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The infrared divergence*)

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Abstract. The long standing problem of the infra-red divergence in quantum electrodynamics is reexamined in the light of the recent progress in this field. A general proof is given of the cancellation of the infrared divergences which arise from real and virtual processes. This proof is valid to all orders of the coupling constant and for all types of scattering events. The problem is solved in the iteration solution as well as in the closed form used by BLOCH and NORDSIECK and by PAULI and FIERZ. The equivalence of the two methods is discussed and the results obtained by BLOCH and NORDSIECK are found to be correct provided radiative corrections are ignored. The ultraviolet divergences found by PAULI and FIERZ in this connection has nothing to do with the infra-red divergence problem. It cannot be regarded as a difficulty in the theory, since the method of calculation employed is no longer valid at high frequencies. The exact calculation (iteration solution) shows that the renormalized theory is completely free of divergences at both low and high frequencies. A plausibility argument for the convergence in certain cases of the renormalized iteration solution is presented.

I. Introduction and summary.

The quantum theory of radiation has for a long time been plagued by divergence difficulties, which tended to discredit the theory. The discovery of the electromagnetic level shift in 1947 led to a reexamination of the divergences and to the discovery that it is possible to extract finite and physically observable quantities from the divergent expressions by the process of renormalization of mass and charge. In this method the divergent quantities are identified with the (infinite) correction to the mass and charge of the electron. After mass and charge renormalization, the remaining terms are all finite to all orders in the fine structure constant α[1]), and they represent the observable quantities in the theory.

This method of eliminating the undesirable consequences of the theory does not apply to the type of divergences which have been known for a long time under the name of infrared (i.r.) divergences. Indeed, all the divergences which are eliminated by the renormali-

*) Part of this work was completed while one of us (J.M.J.) was at the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

zation procedure are of the ultraviolet type; they are caused by the divergence of certain integrals at very high values of the intermediate state four-momenta. In contrast, the i. r. divergences are caused by the low values of these momenta.

The simplest type of i. r. divergence, and one of the earliest to be noted, occurs in the inelastic scattering of electrons with the emission of photons[2]). The cross section for the scattering with the emission of one photon of frequency ω within $d\omega$ is in lowest order of e proportional to $d\omega/\omega$. The total cross section for the energy loss $\Delta\varepsilon$ due to one photon emission is then proportional to

$$\int_0^{\Delta\varepsilon} \frac{d\omega}{\omega}$$

which diverges logarithmically at the lower limit.

The fact that the i. r. divergences are not touched by the renormalization procedure does not in any way affect the consistency of the method. Indeed it was known for a considerable time (long before the invention of renormalization) that the i. r. divergences are of an entirely different character from the ultraviolet divergences. In a fundamental paper, BLOCH and NORDSIECK[3]) were able to show that the i. r. divergence disappears entirely if a rigorous solution of the field equations is employed for that part of the Hamiltonian which couples the low frequency field oscillators to the electron. The i. r. divergence is therefore recognized as one which is caused by the particular features of the iteration solution of ordinary perturbation theory which separates the various processes pertaining to real and virtual photons. One of the main results of BLOCH and NORDSIECK was that the probability for the emission of a *finite* number of soft photons in a scattering process is exactly zero, so that every such process is necessarily accompanied by the emission of an *infinite* number of soft photons.

This result seems at first sight inconsistent with the result of the perturbation theory. That this is not the case was demonstrated by BRAUNBECK and WEINMANN[4]) who showed that the first term in the development of the Bloch-Nordsieck solution in powers of α agrees exactly with the result of the perturbation solution. However, these authors failed to point out that they took into account explicitly the limitation on the energy of the emitted soft photons. This limitation was ignored by BLOCH and NORDSIECK and is responsible for their result $b = 1$ for the correction factor multiplying the cross section. We shall show that the exact result for b differs from 1 by terms of order α.

Although the work of BLOCH and NORDSIECK offers the key to the solution of the i. r. divergence, it was not entirely satisfactory. The flaw was pointed out by PAULI and FIERZ[5]). These authors improved the calculation of the correction factors due to the emission of an unlimited number of soft photons by taking correctly into account the conservation of energy in the process which BLOCH and NORDSIECK had failed to do. The latter authors had obtained the satisfactory result 1 for this correction factor with their incorrect procedure. In other words, they found the cross section for the inelastic scattering with very small energy loss equal to the cross section for elastic scattering, neglecting the radiative interaction entirely. When this same quantity was calculated by PAULI and FIERZ, taking into account the conservation of energy, it was found that this correction factor became equal to zero.

This conclusion of PAULI and FIERZ did not make any sense at all, since it seemed to contradict the results which one would expect on the basis of the correspondence principle. It was all the more puzzling when it was found that the correction factor for this "nearly elastic" scattering vanished because of a new kind of logarithmic divergence at the *high* frequency end of the spectrum. If, as it was usual in those days, one introduced an upper limit for the energy of the photons, PAULI and FIERZ obtained a finite result which seemed to depend on the *value* of this upper limit. In physical terms this would mean that the emission of radio waves by an accelerated electron depended critically on the modification of quantum electrodynamics at ultrahigh-frequencies. This is an incredible result.

That this new type of divergence discovered by PAULI and FIERZ was not caused by the nonrelativistic calculation of these authors was verified by DANCOFF[6]) who showed that even in a relativistic theory this divergence reappears. Moreover DANCOFF's work showed that the result for the soft photon emission during a scattering process depended on the spin of the charged particle.

It was clear to everybody that these results did not make any physical sense. But it was equally clear that a simple modification of the theory which eliminated completely the PAULI-FIERZ divergence was not sufficient to remedy the defect. Any change of the theory must be such that it leaves the result of BLOCH and NORDSIECK for the soft photons essentially unchanged if the theory is to be in accordance with the correspondence principle.

In this connection it was especially pointed out by BETHE and OPPENHEIMER[7]) that the problem of soft photon emission furnishes

a valuable check for any modification of quantum electrodynamics. Using this criterion, BETHE and OPPEHEIMER were able to show that Heitler's form of quantum electrodynamics[8]) was unsatisfactory. This same criterion was used previously[9]) for testing the theory of DIRAC involving negative energy photons[10]) with the same negative conclusion.

It seems clear, therefore, that the renormalization theory cannot be considered complete and satisfactory, if it does not also give the solution to the problem of the i. r. divergence. In this connection it is of interest to recall an observation already made by BRAUNBECK and WEINMANN[4]) and especially emphasized by BETHE and OPPENHEIMER[7]). The i. r. divergence due to the emission of one *real* photon and the i. r. divergence which is contained in the first order radiative correction due to the emission and reabsorption of *virtual* photons exactly cancel. The result of BLOCH and NORDSIECK is a strong indication that this cancellation occurs in all orders of the coupling constant. Special cases of this cancellation were noted by several authors[11]), but attempts to prove it in general remained incomplete[12]).

If it is true that the i. r. divergences of the real and virtual photon processes exactly cancel to any order of the coupling constant, then the i. r. divergence is a spurious phenomenon which only appears because the real and virtual photon processes are artificially separated in the iteration solution. When these processes are recombined, the i. r. divergence disappears. There is then no other divergence in the theory but the renormalizable type.

In this paper we give the complete solution of the infrared problem, outlined above, by showing two things. First, we prove that *the low frequency divergences contained in the real and virtual photon processes cancel to all orders in the fine structure constant* α *and for all types of processes.* Second, we show that *the new type of divergence obtained by* PAULI *and* FIERZ *is indeed no real divergence but is merely caused by an unjustified extension of the approximation on which the Bloch-Nordsieck solution is based.* It is shown here that the rigorous calculation of the collective soft photon emission is only possible for energies of the photons much smaller than m. An argument is presented to show that the effect of the higher order terms in ω/m after renormalization can be roughly estimated by introducing a cut-off energy $\varepsilon \sim m$. With this method the divergence disappears and the reasonable result of BLOCH and NORDSIECK is found to be correct within a very good approximation.

We conclude that the divergence of PAULI and FIERZ has nothing to do with the physical interpretation of the theory. It is not an

indication of an incorrect theory but rather an indication of the limitation of a particular method of calculation. The incorrect physical interpretation of this divergence has caused much of the confusion in the history of this problem[13]).

II. Cancellation of the infrared divergences in the iteration solution.

A. *Real Photon Emission.*

We shall use here and in the following the technique of the Feynman diagrams which allows a simple pictorial representation of the basic processes under discussion.

Consider a *basic* scattering process, involving one ingoing and one outgoing electron. The electron may be either a negaton or a positon; the following arguments are equally valid for both cases. In order to be specific, we shall assume that we are dealing with a

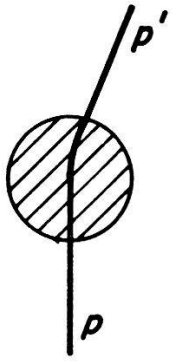

Fig. 1.
The basic process.

negaton (that is a negative electron). The basic process is schematically represented by the diagram of fig. 1. The generalization to processes with more than one electron path will be discussed later. The process may be accompanied by any number of initial or final photons which should be indicated by additional dotted lines entering or leaving the diagram of fig. 1. We have omitted these photon lines, since, as we shall see, their presence is entirely irrelevant to the argument.

The shaded area in fig. 1 stands for a diagram of any arbitrary degree of complexity. This diagram may contain any number of closed loops of electron lines. The contribution of such a general diagram to the matrix element will of course contain every conceivable divergence of the ultraviolet type, which must be removed in accordance with the general theory of renormalization[1]). The shaded area may further contain contributions from an external field interacting with the scattered electron.

The S-matrix element is of the form

$$M = \bar{u}\,(p')\,Q\,(p'\,p)\,u\,(p) \tag{1}$$

where $Q(p'p)$ is a constant times the spinor matrix associated with the shaded area of the basic process. The spinor $u(p)$ is the amplitude of the plane wave for a negaton with momentum p. We use the units cm, $\hbar$, and c, and the metric $-g_{00} = g_{11} = g_{22} = g_{33} = 1$.

We wish to calculate the matrix element M_1 corresponding to the emission of one additional soft photon of momentum $k = (\omega, \boldsymbol{k})$ ($\omega \ll m$) and polarization vector e (cf. fig. 2). The result is

$$M_1 = \frac{ie}{(2\pi)^{3/2}}\,\bar{u}\,(p')\left\{\frac{\boldsymbol{e}(k)}{\sqrt{2\,\omega}}\,\frac{i(\boldsymbol{p'}+\boldsymbol{k})-m}{(p'+k)^2+m^2}\,Q\,(p'-k,p)\right.$$
$$\left. + Q\,(p',p-k)\,\frac{i\,(\boldsymbol{p}-\boldsymbol{k})-m}{(\boldsymbol{p}-k)^2+m^2}\,\frac{\boldsymbol{e}(k)}{\sqrt{2\,\omega}}\right\}u\,(p). \tag{2}$$

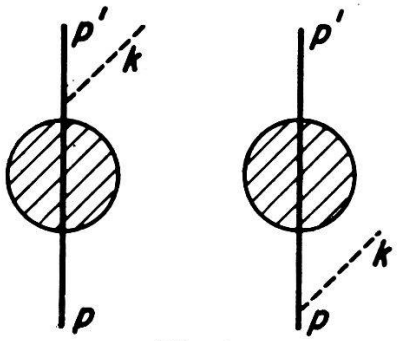

Fig. 2.
Diagrams for the basic process with one photon emitted from external lines.

In this calculation we have ignored the emission of photons from internal lines of M, since such processes do not give rise to i. r. divergences.

The expression (2) can be considerably simplified if we use the fact that $\omega \ll m$ and that the propagation functions operate on free particle functions which satisfy the equations*)

$$(i\boldsymbol{p} + m)\,u(p) = 0$$
$$\bar{u}(p)\,(i\boldsymbol{p} + m) = 0.$$

Retaining only the terms of lowest order in ω/m, we obtain after some rearrangement

$$M_1 = \beta(k)\,M \tag{3}$$

with

$$\beta\,(k) = \frac{e}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,\omega}}\left(\frac{p\cdot e}{p\cdot k} - \frac{p'\cdot e}{p'\cdot k}\right). \tag{4}$$

*) In the following equation, as well as in Eqs. (2), (11), (14), (17), (18) and (19) the bold-face letters indicate the four-vector product with γ^μ (e. g. $\boldsymbol{p} \equiv p_\mu \gamma^\mu$). In all other places they indicate space parts of four vectors.

The total transition probability, and the cross section for the basic process is proportional to the square of the matrix element. Hence, the emission of one soft photon modifies the cross section by the factor

$$b = \frac{e^2}{(2\pi)^3} \sum_{\text{pol}} \int \frac{d^3k}{2\omega} \left| \frac{p \cdot e}{p \cdot k} - \frac{p' \cdot e}{p' \cdot k} \right|^2 . \tag{5}$$

The summation sign indicates the summation over the unobserved polarization states of the emitted photons.

In this form the infrared divergence is manifest as a logarithmic divergence of the k-space integral at low frequencies. There is no divergence at the upper limit of the k-space integration since the k-values are restricted by $\omega \ll m$. For energies ω which do not satisfy this inequality, the formula (4) is no longer valid. The formula (5), when integrated over the sphere $\omega \leqslant \Delta\varepsilon \ll m$, is then the correction factor for the cross section of the basic process due to the emission of one photon with energy $\omega \leqslant \Delta\varepsilon$.

The angular integration in Eq. (5) cannot be carried out in terms of elementary functions[14]). For our purposes, however it is quite sufficient to evaluate this integral in a special coordinate system, for instance the rest system of the incident electron ($\boldsymbol{p} = 0$). In this system the integral (5) is elementary.

We first carry out the polarization sum which gives

$$\sum_{\text{pol}} |p' \cdot e|^2 = \boldsymbol{p}'^2 - \frac{(\boldsymbol{p}' \cdot \boldsymbol{k})^2}{\omega^2} .$$

Using

$$p' \cdot k = \omega(|\boldsymbol{p}'| \cos \vartheta' - \varepsilon')$$

and

$$\alpha = \frac{e^2}{4\pi}$$

we find

$$b = \frac{\alpha}{4\pi^2} \int \frac{d^3k}{\omega^3} \left(\frac{\boldsymbol{p}'^2}{(\varepsilon' - |\boldsymbol{p}'| \cos \vartheta')^2} - \frac{\boldsymbol{p}'^2 \cos^2 \vartheta'}{(\varepsilon' - |\boldsymbol{p}'| \cos \vartheta')^2} \right) . \tag{6}$$

By changing the variable of integration

$$\cos \vartheta' = x$$

and using

$$\frac{|\boldsymbol{p}'|}{\varepsilon'} = \beta' .$$

Eq. (6) reduces to

$$b = \frac{\alpha}{2\pi} \beta'^2 \int \frac{d\omega}{\omega} \int_{-1}^{+1} \frac{1 - x^2}{(1 - \beta' x)^2} \, dx = \frac{\alpha}{\pi} \left(\frac{1}{\beta'} \ln \frac{1 + \beta'}{1 - \beta'} - 2 \right) \int \frac{d\omega}{\omega}$$

$$b = \frac{2\alpha}{\pi} \left(\frac{\tanh^{-1} \beta'}{\beta'} - 1 \right) \int \frac{d\omega}{\omega} . \tag{7}$$

B. *Virtual Photon Processes.*

We now turn to the first radiative corrections of the basic process M shown in fig. 1, by inserting into M one internal photon line in all possible ways. Among the diagrams thus obtained, we select those which exhibit an i. r. divergence.

In order to obtain all contributions, it is essential to recall the fact that both, the second order electron self-energy and the vertex part, exhibit an i.r. divergence after separation of the renormalizable divergences[15]). On the other hand, an internal photon line which is attached to an internal electron line at one or both ends has no i. r. divergence unless it produces a self-energy or a vertex part. The radiative correction diagrams which do in fact have i. r. divergences are shown in fig. 3.

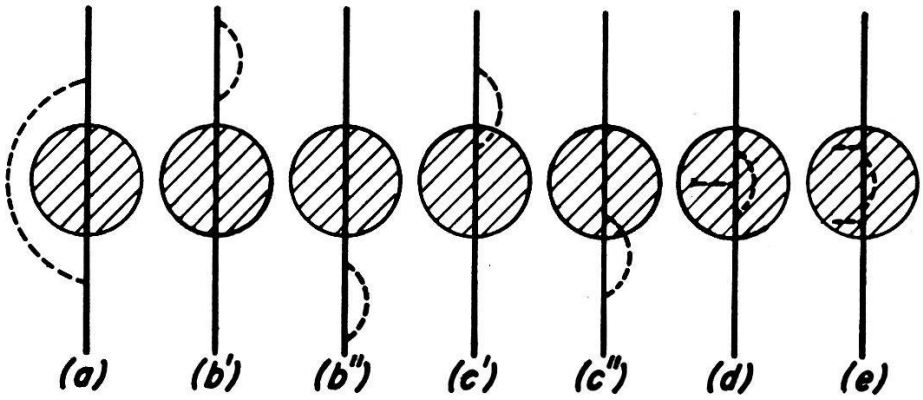

Fig. 3.
Radiative corrections to the basic process which contain infrared divergences.

The meaning of diagrams (a) and (b) is clear. In diagrams (c) we mean to indicate that the added photon line begins on an external line and ends on the *first* internal line of M thereby creating a vertex part at the very edge of the diagram. In (d) the added photon line creates a vertex part and in (*e*) a self-energy part.

As we shall see below, the explicit evaluation of the i. r. divergent parts of these radiative corrections are all of the form

$$M^{(2)} = \varrho M \tag{8}$$

where ϱ is a numerical factor of order α. The correction of order α of the transition probability is obtained from these terms by calculating the cross-terms of M and $M^{(2)}$,

$$| M + M^{(2)} |^2 = | M |^2 (1 + 2\varrho + \cdots)$$

Our aim is to prove the compensation of the divergence in b (Eq. (7)) by the term

$$r = 2\varrho \tag{9}$$

i.e. we want to show that

$$b + r = \text{convergent.} \tag{10}$$

The diagram (a) gives for small $\boldsymbol{k}$

$$\begin{aligned} M^{2)}_{(a)} &= \frac{i e^2}{(2\pi)^4} \int \bar{u}(p')\, \gamma_\mu \frac{i\,(\boldsymbol{p}'-\boldsymbol{k})-m}{(p'-k)^2+m^2}\, Q\, \frac{i(\boldsymbol{p}-\boldsymbol{k})-m}{(p-k)^2+m^2}\, \gamma^\mu \frac{d^4 k}{k^2} \\ &= \frac{i\alpha}{4\pi^3} \int \frac{-i p_\mu'}{p'\cdot k}\, \bar{u}(p')\, Q\, u(p)\, \frac{-i\, p^\mu}{p\cdot k}\, \frac{d^4 k}{k^2} \\ &= \varrho_1 M \end{aligned} \tag{11}$$

with

$$\varrho_1 = -\frac{i\alpha}{4\pi^3} \int \frac{p'\cdot p}{p'\cdot k\; p\cdot k}\, \frac{d^4 k}{k^2}. \tag{12}$$

The integration over k^0 in Eq. (12) has to be carried out over the path which avoids the negative pole on the real axis by a detour in the negative imaginary half plane and the positive pole by a similar detour into the positive imaginary half plane. We can therefore rotate the path in the complex k^0-plane by $\pi/2$ and integrate over the imaginary axis. Replacing k^0 by ik^0 we have

$$\varrho_1 = -\frac{i\alpha}{4\pi^3}\, p'\cdot p \int d^3 k \int_{-\infty}^{+\infty} \frac{i\, dk^0}{(\omega^2+k^{02})\,(|\boldsymbol{p}'|\,\omega\cos\vartheta' - i\varepsilon' k^0)\,(|\boldsymbol{p}|\,\omega\cos\vartheta - i\varepsilon k^0)}$$

$$= -\frac{i\alpha}{4\pi^3}\, \frac{i\pi}{\varepsilon\varepsilon'}\, p\cdot p' \int \frac{d^3 k}{\omega^3} \left[\frac{\beta'\cos\vartheta'}{1-\beta'^2\cos^2\vartheta'} - \frac{\beta\cos\vartheta}{1-\beta^2\cos^2\vartheta}\right] \frac{1}{\beta'\cos\vartheta' - \beta\cos\vartheta}.$$

As in the evaluation of b, we can choose the reference frame defined by $\boldsymbol{p} = 0$, so that $p'\cdot p = -m\varepsilon'$.

$$\varrho_1 = -\frac{\alpha}{4\pi^2} \int \frac{d\omega}{\omega}\, 2\pi \int_{-1}^{+1} \frac{dx}{1-\beta'^2 x^2}$$

$$= -\frac{\alpha}{2\pi}\, \frac{1}{\beta'} \ln\frac{1+\beta'}{1-\beta'} \int \frac{d\omega}{\omega} = -\frac{\alpha}{\pi}\, \frac{\tanh^{-1}\beta'}{\beta'} \int \frac{d\omega}{\omega}.$$

From Eq. (9) follows

$$r_1 \equiv 2\,\varrho_1 = -\frac{2\alpha}{\pi}\, \frac{\tanh^{-1}\beta'}{\beta'} \int \frac{d\omega}{\omega}. \tag{13}$$

Comparing this value with (7), we see that it exactly cancels the first term of Eq. (7)[16]).

For the cancellation of the remaining term we must examine the diagrams (*b*) to (*e*) in fig. 3. Their contribution to the i. r. divergence will be denoted by ϱ_2. Its value need not be calculated by integration of the respective matrix elements, but can be obtained from the general properties of the divergent diagrams. In order to show this we need some of the results contained in the papers of references

15 and 1. We summarize here briefly the pertinent properties but refer to these papers for details.

The self-energy part $\Sigma(p)$ and the vertex part $\Lambda_\mu(p' p)$ are both free of i. r. divergences. However, when the (ultraviolet) divergent parts are separated we introduce an i. r. divergence in the (otherwise) finite part of these expressions. Consequently, the ultraviolet divergent part will also contain an i. r. divergence.

These separations have the form

$$\Sigma(p) = A - (i\boldsymbol{p} + m)^{-1}B + (i\boldsymbol{p} + m)^{-2}\,\Sigma_f(p) \tag{14}$$

and

$$\Lambda_\mu(p', p) = L\gamma_\mu + \Lambda_{\mu f}(p', p)\,. \tag{15}$$

The constants A, B, and L are ultraviolet divergent. The charge renormalization constants B and L are also i. r.-divergent. The theorem of Ward (see ref. 1) says that

$$B = L. \tag{16}$$

The functions free of ultra-violet divergences are denoted by $\Sigma_f(p)$ and $\Lambda_{\mu f}(p', p)$.

From the preceding remarks follows that we have the following relations with respect to the i.r. divergences

$$-B^i + (i\boldsymbol{p} + m)^{-1}\,\Sigma_f^i = 0 \qquad \gamma_\mu L^i + \Lambda_{\mu f}^i = 0 \qquad B^i = L^i. \tag{17}$$

The superscript i means "infrared divergence of". Therefore,

$$\gamma_\mu(i\boldsymbol{p} + m)^{-1}\,\Sigma_f^i + \Lambda_{\mu f}^i = 0\,. \tag{18}$$

This statement can be interpreted to mean that $\Lambda_{\mu f}^i$ cancels *half* of Σ_f^i *from either side* of the electron path. Therefore, the sum of the diagrams (b), (c), (d), and (e) would exactly cancel in their infrared contributions, were it not for the fact that the terms $\Sigma_f(p')$ and $\Sigma_f(p)$ in the external lines (diagrams (c)) vanish because of

$$(i\boldsymbol{p} + m)\,u(p) = 0$$
$$\bar{u}(p)\,(i\boldsymbol{p} + m) = 0.$$

The corresponding parts of the divergences in (c) are therefore not compensated. The missing parts are

$$-\tfrac{1}{2}\,\Sigma_i^f(p')\,(i\boldsymbol{p}' + m) - \tfrac{1}{2}\,\Sigma_f^i(p)\,(i\boldsymbol{p} + m) = B^i. \tag{19}$$

The part B^i has been obtained explicitly[17]). In our notation,

$$B^i = -\frac{\alpha}{\pi}\int\frac{d\omega}{\omega},$$

i.e.

$$r_2 = 2\varrho_2 = \frac{2\alpha}{\pi}\int\frac{d\omega}{\omega}. \tag{20}$$

These diagrams are seen to yield an infrared divergence which exactly compensates the second term in b, Eq. (7). We can combine Eqs. (13) and (20),

$$r = r_1 + r_2 = -\frac{2\alpha}{\pi}\left(\frac{\tanh^{-1}\beta'}{\beta'} - 1\right)\int\frac{d\omega}{\omega} = -b. \tag{21}$$

This completes the proof of the statement (10).

We may add the remark that according to this proof, closed loops can never contribute to the i. r. divergences, because in the case of a closed loop the cancellation of the i. r. divergences between the self-energy and vertex parts is complete. This is in accord with the fact that the contribution of closed loops to soft photon emission vanishes in the limit of vanishing energy[18]). Thus, closed loops contribute neither to b nor to r.

C. *Generalization.*

We shall now briefly indicate how the result thus far obtained can be generalized so as to include all types of processes to any order in α. Let us first consider the basic process of fig. 1 with more than one additional soft photon emission.

For instance, in order to prove that the cancellation occurs for n soft photons, we proceed by induction. We assume that the cancellation is true for the process involving $n-1$ real soft photon emissions. We then consider the diagram corresponding to this process as a new basic diagram. We note that the number of real photons emitted does not enter into the explicit proofs of subsections A and B. Thus, the reasoning of these sections can now be applied to this new basic diagram; it follows that the i. r. divergence from the emission of n real soft photon cancels the i. r. divergence of the radiative correction of corresponding order. Since we have proven this cancellation explicitly for the one photon process, we have established that in the process of fig. 1 no i. r. divergence exists.

Next, we need also a generalization with respect to the direction and number of electron paths. It is clear that the direction of the electron path can be reversed without affecting in any way the result of sections A and B.

The result of these sections can also be extended to the case of more than one electron path. Indeed, it follows that for every open electron path separately, the required cancellations occur. Therefore, the result is general.

We have thus established that the i. r. divergence is entirely due to the unphysical description of certain processes by the iteration solution. The processes of soft photon emission and radiative corrections due to soft virtual photons are evaluated as two quite different processes in the iteration solution, and originate in S-matrix elements of different order in e. Physically, these processes should not be separated. Their separation causes the appearance of i. r. divergences which are thus introduced in an artificial way due to the particular mathematical treatment adopted.

The recombination of soft photon emission and soft photon corrections eliminates the infrared divergences in all processes and in all orders of the approximation.

III. The closed form of the soft photon emission.

In the preceding section we have come to the important conclusion that the model for the mechanism of radiative processes which follows so naturally from the iteration solution in the form of the Feynman diagrams, is actually a very poor model. It breaks down completely for the description of soft photon processes, as we have just seen.

In order to arrive at a better description of this case we evidently must obtain closed expressions which contain the effect of photons of various low energies, and which are independent of the number of the participating photons. This number is not observable, since the "free" and the virtual photons apparently become indistinguishable in the soft photon limit. In fact, there seems to exist no very sharp separation between photons which are observable individually, and photons which are observable only collectively. The latter appear as an energy loss, $\Delta\varepsilon$, whose details cannot be differentiated.

To study these conjectures we ask for the modification of a given process due to this collective soft photon effect. We thereby expect to obtain a dependence on $\Delta\varepsilon$, so that the whole collective effect *vanishes* for $\Delta\varepsilon = 0$. This calculation will be free of infrared divergences, as follows from the above proof. It was first carried out in the fundamental paper by Bloch and Nordsieck[3]).

We shall go beyond the result of these authors as well as that of Pauli and Fierz[5]) by demonstrating that there is no other diver-

gence in this theory. This is in contradiction to the results contained in reference 5 and the resolution of this contradiction is one of the main problems to be solved. In addition, we have made use of the covariant formalism, which allows us to remove several of the restrictions which had to be made in previous work on this subject. Specifically, we do not need to assume that the electrons move with nonrelativistic velocities, nor do we have to restrict ourselves to the first order Born approximation of the scattering problem under discussion. All these restrictions are recognized as irrelevant to the problem on hand and their removal is necessary for a satisfactory treatment of the question.

In fact, our treatment is so general that it applies to any type of process involving any number of open electron paths. As in the preceding section, we shall discuss in detail only the case of electron scattering, but this restriction is merely made for the convenience of presentation. The necessary modifications in the general case will be obvious once the method is clearly understood for the simple special case discussed here.

The method to be used is an adaptation of the transformation of BLOCH and NORDSIECK. It consists of two steps. In the first step we show that the calculation of the soft photon emission is equivalent to the calculation of the S-matrix associated with a certain classical current distribution. In the second step we give the rigorous expression for the S-matrix associated with such a classical current.

Consider the basic process of fig. 1. The single photon emission probability $\beta(k)$ was calculated in Eq. (4). This same value for the photon emission probability amplitude can also be obtained from a classical current distribution

$$s_\mu(x) = \frac{1}{(2\pi)^{5/2}} \int s_\mu(k)\, e^{ikx}\, d^4k\,, \tag{22}$$

provided we choose

$$s_\mu(k) = \frac{ie}{(2\pi)^{3/2}} \left(\frac{p_\mu'}{p' \cdot k} - \frac{p_\mu}{p \cdot k} \right). \tag{23}$$

Indeed, the first order S-matrix element for the emission of a photon of momentum k and polarization e by a classical current is given by

$$\beta(k) = \frac{i}{\sqrt{2\omega}}\, s_\mu(k)\, e^\mu \tag{24}$$

which agrees with Eq. (4) and with the choice (23). We see therefore that for soft photon emission the matrix element of the basic process M behaves just like a classical current. This is correct not only for

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the first order process of single photon emission, but remains true for processes of any order.

Therefore, we can calculate the correction factor β for the emission of any number of soft photons in the process M by calculating the probability amplitude for any number of soft photons created by the classical current density (22) and (23). This problem can be solved rigourously.

We note here in passing that the classical current with the FOURIER components $s_\mu(k)$ as given in Eg. (23) has a simple physical interpretation which can be seen most easily when it is transformed into x-space. The identities

$$\int_{-\infty}^{0} \delta(x-v\tau)\, d\tau = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k\cdot v} e^{ikx}$$

$$\int_{0}^{\infty} \delta(x-v\tau)\, d\tau = \frac{-i}{(2\pi)^4} \int \frac{d^4k}{k\cdot v} e^{ikx}$$

allow us to express the current in x-space in th e form

$$s_\mu(x) = -e\int_{-\infty}^{0} v_\mu\, \delta(x-v\tau)\, d\tau - e\int_{0}^{\infty} v_\mu'\, \delta(x-v'\tau)\, d\tau \tag{25}$$

with

$$v_\mu = \frac{1}{m} p_\mu\,, \qquad v_\mu' = \frac{1}{m} p_\mu'\,.$$

This is the current of a classical point-charge $-e$ moving for $x^0 < 0$ with the constant four-velocity v_μ and for $x^0 > 0$ with the velocity v'_μ.

Before we proceed with the calculation, we consider a lemma which is of importance for the S-matrix of a classical current distribution.

Let $H(\tau)$ be an interaction operator which satisfies the condition

$$[H(\tau), H(\tau')] = i\, C(\tau, \tau') \tag{26}$$

where $C(\tau, \tau')$ is a c-number. The S-matrix for this system differs from the matrix

$$S' = e^{-i\Sigma} \tag{27}$$

$$\Sigma = \int_{-\infty}^{+\infty} H(\tau)\, d\tau \tag{28}$$

only by a phase factor.

The significance of this lemma lies in the fact that the phase factor can be ignored and that the S-matrix can be calculated according to Eq. (27).

An alternative way of expressing this lemma is to state that the time-ordering operation in the iteration solution contributes only a phase factor.

In order to prove it we carry out the τ-dependent canonical transformation

$$\omega'(\tau) = e^{i\Sigma(\tau)}\,\omega(\tau). \tag{29}$$

The new state vector satisfies a transformed SCHRÖDINGER equation

$$i\,\dot{\omega}'(\tau) = H'(\tau)\,\omega'(\tau)$$

with $H'(\tau)$ given by

$$H'(\tau) = e^{i\Sigma(\tau)}\left(H(\tau) - i\frac{d}{d\tau}\right)e^{-i\Sigma(\tau)}.$$

This expression can be written in terms of iterated commutators

$$H'(\tau) = H(\tau) + i[\Sigma(\tau), H(\tau)] - \dot{\Sigma}(\tau) + \frac{i}{2}[\dot{\Sigma}(\tau), \Sigma(\tau)] + \cdots. \tag{30}$$

If we choose $\Sigma(\tau)$ such that it satisfies

$$\left.\begin{aligned}\dot{\Sigma}(\tau) &= H(\tau)\\ \Sigma(-\infty) &= I,\end{aligned}\right\} \tag{31}$$

and

then the omitted commutators in Eq. (30) are all zero and the terms in (30) which are explicitly written out constitute the rigorous expression for $H'(\tau)$. In this case, the transformed interaction operator is given by

$$H'(\tau) = -\frac{1}{2}\int_{-\infty}^{\tau} C(\tau', \tau)\,d\tau' \tag{32}$$

and is a c-number.

We now finish the proof of the lemma by expressing the S-operator of the original problem in terms of the transformation matrix $V(\tau, \tau_0)$ of the transformed problem. This transformation matrix is defined by the relation

$$\omega'(\tau) = V(\tau, \tau_0)\,\omega'(\tau_0)$$

and is given explicitly by

$$V(\tau, \tau_0) = e^{-i\int_{\tau_0}^{\tau} H'(\tau')\,d\tau'}.$$

Using the transformation law Eq. (29) we find for the S-operator

$$S = e^{-i\Sigma(\infty)}\,V(\infty, -\infty)\,e^{i\Sigma(-\infty)}.$$

With Eqs. (31) and the definition

$$\Sigma \equiv \Sigma(\infty) = \int_{-\infty}^{+\infty} H(\tau)\, d\tau$$

we may write

$$S = e^{i\Theta}\, e^{-i\Sigma} \tag{33}$$

where

$$\Theta = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{\tau} C(\tau', \tau)\, d\tau'. \tag{34}$$

This proves the lemma.

Returning now to the problem of soft photon emission, we see that the interaction operator

$$H(\tau) = -\int_{\sigma(\tau)} s_\mu(x)\, a^\mu(x)\, d\sigma$$

with the classical current density (22) and (23) satisfies the condition of our lemma. We therefore write for the S-operator (omitting the irrelevant phase factor) according to Eq. (27)

with

$$\left.\begin{aligned} S &= e^{-i\Sigma} = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!}\, \Sigma^m \\ \Sigma &= -\int d^4x\, s_\mu(x)\, a^\mu(x)\,. \end{aligned}\right\} \tag{35}$$

In order to calculate explicitly a specific matrix element we divide the k-space into finite but small cells Δ_r ($r = 1, \ldots$) and denote by $\boldsymbol{k}_r$ a representative value of the k-vector in Δ_r. We evaluate the matrix element M which connects the photon vacuum state with a state representing precisely n_1 photons in Δ_1, n_2 photons in Δ_2, etc.

The matrix element in question is obtained from those terms in the expansion (35) of S for which $m \leqslant \sum_r n_r$. For, the operator Σ contains a sum of one creation and one annihilation operator for the photons. The power Σ^m must contain at least $\sum_r n_r$ creation operators. Hence $m \geqslant \sum_r n_r$. For a particular value of m which satisfies this condition the contribution of Σ^m to the matrix element can be obtained by the use of the ordering theorem of WICK[19]). According to this theorem one selects from Σ^m all possible factor pairings and replaces each pairing by the value of the contraction symbol. These contraction factors describe the effect of the virtual photon processes.

In the present case there is a considerable simplification as compared to the general case treated by WICK due to the fact that the

current is a c-number and the time ordering process is not needed. One verifies by inspection that as a result the matrix element M can be written as a product

$$M = \prod_{r=1}^{\infty} M_r \tag{36}$$

such that each M_r contains only the contributions from photon operators referring to the cell Δ_r. Furthermore, each contraction of a pair of photon operators $a_\mu(\boldsymbol{k}_r)$, $a_\mu^*(\boldsymbol{k}_r)$ produces a factor

$$\varrho_r \equiv \varrho(\boldsymbol{k}_r) = \frac{\Delta_r}{2\omega_r} s_\mu^*(\boldsymbol{k}_r)\, s^\mu(\boldsymbol{k}_r)\,. \tag{37}$$

If l is the number of contractions from photon operators referring to Δ_r and $m = n_r + 2l$, we obtain from Σ^m a factor ϱ_r^l and also a combinational factor $\frac{1}{2^l}\frac{m!}{n_r!\,l!}$, which is the number of times l ordered pairs can be selected from m distinct objects. When this is summed over l, we obtain the total contribution to M_r in the form

$$M_r = \frac{1}{n_r!}\left(\frac{+i}{\sqrt{2\omega_r}} s_\mu(\boldsymbol{k}_r)\, e^\mu(\boldsymbol{k}_r)\right)^{n_r} \sum_{l=0}^{\infty} (-1)^l \frac{\varrho_r^l}{2^l\, l!}\,. \tag{38}$$

The total transition probability for the emission process is thus

$$P = \sum_{\text{pol}} \prod_{r=1}^{\infty} n_r!\, |M_r|^2\, \Delta_r^{n_r}. \tag{39}$$

The factors $\Delta_r^{n_r}$ are the densities of final states and the factors $n_r!$ take account of the different possible sequences in which n_r distinct photons can be emitted. In this expression we have also summed over the unobserved polarization states of the emitted photons which can be evaluated as follows:

$$\sum_{\text{pol}} |s_\mu(k)\, e^\mu(k)|^2 = |\boldsymbol{s}(k)|^2 - \frac{1}{\omega^2}|\boldsymbol{s}(k)\cdot\boldsymbol{k}|^2 = s^\mu(k)\, s_\mu^*(k)\,. \tag{40}$$

The last equation is a simple consequence of the conservation law for the current in the form $s_\mu(k)\, k^\mu = 0$.

The transition probability (39) becomes, with Eqs. (40), (37), and (36),

$$P = \prod_{r=1}^{\infty} e^{-\varrho_r} \frac{\varrho_r^{n_r}}{n_r!}\,. \tag{41}$$

This result shows that the final photons are emitted according to the Poisson distribution law into each volume element in k-space, and that therefore the average number of photons emitted into Δ_r is just exactly ϱ_r (Eq. (36)).

We also see that the average total number of emitted photons is

$$\overline{N} = \sum_r \varrho_r = \sum_r \frac{\Delta_r}{2\,\omega_r}\, s^*_\mu(k)\, s^\mu(k)\,. \tag{42}$$

When this expression is transformed into an integral by letting $\Delta_r \to 0$ and the current (23) is substituted, we obtain precisely the integral for b in Eq. (5), which was shown to be logarithmically divergent. It follows that the average number of photons emitted into all phase space is infinite. This divergence is now no longer a difficulty since it appears as an infinity for the total number of photons emitted rather than as an infinite transition probability[20]).

The result up to this point is in complete agreement with the conclusions of BLOCH and NORDSIECK, except for the fact that the validity of this result is now secured for much more general cases.

We turn to the question of the effect of soft photon emission on the scattering cross section of the basic process (fig. 1). From the preceding conclusion, it is evident that the cross section σ' which includes the emission of soft photons is related to the cross section σ without such photons by a multiplicative numerical factor. The value of this factor depends on the energy resolution $\Delta\varepsilon$ which can be detected in the observation of the scattering process. In fact, we find from (41)

$$\sigma' = b(\Delta\varepsilon)\,\sigma \tag{43}$$

where

$$b(\Delta\varepsilon) = \sum_{(n_r)}{}' \prod_{r=1}^{\infty} e^{-\varrho_r} \frac{\varrho_r^{n_r}}{n_r!}\,. \tag{44}$$

The summation sign is written with a prime. This symbol indicates a restriction on the summation to such values of ω_r, that the total energy lost to the soft photons is less than $\Delta\varepsilon$*),

$$\sum_{r=1}^{\infty} n_r\,\omega_r < \Delta\varepsilon\,. \tag{45}$$

*) In general, there should also be a restriction on the momenta

$$\sum_{r=1}^{\infty} n_r\,\boldsymbol{k}_r < \Delta\boldsymbol{p}$$

corresponding to an experimental inaccuracy $\Delta\boldsymbol{p}$ in the measurement of momentum balance. However, the error in the energy measurement, $\Delta\varepsilon/\varepsilon$, implies an error of momentum $|\Delta p/p|$ of at least the order $\Delta\varepsilon/\varepsilon$. Therefore, only when the error in momentum is much smaller than the error in energy need Δp be taken into account. In this case the equations become more complicated, but little is gained in the understanding of the basic problem. Furthermore, the presence of external fields always precludes a measurement of the momentum balance. We shall therefore ignore the momentum restriction in the following.

This is an essential point in the following discussion. BLOCH and NORDSIECK[3]) had omitted this restriction and obtained the desirable value $b = 1$. PAULI and FIERZ did include this restriction[5]) and obtained the physically incomprehensible result $b = 0$. We shall include this restriction and show that the latter result is incorrect and that in fact $b = 1$ to a high degree of approximation.

We can restore the unrestricted summation by introducing the discontinuous function $I(n_r)$ with the property

$$I(n_r) = \left\{ \begin{array}{l} 1 \text{ for } \sum_r n_r \omega_r < \Delta\varepsilon \\ 0 \text{ for } \sum_r n_r \omega_r > \Delta\varepsilon\,. \end{array} \right\} \tag{46}$$

A useful analytical expression for this function is

$$I(n_r) = \int_0^{\Delta\varepsilon} \delta\left(\sum_r n_r \omega_r - x\right) dx = \frac{1}{2\pi} \int_0^{\Delta\varepsilon} dx \int_{-\infty}^{\infty} d\sigma\, e^{i\left(\sum_r n_r \omega_r - x\right)\sigma}\,.$$

The restriction in the summation (44) can now be dropped and we obtain

$$b(\Delta\varepsilon) = \frac{1}{2\pi} \int_0^{\Delta\varepsilon} dx \int_{-\infty}^{\infty} d\sigma\, e^{G(\sigma) - i x \sigma} \tag{47}$$

with

$$e^{G(\sigma)} = \sum_{(n_r)} \prod_{r=1}^{\infty} e^{-\varrho_r} \frac{(\varrho_r e^{i\omega_r \sigma})^{n_r}}{n_r!}\,. \tag{48}$$

The summation is now unrestricted and therefore the product and the sum can be interchanged. Each individual sum for any r is of the form

$$\sum_n \frac{(\varrho e^{i\omega\sigma})^n}{n!} = e^{\varrho e^{i\omega\sigma}}\,.$$

Therefore, we obtain for $G(\sigma)$

$$G(\sigma) = \sum_r \varrho_r (e^{i\omega_r \sigma} - 1)\,. \tag{49}$$

This function can be reduced to an integral in the limit $\Delta_r \to 0$

$$G(\sigma) = \int \frac{d^3 k}{2\omega} s^*_\mu(k)\, s^\mu(k)\, (e^{i\omega\sigma} - 1)\,. \tag{50}$$

It is now easy to see how the i. r. divergence has disappeared. For, when we substitute the expression (23) in Eq. (50), we obtain

$$G(\sigma) = \frac{\alpha}{(2\pi)^2} \int \frac{d^3 k}{\omega} \left(\frac{p'^2}{(p'\cdot k)^2} + \frac{p'^2}{(p\cdot k)^2} - \frac{2\, p\cdot p'}{(p'\cdot k)\,(p\cdot k)} \right) (e^{i\omega\sigma} - 1)\,. \tag{51}$$

The factor $e^{i\omega\sigma} - 1$ approaches $i\omega\sigma$ for $\omega \to 0$ and eliminates the singularity at the origin. One can trace the term $e^{i\omega\sigma}$ to the real processes and the term -1 to the virtual processes, so that the cancellation between these two types of processes is here exhibited explicitly, and is complete in the limit $\omega \to 0$. After the angular integration the expression (51) becomes

$$G(\sigma) = \alpha C \int_0 \frac{d\omega}{\omega} (e^{i\omega\sigma} - 1) \tag{52}$$

where C is a positive numerical factor of order one which depends on $\boldsymbol{p}$ and $\boldsymbol{p}'$. In the coordinate system in which the initial electron is at rest ($\boldsymbol{p} = 0$), C was evaluated to be

$$C = \frac{2}{\pi}\left(\frac{\tanh^{-1}\beta'}{\beta'} - 1\right). \tag{53}$$

We have purposely avoided writing an uper limit in (52) because this limit needs careful discussion. If we integrate in (52) simply to $+\infty$, we obtain the result of PAULI and FIERZ:

$$G(\sigma) = -\infty, \; b = 0.$$

At this point we must recall the restriction for the validity of the expression for b. The derivation of b in the form (47) is based on the formula (3) which allowed us to replace the matrix element M_1 (Eq. (2)) by the matrix element of the classical current density (23). The validity for (3) is only assured for $\omega \ll m$ as was stated explicitly in connection with this equation.

For values of ω which do not satisfy $\omega \ll m$, we would have to work with the rigorous formula (2) for the real emission processes and the complete expressions corresponding to the diagrams of fig. 3 for the virtual processes. In this case it is of course impossible to sum the contributions from virtual photon processes to all orders of α as we did in the case $\omega \ll m$.

Fortunately, the complete expression of the rigorous solution is not needed for the clarification of the problem caused by the PAULI-FIERZ divergence. We need only recall that the divergences in the high frequency limits of virtual processes are all of the type which can be removed by the renormalization of mass and charge[1]). The remaining finite terms are small corrections of order α.

While it is thus assured that the contribution of the radiative corrections from each order of α is finite we do not know whether the *sum* of the contributions from all orders is finite. This involves the difficult and as yet unsolved question whether the infinite series of

the renormalized radiative corrections converges. We shall return to this problem at a later point.

If we wish to obtain quantitative results for the correction factor $b(\Delta\varepsilon)$, we must take account of the limitation involved in the derivation of (44) and restrict the expression to values of ω, such that $\omega \ll m$. We can do this in a crude way by introducing a cut-off at an energy value $\varepsilon \cong m$ throughout the calculation. The integral in (52) is then to be extended only up to the value $\omega = \varepsilon$.

We emphasize here that the need for a cut-off in the evaluation of ε is not a principle difficulty. It is merely caused by the limitation of the approximation, $\omega \ll m$. With this crude method we cannot expect to obtain a result which can be trusted in detail. However, there are certain features of the result which are essentially independent of the cut-off and which will be very likely characteristic for the correct result also.

In order to illustrate this point we shall study the expression for $b(\Delta\varepsilon)$ for two different cut-offs. We write for (52)

$$G(\sigma) = \alpha C \int_0^\infty \frac{d\omega}{\omega} (e^{i\omega\sigma} - 1)\, g(\omega) \tag{54}$$

where $g(\omega)$ is some form factor.

We discuss the following two cases[21])

$$g(\omega) = \begin{cases} 1 \text{ for } \omega < \varepsilon \\ 0 \text{ for } \omega > \varepsilon \end{cases} \tag{55a}$$

$$g(\omega) = e^{-\omega/\varepsilon}. \tag{55b}$$

Case (a) corresponds to a sharp cut-off at $\omega = \varepsilon$. In this case,

$$b(\Delta\varepsilon) = \frac{1}{2\sigma} \int_0^{\Delta\varepsilon} dx \int_{-\infty}^{+\infty} d\sigma \exp\left\{-ix\sigma + \alpha C \int_\sigma^\varepsilon \frac{d\omega}{\omega} (e^{i\omega\sigma} - 1)\right\}. \tag{56}$$

It then follows that, independently of ε,

$$b(0) = 0. \tag{57}$$

The probability for elastic scattering is strictly zero. This result is in complete agreement with BLOCH and NORDSIECK.

We did not succeed in evaluating the integrals (56) in closed form. The first few coefficients of the power series development in α can be evaluated analytically with the result:

$$b(\Delta\varepsilon) = 1 - \alpha C \ln \frac{\varepsilon}{\Delta\varepsilon} - \frac{1}{2}\left(\frac{\pi^2}{6} - \left(\ln \frac{\varepsilon}{\Delta\varepsilon}\right)^2\right) \alpha^2 C^2 + \cdots. \tag{58}$$

This series shows that for all but the smallest values of $\Delta\varepsilon$ this correction factor is nearly equal to one. In fact the first order term $\alpha C \ln \varepsilon/\Delta\varepsilon$ is of the same order of magnitude as the correction factor due to the emission of one photon with total energy loss $\Delta\varepsilon$ combined with the radiative correction of first order. In the limit $\Delta\varepsilon \to 0$ this series expansion diverges. We find, therefore, that the correction factor $b(\Delta\varepsilon)$ departs appreciably from unity only for

$$\Delta\varepsilon < \varepsilon\, e^{-\frac{1}{\alpha C}}. \tag{59}$$

The cut-off function (55b) yields, when inserted into Eq. (54),

$$b(\Delta\varepsilon) = \frac{1}{2\pi}\int_0^{\Delta\varepsilon} dx \int_{-\infty}^{+\infty} d\sigma \exp\left\{-i x\sigma + \alpha C \int_0^{\infty} \frac{d\omega}{\omega}\left(e^{\left(i\sigma - \frac{1}{\varepsilon}\right)\omega} - e^{-\frac{\omega}{\varepsilon}}\right)\right\}. \tag{60}$$

This integral can be evaluated in closed form. Since the method of evaluation is discussed in detail in reference 5, we shall give here only the result

$$b(\Delta\varepsilon) = \frac{1}{\Gamma(\alpha C)}\int_0^{\Delta\varepsilon/\varepsilon} e^{-z} z^{(\alpha C - 1)} dz. \tag{61}$$

The integral is the incomplete Gamma function

$$\Gamma(x, y) = \int_0^{y} e^{-z} z^{x-1} dz, \qquad \Gamma(x, \infty) = \Gamma(x),$$

which has the development

$$\frac{\Gamma(x, y)}{\Gamma(x)} = \frac{y^x}{\Gamma(x+1)}\left(1 - \frac{y}{1!}\frac{x+1}{x+2} + \frac{y^2}{2!}\frac{x+1}{x+3} - + \cdots\right). \tag{62}$$

For $\Delta\varepsilon/\varepsilon \ll 1$ it is sufficient to retain the first order term only,

$$b(\Delta\varepsilon) \cong \frac{1}{\Gamma(1+\alpha C)}\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^{\alpha C}. \tag{63}$$

For $\Delta\varepsilon = 0$ we find again

$$b(0) = 0.$$

The factor $\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^{\alpha C}$ reduces to $\frac{1}{2}$ for

$$\Delta\varepsilon = \varepsilon\, 2^{-\frac{1}{\alpha C}}. \tag{64}$$

This condition differs very little from the condition (59) which was found for the case (a). For $\Delta\varepsilon$ substantially larger than $\varepsilon\, 2^{-1/\alpha C}$ the expression for b can be written as a power series in α.

$$b(\Delta\varepsilon) \cong 1 - \alpha C \ln\frac{\varepsilon}{\gamma\,\Delta\varepsilon} + O(\alpha^2) \tag{65}$$

where γ is Euler's constant:

$$\ln \gamma = 0{,}5772\ldots$$

In this case we find again the result that $b(\Delta\varepsilon)$ differs from 1 only by a term of order α.

In order to investigate this point further, we compare the expansions (58) and (65) with the result of the iteration solution. In this solution the successive terms in α correspond each to a combination of radiative corrections and the emission of a finite number of soft photons. As was shown in the previous section, each term in α is free of the infrared divergence. The successive terms in the expansions (58) and (65) approximate the exact results of the iteration solution. In fact, it is always possible to choose the cut-off ε in such a way that the expansion (58), say, coincides exactly with the iteration solution. ε must then be a function of n, the power of α, and also depends on the particular process under consideration.

The series (58) can then be written

$$b(\Delta\varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha C)^n}{n!} \left[\ln\left(\frac{\varepsilon_n}{\Delta\varepsilon}\right)\right]^n. \tag{66}$$

It seems reasonable to assume that the upper bound of ε_n is within the radius of convergence of the infinite sum (58), which would imply the convergence of the renormalized iteration solution*).

The two methods of calculation discussed in the preceding and the present section, respectively, are, in a certain sense, equivalent. The cross section for any process, as calculated for the *bare* electron, can be corrected to represent the *physical* electron in one of two different ways. Either the radiative corrections and the associated soft photon emission can be calculated to arbitrarily high order in α by the iteration solution, or the cross section can be multiplied by the factor $b(\Delta\varepsilon)$. The two methods of calculation are complementary, in that the first method is exact in each power of α, but necessarily restricted to a finite number of terms, whereas the second method includes all powers of α, i. e. an infinite number of soft photons, but is only approximate in its account of the intermediate states of high energy.

The two procedures show the disappearance of the infrared divergence from two different aspects, of which the method discussed in

*) Certain cases, e. g. those in which the system has bound states, are of course excluded a priori, since then the iteration solution is known to diverge.

the present section is of particular interest. We refer to the properties of $b(\Delta\varepsilon)$ which are independent of the cut-off:

(1) $b(0) = 0$.

(2) For all finite values of $\Delta\varepsilon$ b differs from 1 by terms of order α only, in contradistinction to the result of PAULI and FIERZ.

We conclude further that the result of BLOCH and NORDSIECK which is obtained by disregarding the energy conservation in the soft photon emission process is very nearly correct, radiative corrections being negligible in most cases.

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