators which occur on the right are c-numbers [Eq. (1)] and the various ψ-factors can be freely shifted. Thus the four terms cancel in pairs.

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[1]) The reason why (2) does not follow from the Action Principle is that the canonically conjugate momenta $\pi_\mu = -\overline{\psi}\,\gamma_\mu$ are not dynamically independent of the ψ for a spinor field.

Next we exploit the requirement of the relativistic invariance in the following form. A Lorentz transformation

$$x \rightarrow x' = L\,x \tag{42}$$

under which the spinor field ψ is transformed into a field ψ' defined by

$$\psi'(L\,x) = S\,\psi(x) \tag{43}$$

gives rise to a unitary transformation U defined by the relation

$$\psi'(x) = U\,\psi(x)\,U^{-1}. \tag{44}$$

For infinitesimal transformations Eq. (25) leads to the integrals of momentum and angular momentum. It follows then with the help of (15) that

$$C(x', y') = S\,C(x, y)\,S^{-1}. \tag{45}$$

For the particular transformation which corresponds to a displacement $S = 1$, $x' = x + a$, $y' = y + a$

$$C(x + a, y + a) = C(x, y)\,. \tag{46}$$

Thus C is a function of the difference $x - y$ only

$$C(x, y) = C(x - y)\,. \tag{47}$$

Finally from the field equation

$$(\gamma\,\partial + m)\,\psi(x) = 0 \tag{48}$$

follows

$$(\gamma\,\partial + m)\,C(x - y) = 0 \tag{49}$$

which shows that C is of the form

$$C(x - y) = i\,\varrho(\gamma\,\partial - m)\,D'(x - y) \tag{50}$$

(ϱ arbitrary)

and the function $D'(x)$ satisfies

$$(\partial^\mu\,\partial_\mu + m^2)\,D' = 0 \text{ for all } x. \tag{51}$$

Thus D' is a linear superposition of the D and the D_1 function. The D_1-function is ruled out for the following reason. For time inversion

($x_0 = - x_0$, $x_k' = x_k$, $k = 1, 2, 3$) the D_1-function does not change the sign

$$D_1(x') = D_1(x) \tag{52}$$

while for this same transformation

$$D(x') = - D(x). \tag{53}$$

This sign change is necessary if the commutation rules are to remain invariant under the operation of time inversion as was shown in the preceding section. Thus we have

$$D' = D \tag{54}$$

where D is given by

$$D(x) = \frac{1}{(2\pi)^3} \int e^{i \boldsymbol{k} \boldsymbol{x}} \frac{\sin \omega x^0}{\omega} d^3 k\,,$$
$$\omega = + \sqrt{\boldsymbol{k}^2 + m^2}. \tag{55}$$

In order to establish also $0 \leqslant \varrho \leqslant 1$ (38) it is convenient to express the commutation rules in terms of the emission and absorption operators a, b defined by

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_{r=1}^{2} \int d^3 k \left(a_r(\boldsymbol{k})\, u_r\, e^{ikx} + b_r^*(\boldsymbol{k})\, v_r\, e^{-ikx}\right). \tag{56}$$

The amplitudes of the plane waves u and v are solutions of the equations

$$\begin{aligned} K_+ u_r &= 0 \\ K_- v_r &= 0 \end{aligned} \qquad (r = 1, 2) \tag{57}$$

$$K_\pm = \pm i k \gamma + m \tag{58}$$

and are normalized according to

$$(\bar{u}_r\, u_s) = - (\bar{v}_r\, v_s) = \frac{m}{\omega}\,. \tag{59}$$

From (59) follow the relations

$$\left.\begin{aligned} \sum_{r=1}^{2} u_r \bar{u}_r &= \frac{1}{2\omega} K_+ \\ \sum_{r=1}^{2} v_r \bar{v}_r &= - \frac{1}{2\omega} K_-. \end{aligned}\right\} \tag{60}$$

The commutation rules (1), (37) are then equivalent to

$$\{a_r(\boldsymbol{k}),\, a_s(\boldsymbol{k}')\} = \{b_r(\boldsymbol{k}),\, b_s(\boldsymbol{k}')\} = 0 \tag{61}$$

$$\{a_r(\boldsymbol{k}),\, a_s^*(\boldsymbol{k}')\} = \{b_r(\boldsymbol{k}),\, b_s^*(\boldsymbol{k}')\} = \delta_{rs}\, \delta(\boldsymbol{k} - \boldsymbol{k}') \tag{62}$$

and

$$\{a_r(\boldsymbol{k}),\, b_s^*(\boldsymbol{k}')\} = \varrho\, \delta_{rs}\, \delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{63}$$

We can choose the phase in ψ so that ϱ is real and positive; this we shall do in the following. If $\varrho \neq 0$ we can construct for any r and $\boldsymbol{k}$ an operator

$$M = a - \frac{1}{\varrho} b\,. \tag{64}$$

The operator $M + M^*$ is then hermitian and the square is given by

$$(M + M^*)^2 = \{M,\, M^*\} = \frac{1}{\varrho^2} - 1 \geqslant 0\,. \tag{65}$$

Thus $0 \leqslant \varrho \leqslant 1$ q.e.d.

We see also that for $\varrho = 1$, both

$$(M + M^*)^2 = (M - M^*)^2 = 0\,. \tag{66}$$

Thus M is the nul-matrix and consequently

$$a = b\,. \tag{67}$$

This is the Majorana case.

Finally we can construct explicitly matrices which satisfy the relations (61), (62), (63). Putting for each r, $\boldsymbol{k}$

$$a' = \alpha\, a - \beta\, b \quad b' = \alpha\, b - \beta\, a$$

with

$$\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{1-\varrho}} + \frac{1}{\sqrt{1+\varrho}}\right); \qquad \beta = \frac{1}{2}\left(\frac{1}{\sqrt{1-\varrho}} - \frac{1}{\sqrt{1+\varrho}}\right)$$

then these operators a', b' satisfy the relations (61), (62). But instead of (63) we obtain

$$\{a_r'(\boldsymbol{k}),\quad b_s'^*(\boldsymbol{k})\} = 0\,.$$

Thus the construction of matrices which satisfy relations (61), (62), (63) is reduced to the standard case of Jordan and Wigner. Since the representation in the latter case is uniquely determined by the commutation rules this is also true for the present case.

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3. Conclusion.

We have been able to show that the commutation rules (1) determine to a large extent the remaining anticommutators. The freedom which is left is characterized by a parameter ϱ, $0 \leqslant \varrho \leqslant 1$. The limiting cases $\varrho = 0$ and $\varrho = 1$ represent the already known special cases of an electron positron field and a Majorana field. Since the commutation rules are invariant under canonical transformations the theories for different ϱ are not equivalent.

Since the commutation rules in the cases $\varrho \neq 0$ are not invariant under phase transformations, there exists no gauge invariant interaction with the electromagnetic field. These theories could therefore only be used for the description of neutral particles.