Zeitschrift: Helvetica Physica Acta
Band: 28 (1955)
Heft: I

Artikel: The contribution of non-static forces to the binding-energy of the deuteron
Autor: Cerulus, Frans
DOI: https://doi.org/10.5169/seals-112601

The Contribution of Non-Static Forces to the Binding-Energy of the Deuteron

by **Frans Cerulus**, Basel*).

(20. XII. 1954.)

Summary: The Bethe-Salpeter equation for the bound states of a neutron and a proton, having a scalar interaction with a charged meson-field, is set up. Keeping the binding energy fixed the eigenvalue for the coupling constant g is determined, taking into account the g^2 and the g^4 terms in the equation. This is done by means of a perturbation method, starting from the static approximation, and showing that the non-static forces contribute an important part to the binding. For a 2,18 Mev binding energy there results $g^2 = 0{,}54\ g_0^2$ where g_0 is the value obtained from the static approximation.

Introduction.

The equation for a function describing bound states in a relativistically invariant way, was first given by BETHE and SALPETER[1]), and derived from field-theory by GELL-MAN and LOW[2]). (We shall henceforth call it B.-S. equation.) Unfortunately it is hitherto known explicitly in the form of an expansion of doubtful convergence only. An investigation of the first term of this expansion, however, shows it to yield the static approximation as we known it from classical meson-theory[3]), provided possible retardation effects are disregarded. This fact might give us some confidence in the validity of the expansion, as the static approximation has some, at least qualitatively, desirable features. At the same time, it gives us the opportunity of carrying out a quantitative comparison between the results from the static approximation and those from a more refined theory to which the former is only a first approximation.

In view of the mathematical intricacies of the problem, it is unavoidable to start from a particular model of a bound state in order to do any actual calculations.

*) Present address: Natuurkundig Laboratorium der Universiteit, Rosier 6, Gent (Belgium).

Results will consequently have only a broader significance than that of corrolaries from some mathematical assumptions if they are derived from physically meaningful features of the model. It might be noted that, as the mathematics of the B.-S. eq. are not altogether clear, such corrolaries do have some interest. We will, however, attach ourselves to the physical aspects of the theory, and make fully use of arguments from physics in the course of our calculations.

As a model we take a deuteron-like structure consisting of a neutron and a proton having a scalar, symmetrically charged interaction with a meson-field. We will compare the value of the coupling constant for a given binding energy, as derived from the first two terms of the expansion of the B.-S. eq., with the value derived from the static approximation.

Similar investigations have been published by LEVY[4]) and WENTZEL[5]) for the scalar neutral case. The range of validity of some of their approximations is not clearly defined; when we encounter similar problems we will endeavour to show under what circumstances our results are valid. But the important difference lies in the fact that non-static forces behave differently in the neutral and in the charged case, giving in the latter a substantial contribution to the binding energy.

Our notation closely follows that of ref.[4]).

The B.-S. Equation in the g^4-Approximation.

We start from the general B.-S. equation:

$$\chi(1,2) = \frac{1}{4} \int S_c'(1,3)\, S_c''(2,4)\, G(3,4;5,6)\, \chi(5,6)\, d\omega(3,4,5,6) \qquad (1)$$

as derived by GELL-MAN and LOW[2]). We have written 1, 2, ... instead of x_1, x_2, ... The primes (′) and (″) mean that the operators so marked operate only on the first (or second) particle. The coefficient 1/4 is due to a different definition of S_c; we use the definitions of PAULI[6]) and DYSON[7]) throughout with $\hbar = c = 1$.

We shall calculate now the first two terms of the expansion of G in terms derived from irreducible graphs. The first two of these are shown in fig. 1.

As hamiltonian density of interaction we take

$$H_i = g \sum_{\alpha=1}^{3} \bar{\psi}(x)\, \tau_\alpha\, \psi(x)\, \varphi_\alpha(x)$$

i. e. a scalar, charge-symmetric interaction; $\psi(x)$ is a nucleon-field

operator, φ_α ($\alpha = 1, 2, 3$) an operator of the charged meson-field; $\psi(x)$ incorporates an isotopic spin part.

The first term of G, the so-called ladder approximation, is

$$-g^2\,\delta(3-5)\;\delta(4-6)\sum_{\alpha,\beta}\tau'_\alpha\,\varphi_\alpha(5)\;\tau''_\beta\,\varphi_\beta(6)\,.$$

The second term, derived from (b), is

$$g^4\sum_{\alpha,\beta}\sum_{\gamma,\delta}\tau'_\alpha\,\psi'(3)\;\bar{\psi}'(5)\;\tau'_\beta\tau''_\gamma\,\psi''(4)\;\bar{\psi}''(6)\;\tau''_\delta\,\varphi_\alpha(3)\;\varphi_\beta(5)\;\varphi_\gamma(4)\;\varphi_\delta(6)\,.$$

The operators are now taken in pairs, according to the graph, and these pairs replaced by their vacuum expectation value, the rules being the same as those applied in the S-matrix calculations.

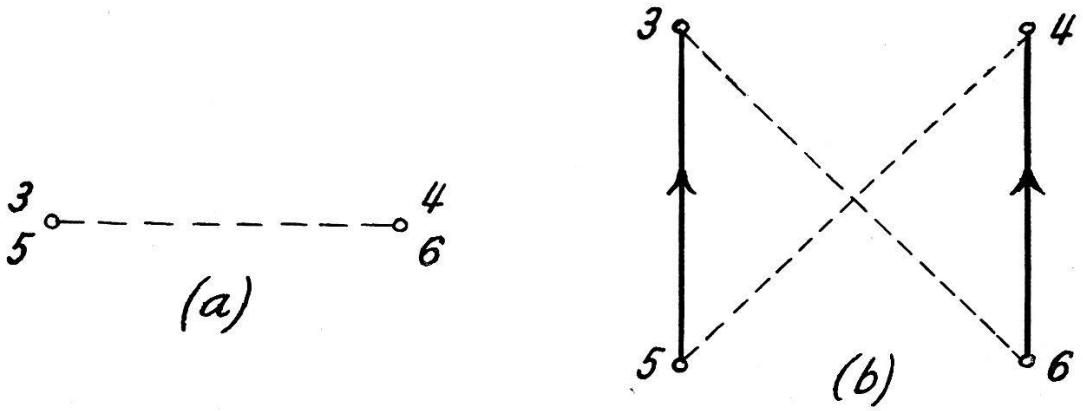

Fig. 1.

Taking only these two terms of G into account we obtain the g^4-approximation to the B.S. equation:

$$\chi(1,2) = -g^2\frac{1}{4}\int S'_c(1-3)\,S''_c(2-4)\,\delta(3-5)\,\delta(4-6)\sum_\alpha\tau'_\alpha\tau''_\alpha\,\frac{1}{2}\,\Delta_c(5-6)\times$$

$$\times\,\chi(5,6)\,d\omega(3,4,5,6) +$$

$$+g^4\frac{1}{4}\int S'_c(1-3)\,S''_c(2-4)\sum_{\alpha,\beta}\sum_{\gamma,\delta}\tau'_\alpha\left(-\frac{1}{2}\right)S'_c(3-5)\,\tau'_\beta\tau''_\gamma\left(-\frac{1}{2}\right)\times$$

$$\times S''_c(4-6)\,\tau''_\delta\,\delta_{\alpha\delta}\cdot\delta_{\beta\gamma}\,\frac{1}{2}\,\Delta_c(4-5)\,\chi(5,6)\,d\omega(3,4,5,6)\,. \qquad (2)$$

In constructing the vacuumpropagators $S'_c(3—5)$ and $S''_c(4—6)$ we have to sum over the two charge states. The isotopic spin part of these S_c is therefore unity, and we can take the isotopic spin operators together.

We use the identity

$\sum_{\alpha,\beta} \tau'_\alpha \tau'_\beta \tau''_\alpha \tau''_\beta = 3+2(\vec{\tau}', \vec{\tau}'')$ and we define the symbol $T \equiv (\vec{\tau}', \vec{\tau}'')$.

The equation is then:

$$\chi(1,2) = -g^2 \frac{1}{2^3} \int S'_c(1-3)\, S''_c(2-4)\, \Delta_c(3-4)\, T\, \chi(3,4)\, d\omega(3,4) +$$
$$+ g^4 \frac{1}{2^6} \int S'_c(1-3)\, S''_c(2-4)\, S'_c(3-5)\, S''_c(4-6)\, \Delta_c(3-6) \times$$
$$\times \Delta_c(4-5)\,[3+2\,T]\,\chi(5,6)\, d\omega(3,4,5,6)\,. \qquad (3)$$

In a neutral scalar theory we would get the corresponding equation by replacing T by 1 in the g^2-term,
$3 + 2\,T$ by 1 in the g^4-term.

The g^4-B.-S. Equation in Momentum Space.

Let us denote, in general, by $f(p)$ the 4-dimensional Fourier-transformation of a function $f(x)$ of $\vec{x}$, $x_0 \equiv t$

$$f(x) = \int e^{i(\vec{p},\vec{x}) - i p_0 t} f(p)\, d\vec{p}\, dp_0.$$

We assume the center of mass of the two bound particles to be at rest, and we call W the total energy of the system. Let $\vec{p}$ be the momentum of the first nucleon (and therefore $-\vec{p}$ that of the second) and $W/2 + p_0$ its energy, in the state described by $\chi(1,2)$.

The easiest way to write Eq. (3) in the momentum-energy representation is to make use of the Feynman rules, starting directly from the graphs, after having satisfied the momentum- and energy-conservation laws by inspection.

For the sake of convenience, in order to write down at once the numerical factors in front of each term, we will take the $-\frac{2i}{(2\pi)^4}$ out of the usual definition of S_c and Δ_c, and we will call the functions so defined S and Δ.

Next to every line of the graph we write the corresponding momentum and energy (cf. fig. 2).

An undotted line, with $\vec{p}$, p_0, gives a factor

$$-\frac{2i}{(2\pi)^4} \frac{i(\vec{\gamma}, \vec{p}) - \gamma_4 p_0 - m}{\vec{p}^2 + (m - i\eta_p)^2 - p_0^2} \equiv -\frac{2i}{(2\pi)^4} S(\vec{p}, p_0)\,. \quad *)$$

*) The imaginary part $-i\eta_p$ in the denominator serves only to indicate the right way of integrating around the poles. The index will be found useful when carrying out successive integrations over 4th coördinates, as e. g. in the calculations for the formulae on p. 79.

A dotted line gives a factor

$$-\frac{2i}{(2\pi)^4}\frac{1}{\vec{p}^{\,2}+(m-i\varepsilon_p)^2-p_0^2}\equiv-\frac{2i}{(2\pi)^4}\Delta(\vec{p},p_0).$$

A vertex point gives a factor $(2\pi)^4$.

The numerical factors that might arise in the transformation from $\chi(x_1, x_2)$ to $\chi(p_1, p_2)$ occur on both sides of the B.-S. equation, and may thus be left out.

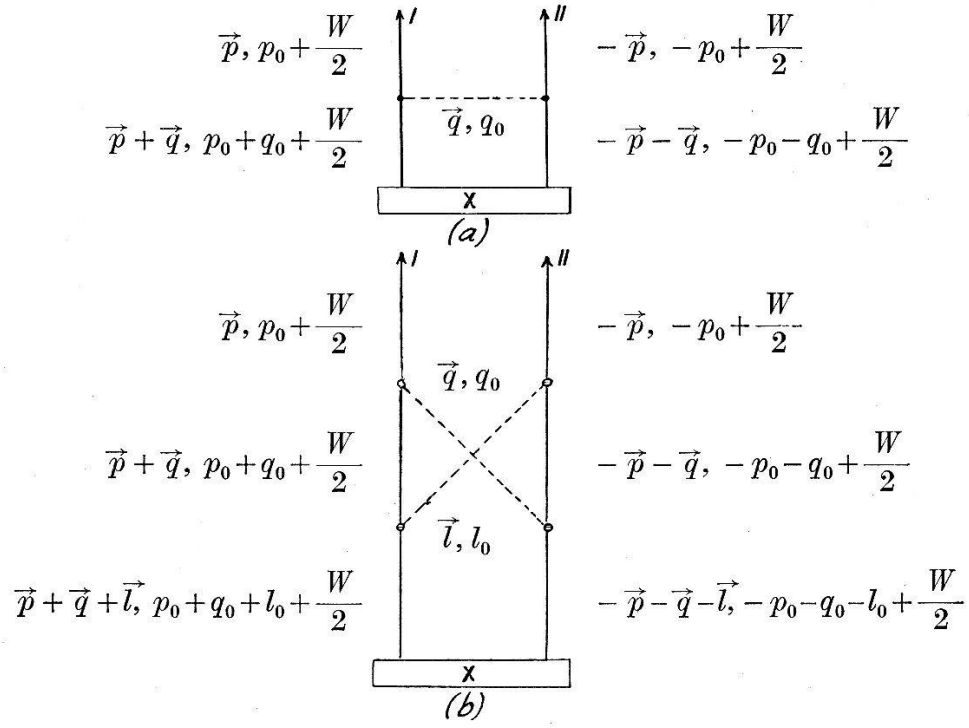

Fig. 2.

The result is:

$$\begin{aligned}\chi\left(\vec{p},p_0+\frac{W}{2};-\vec{p},-p_0+\frac{W}{2}\right)=&-\frac{i}{(2\pi)^4}g^2S'\left(\vec{p},p_0+\frac{W}{2}\right)\times\\&\times S''\left(-\vec{p},-p_0+\frac{W}{2}\right)\int\Delta(\vec{q},q_0)\,T\chi\left(\vec{p}+\vec{q},\,p_0+q_0+\frac{W}{2};\right.\\&\left.-\vec{p}-\vec{q},-p_0-q_0+\frac{W}{2}\right)d\vec{q}\,dq_0-\\&-\frac{1}{(2\pi)^8}g^4S'\left(\vec{p},p_0+\frac{W}{2}\right)S''\left(-\vec{p},-p_0+\frac{W}{2}\right)\times\\&\times\int S'\left(\vec{p}+\vec{q},\,p_0+q_0+\frac{W}{2}\right)S''\left(-\vec{p}-\vec{l},-p_0-l_0+\frac{W}{2}\right)\times\\&\times\Delta(\vec{q},q_0)\,\Delta(\vec{l},l_0)\,[3+2T]\,\chi\left(\vec{p}+\vec{q}+\vec{l},\,p_0+q_0+l_0+\frac{W}{2};\right.\\&\left.-\vec{p}-\vec{q}-\vec{l},-p_0-q_0-l_0+\frac{W}{2}\right)d\vec{q}\,dq_0\,d\vec{l}\,dl_0\,.\end{aligned}\tag{4}$$

It is known[2]) that $\chi(x_1, x_2)$ may be split into two factors, the first

depending only on the center-of-mass coordinates, the second on the relative coordinates:

$$\chi(x_1, x_2) = e^{-iW\frac{t_1+t_2}{2}} \int e^{i(\vec{p},\vec{x}_1-\vec{x}_2)-ip_0(t_1-t_2)} A(\vec{p}, p_0)\, d\vec{p}\, dp_0$$

where $A(\vec{p}.\, p_0)$ is the Fouriertransformation of the second factor.

(4) leads to the following equation for $A(\vec{p}, p_0)$:

$$\begin{aligned} A(\vec{p}, p_0) = &-\frac{i}{(2\pi)^4} g^2 S'\left(\vec{p}, p_0+\frac{W}{2}\right) S''\left(-\vec{p}, -p_0+\frac{W}{2}\right) \int \Delta(\vec{q}, q_0) \times \\ &\times T A(\vec{p}+\vec{q}, p_0+q_0)\, d\vec{q}\, dq_0 - \\ &-\frac{1}{(2\pi)^8} g^4 S'\left(\vec{p}, p_0+\frac{W}{2}\right) \times \\ &\times S''\left(-\vec{p}, -p_0+\frac{W}{2}\right) \int S'\left(\vec{p}+\vec{q}, p_0+q_0+\frac{W}{2}\right) \times \\ &\times S''\left(-\vec{p}-\vec{l}, -p_0-l_0+\frac{W}{2}\right) \Delta(\vec{q}, q_0)\, \Delta(l, l_0)\, [3+2T] \times \\ &\times A(\vec{p}+\vec{q}+\vec{l}, p_0+q_0+l_0)\, d\vec{q}\, dq_0\, d\vec{l}\, dl_0 . \end{aligned} \tag{5}$$

This will be the starting-point of our investigation.

The B.-S. Equation for unrelativistic Nucleon-Momenta.

We assume that, at least for the ground-state of the deuteron, the relative momenta of the nucleons lie practically always in the unrelativistic region, i. e. $\langle p \rangle \ll m$ (m is the nucleon mass).

This assumption is suppported by an elementary qualitative argument from classical quantum mechanics, viz., if the binding may be described by a central potential that vanishes sufficiently strongly at infinity, the eigenfunction of the relative coördinates, say ψ, has an asymptotic behaviour such that

$$e^{\sqrt{mE}\, r}\, \psi(r) \to \frac{1}{r} \qquad \text{for } r \to \infty \tag{6}$$

where E is the binding energy (counted positively). Near the origin ψ behaves as

$$\frac{1}{r} \sin\left(\frac{\pi}{2} r\, \mu\right) \tag{7}$$

(where $1/\mu$ is the range of the nuclear forces).

If we expand into plane waves the functions obtained by fitting (6) and (7) to one another at $r = 1/\mu$, we obtain a distribution for $\vec{p}$; this exhibits maxima at $|\vec{p}| \approx \sqrt{mE}$ and $|\vec{p}| \approx \pi/2\ \mu$ and decreases afterwards as p^{-2}.

Consequently the approximation we are going to use is valid if

$$\sqrt{mE} \ll m \quad \text{and} \quad \mu \ll m . \tag{8}$$

The mesons we shall of course treat by relativistic methods, and we will more specifically look into the effects of retardation on the eigenvalue problem, as contrasted to the so-called static approximation.

Our assumption will allow us to simplify the equation (5) to a great extent, because the components of the spinor-field can now be separated into small and large ones. By neglecting the small components we introduce at most an error of the order of $(v/c)^2$. And as we use a scalar theory we need only consider one large component.

First of all $A(\vec{p}, p_0)$ is thus reduced from a 16-component spinor to a scalar function.

Secondly the operators in the numerator of S are reduced to scalars in the following manner:

$$i(\vec{\gamma}, \vec{p}) - \gamma_4 p_0 - m \rightarrow \frac{p^2}{2m} - p_0 - m$$

p_0 is the deviation from the mean value, $W/2$, of the energy of one nucleon, and is therefore of the order of its kinetic energy, so

$$| p_0 | \approx \frac{p^2}{2m}$$

Putting $m - W/2 = E/2$, and in view of (8)

$$S\left(\vec{p}, p_0 + \frac{W}{2}\right) \rightarrow \frac{-2m}{\vec{p}^2 + mE - p_0 W - i\eta_p}.$$

We introduce the symbols

$$\lambda \equiv \frac{g^2}{(2\pi)^3} \qquad \Lambda_p \equiv \frac{\vec{p}^2}{2m} + \frac{E}{2} - i\eta_p \quad *)$$
$$\omega_q^2 \equiv \vec{q}^2 + (\mu - i\varepsilon_q)^2. \tag{9}$$

Eq. (5), as a result of our assumptions (8), reduces to

$$A(\vec{p}, p_0) = -i\frac{\lambda}{2\pi}\frac{1}{\Lambda_p^2 - p_0^2}\int \frac{TA(\vec{p}+\vec{q}, p_0+q_0)}{\omega_q^2 - q_0^2} d\vec{q}\, dq_0 -$$
$$- \frac{\lambda^2}{(2\pi)^2}\frac{1}{\Lambda_p^2 - p_0^2}\int \frac{(3+2T)\, A(\vec{p}+\vec{q}+\vec{l}, p_0+q_0+l_0)}{(\omega_q^2 - q_0^2)(\omega_l^2 - l_0^2)(\Lambda_{p+q} - p_0 - q_0)(\Lambda_{p+l} + p_0 + l_0)} \times$$
$$\times d\vec{q}\, dq_0\, d\vec{l}\, dl_0. \tag{10}$$

We have to calculate from this integral equation the lowest eigenvalue of λ as a function of the parameters m, μ and E which have to satisfy (8). (We will not go into the much more arduous problem of finding the eigenfunction associated with λ.) We propose to do

*) Λ_p has the opposite sign as in the work of LÉVY[4]); it is positive for all $\vec{p}$.

this by a classical perturbation calculation, starting from a suitable trial-function for $A(\vec{p}, p_0)$. For this purpose we will take the static approximation. As the latter strictly agrees with classical, non relativistic meson-theory of nuclear interactions our choice has the additional advantage of showing clearly any departure from non-relativistic theory.

The Static Approximation.

This is obtained from (10) by dropping the λ^2-term altogether, and neglecting the q_0-dependence of the kernel in the first term. The physical meaning of this is well-known: we thus assume an instantaneous interaction between nucleons, replacing

$$\Delta_c(\vec{x}, t) \text{ by } \delta(t) \int_{-\infty}^{+\infty} \Delta_c(\vec{x}, t)\, dt .$$

This leads to

$$A_0(\vec{p}, p_0) = -i \frac{\lambda_0}{2\pi} \frac{1}{\Delta_p{}^2 - p_0{}^2} \int \frac{T A_0(\vec{p}+\vec{q}, p_0+q_0)}{\omega_q{}^2} d\vec{q}\, dq_0 \qquad (11)$$

The p_0-dependence is obvious, and (11) yields

$$A_0(\vec{p}, p_0) = a(\vec{p}) \cdot b(\vec{p}, p_0) \qquad (12)$$

with

$$b = \frac{2\Delta_p}{\Delta_p{}^2 - p_0{}^2} \qquad (13)$$

and $a(\vec{p})$ satisfying

$$a(\vec{p}) = \frac{\lambda_0}{2\Delta_p} \int \frac{T a(\vec{p}+\vec{q})}{\omega_q{}^2} d\vec{q} . \qquad (14)$$

The latter equation we write concisely as

$$a(\vec{p}) = \lambda_0 K_0\, a(\vec{p}) \qquad (15)$$

denoting by K_0 the operation

$$\frac{1}{2\Delta_p} \int dq \frac{T}{\omega_q{}^2} \qquad (16)$$

It is known that (14) is nothing but the momentum space representation of the Schrödinger equation for the relative coördinates with Yukawa potential:

$$\left(-\frac{1}{m} \Delta - 2\pi^2 \lambda_0 \frac{e^{-\mu|\vec{x}|}}{|\vec{x}|} T + E \right) \psi(\vec{x}) = 0 \qquad (17)$$

where

$$\psi(\vec{x}) = (2\pi)^{-3/2} \int e^{i(\vec{p}, \vec{x})} a(\vec{p})\, d\vec{p} .$$

The Matrixelement for the Eigenvalue.

In order to get a better view of the method we give Eq. (10) the concise form

$$A = (\lambda K + \lambda^2 L)\, A \tag{18}$$

where K und L stand for the linear integral operators producing the first and second terms on the r.h.s. of Eq. (10). Suppose we have an eigenvalue problem, given by an equation

$$A = \Omega(\lambda)\, A$$

derived from an operator Ω containing a parameter λ, and by certain boundary conditions.

If we know an eigenfunction A we get immediately the corresponding eigenvalue λ by constructing the expectation value

$$\langle A^* | \Omega(\lambda) | A \rangle = \langle A^* | A \rangle \tag{19}$$

which gives us implicitly λ as a function of the parameters of the problem.

Suppose we do not know A exactly, but only an approximation $(A + \varepsilon)$. If we use $(A + \varepsilon)$ instead of A in calculating the matrixelement of $(\lambda K + \lambda^2 L)$ we shall get an approximated value for λ, say $\lambda + \eta$:

$$\langle (A+\varepsilon)^* |\, (\lambda+\eta)\, K + (\lambda+\eta)^2 L \,|\, (A+\varepsilon) \rangle = \langle A^* + \varepsilon^* |\, A + \varepsilon \rangle .$$

Making use of (18), we see that the only term linear in η and not containing ε

$$\langle A^* |\, \eta K + 2\lambda\eta L \,|\, A \rangle$$

must be equal to an expression at least linear in ε so that in general η will be of the order of ε; but if ε is nearly orthogonal on A, η will be almost of the order of ε^2. A first choice for $A + \varepsilon$ might be the static approximation A_0. We will see later on that this choice leads to errors of the order v/c, so that we want to take a better trialfunction. Due to the special character of our problem which permits to solve the simplified equation

$$A = \lambda K A \tag{20}$$

this may easily be obtained.

Solution of the Ladder Approximation by an Iteration Method.

Let, in an eigenvalue problem, the equation be

$$K\psi = k\psi \tag{21}$$

where K is an operator having a complete orthogonal set of eigenfunctions ψ_i with corresponding eigenvalues k_i.

Let there exist furthermore a greatest eigenvalue k_0

$$|k_0| > |k_j| \qquad (j \neq 0) \tag{22}$$

Then ψ_0 may be calculated to any degree of accuracy.

We suppose the ψ_i normalised to unity, and the k_i numbered in order of decreasing absolute value.

We choose a function ψ with the single restriction that it can be expanded in a series of ψ_i, in the sense that if

$$\psi = \sum_i c_i \psi_i \tag{23'}$$

then

$$\int |\psi|^2 d\tau = \sum_i |c_i|^2 = \text{finite number} \tag{23''}$$

holds.

On this function ψ we operate n times with the operator $(1/k_0)K$; a function which we will call $\psi^{(n)}$ results

$$\psi^{(n)} = \frac{1}{k_0^n} K^n \psi$$

or, in view of (23')

$$\psi^{(n)} = \sum_{i=0} \left(\frac{k_i}{k_0}\right)^n c_i \psi_i .$$

This means that

$$\begin{aligned} \int |\psi^{(n)} - c_0 \psi_0|^2 d\tau &= \sum_{i=1} \left|\frac{k_i}{k_0}\right|^{2n} |c_i|^2 \\ &\leqq \left|\frac{k_1}{k_0}\right|^{2n} \sum_{i=1} |c_i|^2 \\ &\leqq \left|\frac{k_1}{k_0}\right|^{2n} \left(\int |\psi|^2 d\tau - \int |c_0 \psi_0|^2 \, d\tau\right). \end{aligned} \tag{24}$$

Which proves that the sequence of $\psi^{(n)}$ converges towards $c_0 \psi_0$, in the sense of (23).

Now, Eq. (20) is of the type (21), with $k = 1/\lambda$. It may be taken, in analogy to (15), to describe bound states. Keeping the binding-energy E fixed, the eigenvalue $k_i = 1/\lambda_i$ belongs to a system with $i + 1$ stationary states, the highest of which has the prescribed value E of the binding-energy.

But we are interested only in the eigenfunction corresponding to $k_0 = 1/\lambda_0$, describing a system with only one level.

It is obvious that λ_0 must be smaller than all other eigenvalues, as it is a measure for the strength of the nuclear forces. For the problem in hand, a solution of the simplified equation (15) or (17) gives (for $E = 0$)[8])

$$\frac{\lambda_0}{\lambda_1} = \frac{1{,}68}{6{,}45} = \frac{1}{3{,}84}.$$

For the eigenfunction this leads to errors of at most 1/3,8 and 1/15 in first and second approximation, respectively.

But, as a matter of fact, $(K_0 - K)\,A$ is in our case orthogonal with respect to K_0, at least up to terms of order $(v/c)^2$, so that the errors on expectation values calculated with the help of the first approximation will be more like 1/15. This fact may be seen from the explicit forms of K_0 and K [cf. Eq. (29)].

The first approximation

$$A_1 = \lambda K A_0 \tag{25}$$

will thus already be quite satisfactory.

The Formula for the corrected Eigenvalue.

Substituting A_1 for A in (18) we obtain

$$\lambda K A_0 = \lambda^2 K^2 A_0 + \lambda^3 L K A_0 \tag{26}$$

As the p_0-dependence of A_0 is known we may as well integrate both sides of Eq. (26) over p_0, and keep only an integral equation in $\vec{p}$:

$$\overline{K}\,a = \lambda\,\overline{K^2}\,a + \lambda^2\,\overline{LK}\,a \tag{27}$$

where the dash over the operators indicates the p_0-integration, e.g.

$$\overline{K} \equiv \int K\,b(\vec{p}, p_0)\,dp_0 .$$

In order to compare this with the result, λ_0, of the static approximation we will write in K the static interaction explicitly

$$K \equiv K_0 + K_1$$

where K_0 [cf. (16)] is independent from p_0.

Multiplying (27) on the left with $a^*(\vec{p}) = a(\vec{p})$ and integrating over $\vec{p}$ leads then to the following equation between matrixelements.

$$\langle K_0 \rangle + \langle \bar{K}_1 \rangle = \lambda \langle K_0^2 + 2 K_0 \bar{K}_1 + \bar{K}_1^2 \rangle + \lambda^2 \langle \bar{L} K_0 + \overline{L K_1} \rangle$$

where e.g.

$$\langle \bar{K}_1 \rangle \equiv \int a(\vec{p}) \bar{K}_1 a(\vec{p}) \, d\vec{p} \, .$$

We shall now have to make some assumptions on the relative magnitude of the various terms. We do this by a perturbation method procedure, putting

$$\lambda = \lambda_0 + \lambda_1 \, ^*)$$

and assuming

$$\lambda_1 \ll \lambda_0 \qquad \langle \bar{K}_1 \rangle \ll \langle K_0 \rangle \qquad \langle \bar{L} \rangle \ll \langle K_0 \rangle \, .$$

The justification of this will become apparent when the explicit form of the K and L will be given.

If we drop all terms of third or higher order in the small quantities we get

$$\begin{aligned} \langle K_0 \rangle + \langle \bar{K}_1 \rangle = {} & \lambda_0 \langle K_0^2 \rangle + \lambda_1 \langle K_0^2 \rangle + 2 \lambda_0 \langle K_0 \bar{K}_1 \rangle + \\ & + 2 \lambda_1 \langle K_0 \bar{K}_1 \rangle + \lambda_0 \langle \bar{K}_1^2 \rangle + \\ & + \lambda_0^2 \langle \bar{L} K_0 \rangle + 2 \lambda_0 \lambda_1 \langle \bar{L} K_0 \rangle + \lambda_0^2 \langle \overline{L K_1} \rangle \, . \end{aligned}$$

This may be simplified by noticing that, for any K or L

$$\lambda_0 \langle K K_0 \rangle = \langle K \rangle$$

because of (15). Hence:

$$\frac{\lambda_1}{\lambda_0} = - \frac{\langle \bar{K}_1 \rangle + \lambda_0 [\langle \bar{K}_1^2 \rangle + \langle \bar{L} \rangle]}{\langle K_0 \rangle + 2 \langle \bar{K}_1 \rangle} - \frac{2 \lambda_1 \langle \bar{L} \rangle + \lambda_0^2 \langle \overline{L K_1} \rangle}{\langle K_0 \rangle + 2 \langle \bar{K}_1 \rangle} \, . \qquad (28)$$

Explicit Calculation of the various Expectation Values.

From (16):

$$K_0 a = \frac{1}{2 \Lambda_p} \int \frac{T a(\vec{p} + \vec{q})}{\omega_q^2} \, dq \, .$$

From Eq. (10), (18), and (16):

$$K_1 a b = - i \, \frac{1}{2 \pi} \, \frac{1}{\Lambda_p^2 - p_0^2} \left\{ \int \frac{T a(\vec{p} + \vec{q}) \, b(\vec{p} + \vec{q}, \, p_0 + q_0)}{\omega_q^2 - q_0^2} \, d\vec{q} \, dq_0 - \right.$$

$$\left. - \int \frac{T a(\vec{p} + \vec{q}) \, b(\vec{p} + \vec{q}, \, p_0 + q_0)}{\omega_q^2} \, d\vec{q} \, dq_0 \cdot \right\}$$

*) This λ_1 has of course no connection with the one on p. 77.

Hence

$$\overline{K}_1 a = -\frac{1}{2\Lambda_p}\int \frac{T[\Lambda_p+\Lambda_{p+q}]\,a(\vec{p}+\vec{q})}{\omega_q{}^2[\Lambda_p+\Lambda_{p+q}+\omega_q]}\,d\vec{q}\,. \tag{29}$$

We calculate $\overline{K_1^2}$ by using the following identity:

$$\overline{K_1^2} = \overline{K^2} - \overline{K}^2 + \overline{K}_1^2.$$

Iterating K once gives

$$K^2 ab = -\frac{1}{4\pi^2}\frac{1}{\Lambda_p{}^2 - p_0{}^2}\times$$

$$\times\int \frac{2\,\Lambda_{p+q+l}\,T^2 a(\vec{p}+\vec{q}+\vec{l})\,d\vec{q}\,d\vec{l}\,dq_0\,dl_0}{(\omega_q{}^2-q_0{}^2)\,[\Lambda^2{}_{p+q}-(p_0+q_0)^2]\,[\omega_l{}^2-l_0{}^2]\,[\Lambda^2{}_{p+q+l}-(p_0+q_0+l_0)^2]}\,.$$

Hence

$$\overline{K^2}\,a = \frac{1}{4\Lambda_p}\times$$

$$\times\int \frac{(\Lambda_p+\Lambda_{p+q}+\Lambda_{p+q+l}+\omega_q+\omega_l)\,T^2 a(\vec{p}+\vec{q}+\vec{l})\,d\vec{q}\,d\vec{l}}{\omega_q\,\omega_l\,\Lambda_{p+q}(\Lambda_p+\Lambda_{p+q}+\omega_q)\,(\Lambda_p+\Lambda_{p+q+l}+\omega_q+\omega_l)\,(\Lambda_{p+q}+\Lambda_{p+q+l}+\omega_l)}\,.$$

And further iterating $\overline{K}$:

$$\overline{K}^2 a = \frac{1}{4\Lambda_p}\int \frac{T^2 a(\vec{p}+\vec{q}+\vec{l})\,d\vec{q}\,d\vec{l}}{\omega_q(\Lambda_{p+l}+\Lambda_{p+q+l}+\omega_q)\,\Lambda_{p+l}\,\omega_l(\Lambda_p+\Lambda_{p+l}+\omega_l)}\,.$$

By iterating (29) once we get:

$$\overline{K}_1^2 a = \frac{1}{4\Lambda_p}\int \frac{(\Lambda_{p+l}+\Lambda_{p+q+l})\,(\Lambda_p+\Lambda_{p+l})\,T^2 a(\vec{p}+\vec{q}+\vec{l})\,d\vec{q}\,d\vec{l}}{\omega_q{}^2(\Lambda_{p+l}+\Lambda_{p+q+l}+\omega_q)\,\Lambda_{p+l}\,\omega_l{}^2(\Lambda_p+\Lambda_{p+l}+\omega_l)}\,.$$

Before going further we shall examine these results, making use of our assumptions (p. 72) as to the momentum distributions; as we have hitherto dropped systematically all terms in $(v/c)^2 \approx (\mu/m)^2$ there is no need to retain such terms here. We assume—which will prove to be self-consistent for small μ/m—that

$$\left|\frac{\lambda_1}{\lambda_0}\right| = \text{const.} \times \frac{\mu}{m} \qquad \text{and} \quad \left|\frac{\lambda_1}{\lambda_0}\right|^2 \ll \left|\frac{\lambda_1}{\lambda_0}\right|.$$

From (14) we may infer that $\lambda_0 \approx \mu/m$ [cf. [4]), p. 80].

It follows that we should discard all terms of order v/c (or μ/m) in $\langle\overline{K_1^2}\rangle$ and $\langle\overline{L}\rangle$, and drop the $\langle\overline{K_1}\rangle$ in the denominator.

In particular, if all factors of the form $\Lambda + \omega$ are replaced by ω, in any term of (28), the ensuing error is of the order of μ/m times this term. In the limit $\Lambda + \omega \to \omega$

$$(\overline{K^2} - \overline{K}^2)\,a = \frac{1}{4\Lambda_p}\int \frac{T^2 a(\vec{p}+\vec{q}+\vec{l})}{\omega_q{}^2\,\omega_l{}^2(\omega_q+\omega_l)}\,d\vec{q}\,d\vec{l}\,.$$

Taking for $\omega_q + \omega_l$ a mean value $2\sqrt{2}\,\mu$, this is

$$\frac{\langle \Lambda_{p+q}\rangle_{\text{Av.}}}{2\sqrt{2}\,\mu} K_0\, a$$

and contributes a term of order $1/(2\sqrt{2}) \cdot \mu/m$ to the ratio λ_1/λ_0. The approximation $\Lambda + \omega \approx \omega$ is thus justified. The term

$$\langle \overline{K}_1 \rangle = -\int \frac{a(\vec{p})\, T\, a(\vec{p}+\vec{q})}{2\,\omega_q^{\,3}}\, d\vec{q}\, d\vec{p} - \int \frac{a(\vec{p})\, \Lambda_{p+q}\, T\, a(\vec{p}+\vec{q})}{2\,\Lambda_p\, \omega_q^{\,3}}\, d\vec{p}\, d\vec{q} \qquad (30)$$

will be of the order

$$-\left\langle \frac{\Lambda_p}{\omega_q} \right\rangle_{\text{Av.}} \langle K_0 \rangle - \left\langle \frac{\Lambda_{p+q}}{\omega_q} \right\rangle_{\text{Av.}} \langle K_0 \rangle \approx -\frac{3}{2}\frac{\mu}{m} \langle K_0 \rangle$$

so that this procedure is justified here too. And by a similar argument the term $\overline{K}_1^2$ may be neglected altogether.

The matrixelement $\langle \overline{L} \rangle$ may be computed from the second term on the r.h.s. of (10) and yields a complicated expression. If we neglect systematically terms like Λ/ω against unity this is reduced to

$$\overline{L}\, a = \frac{1}{4\,\Lambda_p} \int \frac{1}{\omega_q^{\,2} - \omega_l^{\,2}} \left(\frac{1}{\omega_l^{\,3}} - \frac{1}{\omega_q^{\,3}} \right) (3 + 2\,T)\, a(\vec{p}+\vec{q}+\vec{l})\, d\vec{q}\, d\vec{l} =$$

$$= -\frac{1}{4\,\Lambda_p} \int \frac{(3+2\,T)\, a(\vec{p}+\vec{q}+\vec{l})}{\omega_q^{\,2}\, \omega_l^{\,2}(\omega_q + \omega_l)}\, d\vec{q}\, d\vec{l} + \frac{1}{2\Lambda_p} \int \frac{(3+2\,T)\, a(\vec{p}+\vec{q}+\vec{l})}{\omega_q^{\,3}\, \omega_l^{\,2}}\, d\vec{q}\, d\vec{l}.$$

As $T^2 = 3 - 2\,T$, $\overline{K}_1^2$ and $\overline{L}$ partially cancel each other*):

$$(\overline{K}_1^2 + \overline{L})\, a = -\frac{2}{2\,\Lambda_p} \int \frac{T\, a(\vec{p}+\vec{q}+\vec{l})}{\omega_q^{\,2}\, \omega_l^{\,2}(\omega_q + \omega_l)}\, d\vec{q}\, d\vec{l} +$$

$$+ \frac{1}{2\,\Lambda_p} \int \frac{(3+2\,T)\, a(\vec{p}+\vec{q}+\vec{l})}{\omega_q^{\,3}\, \omega_l^{\,2}}\, d\vec{q}\, d\vec{l}. \qquad (31)$$

Our model, with scalar interaction, makes sense only for charge-symmetric states, as only these give an attractive potential in the static approximation. We put therefore $T = 1$.

The second term in (31) is then reduced to

$$\frac{1}{\lambda_0} \frac{5}{2\,\Lambda_p} \int \frac{2\,\Lambda_{p+q}\, a(\vec{p}+\vec{q})}{\omega_q^{\,3}}\, d\vec{q}$$

which combines with $\overline{K}_1 a$ from (30) to form two terms

$$\frac{9}{2\,\Lambda_p} \int \frac{\Lambda_{p+q}\, a(\vec{p}+\vec{q})}{\omega_q^{\,3}}\, d\vec{q} - \frac{1}{2} \int \frac{a(\vec{p}+\vec{q})}{\omega_q^{\,3}}\, d\vec{q}\,.$$

*) In a neutral theory they would cancel exactly. That this is not the case in a symmetric theory, as LÉVY assumed, was pointed out without proof by WENTZEL, loc. cit.

From our order of magnitude estimates it is obvious that the second term in the r.h.s. of (28) is of order $(\mu/m)^2$. We have, collecting terms,

$$\frac{\lambda_1}{\lambda_0} = -\frac{\frac{9}{2}\int a(\vec{p})\,\frac{\Lambda_{p+q}}{\Lambda_p\,\omega_q{}^3}\,a(\vec{p}+\vec{q})\,d\vec{p}\,d\vec{q} - \int a(\vec{p})\,\frac{1}{2\,\omega_q{}^3}\,a(\vec{p}+\vec{q})\,d\vec{p}\,d\vec{q}}{\frac{1}{2}\int a(\vec{p})\,\frac{1}{\Lambda_p\,\omega_q{}^2}\,a(\vec{p}+\vec{q})\,d\vec{p}\,d\vec{q}} +$$

$$+\frac{\lambda_0\int a(\vec{p})\,\frac{1}{\Lambda_p\,\omega_q{}^2\,\omega_l{}^2(\omega_q+\omega_l)}\,a(\vec{p}+\vec{q}+\vec{l})\,d\vec{p}\,d\vec{q}\,d\vec{l}}{\frac{1}{2}\int a(\vec{p})\,\frac{1}{\Lambda_p\,\omega_q{}^2}\,a(\vec{p}+\vec{q})\,d\vec{p}\,d\vec{q}}\,. \qquad (32)$$

Numerical Evaluation of λ_1/λ_0.

From the approximate solution of the Schrödinger equation (17)[8]) we derive an approximate expression for $a(\vec{p})$

$$a(\vec{p}) = \sum_{i=1}^{2} \varepsilon_i \frac{1}{\alpha_i{}^2+p^2}$$

with

$$\varepsilon_1 = 1 \qquad \alpha_1 = \sqrt{m\,E}$$
$$\varepsilon_2 = -1 \qquad \alpha_2 = \sqrt{m\,E} + \mu\,.$$

Then

$$\Lambda_{p+q}\,a(\vec{p}+\vec{q}) = \frac{1}{2\,m}\left[1 - \frac{(\vec{q}+\vec{p})^2+\alpha_1{}^2}{(\vec{q}+\vec{p})^2+\alpha_2{}^2}\right] = \frac{\mu^2+2\,\mu\,\alpha_1}{2\,m}\,\frac{1}{\alpha_2{}^2+(\vec{p}+\vec{q})^2}\,.$$

The first term in the numerator of (32), we may call it A, is therefore

$$A = \frac{9}{2}\,(\mu^2+2\,\mu\,\alpha_1)\int\left(\frac{1}{\alpha_1{}^2+p^2} - \frac{1}{\alpha_2{}^2+p^2}\right)\frac{d\vec{p}\,d\vec{q}}{(\alpha_1{}^2+p^2)\,(\mu^2+q^2)^{3/2}\,[\alpha_2{}^2+(\vec{p}+\vec{q})^2]}\,.$$

The integral

$$I_i = \int[(\alpha_i^2+p^2)\,(\alpha_1{}^2+p^2)\,[\alpha_2{}^2+(\vec{p}+\vec{q})^2]\,(\mu^2+q^2)^{3/2}]^{-1}\,d\vec{p}\,d\vec{q}$$

can be calculated rigorously. The angular variables occur only in the factor

$$2\,\pi\int_0^{\pi}[\alpha_2^2+(\vec{p}+\vec{q})^2]^{-1}\sin\vartheta\,d\vartheta = \frac{\pi}{p\,q}\log\frac{\alpha_2{}^2+(p+q)^2}{\alpha_2{}^2+(p-q)^2}$$

and the logarithm may be written as an integral

$$= \frac{2\,\pi}{p\,q}\int_{|p-q|}^{p+q}\frac{l}{\alpha_2{}^2+l^2}\,dl\,.$$

*

Hence

$$I_i = 8\pi^2 \int_0^\infty dp \int_0^\infty dq \int_{|p-q|}^{p+q} dl \, \frac{p\,q\,l}{(\alpha_i^2+p^2)\,(\alpha_2^2+l^2)\,(\mu^2+q^2)\,(\alpha_1^2+p^2)\sqrt{\mu^2+q^2}} .$$

This triple integral may be calculated by writing the single factors in the integrand as Laplace transformations. This leads to

$$I_i = \frac{16\pi^2}{\alpha_1} \int_0^\infty \cdots \int_0^\infty \frac{1}{x_2} \exp\left[-(\alpha_i x_1 + \alpha_2 x_2 + \mu x_3 + \alpha_1 x_4 + q x_5\right] \times$$
$$\times \sin(p x_1) \sin(p x_2) \cos(p x_4) \sin(q x_2) \sin(q x_3) \times$$
$$\times J_0(\mu x_5)\, d(p\,q\,x_1\,x_2\,x_3\,x_4\,x_5) .$$

Using the relation

$$\int_0^\infty J_0(\mu x) \frac{x}{x^2+c^2}\, dx = K_0(\mu c) . \quad *)$$

We may write

$$I_i = \frac{2\pi^3}{\alpha_i} \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x_2} e^{-\alpha_2 x_2 - \mu x_3 - \alpha_1 x_1} \left[e^{-\alpha_i |x_2 - x_4|} \operatorname{sgn}(x_2 - x_4) + e^{-\alpha_i (x_2 + x_4)}\right] \times$$
$$\times \left[K_0(\mu |x_2 - x_3|) - K_0[\mu (x_2 + x_3)]\right] d x_2\, d x_3\, d x_4 .$$

The x_4-integration gives a different result for $i = 1$ and $i = 2$:

$$I_1 = \frac{2\pi^3}{\alpha_1} \int_0^\infty \int_0^\infty e^{-\alpha_2 x_2 - \mu x_3 - \alpha_1 x_2} \left\{K_0(\mu |x_2 - x_3|) - K_0[\mu (x_2 + x_3)]\right\} dx_2\, dx_3 .$$

This is a two-dimensional Laplace transformation, which is easily expressed as function of the one-dimensional transformation of K_0, say $\mathfrak{L} K_0$

$$\mathfrak{L} K_0 = \frac{1}{\sqrt{u^2-1}} \log(u + \sqrt{u^2-1})$$

$$I_1 = \frac{4\pi^3}{\mu^3} \frac{\mu}{\alpha_1} \left(\frac{\frac{\alpha_1+\alpha_2}{\mu} \log\left(\frac{\alpha_1+\alpha_2}{\mu} + \sqrt{\left(\frac{\alpha_1+\alpha_2}{\mu}\right)^2 - 1}\right)}{\left[\left(\frac{\alpha_1+\alpha_2}{\mu}\right)^2 - 1\right]^{3/2}} - \frac{1}{\left(\frac{\alpha_1+\alpha_2}{\mu}\right)^2 - 1} \right) .$$

For $i = 2$ we have

$$I_2 = \frac{4\pi^3}{\alpha_2^2 - \alpha_1^2} \int_0^\infty \int_0^\infty \frac{1}{x_2} e^{-\alpha_2 x_2} \left(e^{-\alpha_1 x_2} - e^{-\alpha_2 x_2}\right) e^{-\mu x_3} \times$$
$$\times \left\{K_0(\mu |x_2 - x_3|) - K_0[\mu (x_2 + x_3)]\right\} dx_2\, dx_3 .$$

*) Watson, Theory of Bessel Functions, Cambridge 1948, p. 425.

This integral, because of the $1/x_2$ in the integrand may be calculated from I_1 by an additional integration.

$$I_2 = \frac{8\pi^3}{\mu^3}\frac{\mu^2}{\alpha_2^2-\alpha_1^2}\left\{\left[\frac{(\alpha_1+\alpha_2)^2}{\mu^2}-1\right]^{-1/2}\log\left(\frac{\alpha_1+\alpha_2}{\mu}+\sqrt{\frac{(\alpha_1+\alpha_2)^2}{\mu^2}-1}\right)-\right.$$
$$\left.-\left[\frac{4\alpha_2^2}{\mu^2}-1\right]^{-1/2}\log\left(\frac{2\alpha_2}{\mu}+\sqrt{\frac{4\alpha_2^2}{\mu^2}-1}\right)\right\}.$$

The term we set out to calculate was

$$A = \frac{9}{2}\mu^2\left(1+2\frac{\alpha_1}{\mu}\right)(I_1-I_2).$$

The second term, B say, in (32) gives, with the same methods

$$B = \frac{4\pi^3}{\mu}\sum_{i,j}\varepsilon_i\varepsilon_j\frac{1}{\left|\left(\frac{\alpha_i+\alpha_j}{\mu}\right)^2-1\right|^{1/2}}F\left(\frac{\alpha_i+\alpha_j}{\mu}\right)$$

$$\text{with } F(u) = \begin{cases}\arccos u & u<1\\ \log(u+\sqrt{u^2-1}) & u>1.\end{cases}$$

The third term is more complicated because of the factor $(\omega_q+\omega_l)^{-1}$ in the integrand. We shall obtain a sufficiently accurate value if we set

$$(\omega_q+\omega_l)^{-1} \approx \frac{1}{2\omega_q}.$$

This gives

$$C = \lambda_0\int a(\vec{p})\frac{1}{\Lambda_p\,\omega_q^2\,\omega_l^2(\omega_q+\omega_l)}a(\vec{p}+\vec{q}+\vec{l})\,d\vec{p}\,d\vec{q}\,d\vec{l} \approx \frac{2}{9}A.$$

The denominator of (32), say D, takes the form

$$D \equiv \langle K_0\rangle = \frac{2\pi^4}{\mu}\frac{m}{\alpha_1}\left[\frac{1}{\frac{2\alpha_1}{\mu}+1}-\frac{1}{\frac{\alpha_1+\alpha_2}{\mu}+1}\right]-$$
$$-\frac{4\pi^4}{\mu}\frac{m}{\mu}\frac{\mu^2}{\alpha_2^2-\alpha_1^2}\log\frac{(\alpha_1+\alpha_2+\mu)^2}{(2\alpha_1+\mu)(2\alpha_2+\mu)}.$$

If we take as values of the parameters

$$m = 940 \text{ Mev} \qquad \mu = 140 \text{ Mev} \qquad E = 2{,}18 \text{ Mev}$$

we get

$$I_1 = \frac{\pi^3}{\mu^3}2{,}63 \qquad I_2 = \frac{\pi^3}{\mu^3}0{,}788$$

$$A = \frac{\pi^3}{\mu}13{,}63 \qquad B = \frac{\pi^3}{\mu}0{,}592 \qquad D = \frac{\pi^4}{\mu}6{,}96.$$

Hence

$$\frac{\lambda_1}{\lambda_0} = \frac{-A+B+2/9\,A}{D} = -0{,}46. \tag{35}$$

The corrected eigenvalue is therefore

$$\lambda = 0{,}54\ \lambda_0 .$$

From[4]) or[8]) we get $\lambda_0 = 0{,}0184$ and consequently

$$\lambda = \frac{g^2}{(2\pi)^3} = 0{,}0099 .$$

Conclusions.

The value of $\lambda = \lambda_0 + \lambda_1$ which results is so small that–at least in the static approximation–no bound state could exist.

As we assumed in the course of the calculations that $\lambda_1 \ll \lambda_0$, it might be objected that the result (35) cannot be very accurate. However, it is beyond doubt that the static approximation is rather a poor one and that conclusions drawn from it can only claim a qualitative value. It gives in particular a value for the (positively counted) binding energy that is substantially too low. It is further clear–as was pointed out by LÉVY for the neutral case–that one is not justified to treat the ladder approximation in an exact way and to neglect the next higher graph. The contributions from both these graphs combine to give forces that are comparable to the static force.

I am most grateful to prof. M. FIERZ who suggested the subject and actively supervised the elaboration of this paper. His extensive knowledge proved invaluable.

I also owe thanks to prof. P. HUBER, director of the "Physikalisches Institut der Universität" in Basle, for his kind hospitality at this institute.

References.

[1]) H. A. BETHE and E. SALPETER, Phys. Rev. **84**, 1282 (1951).
[2]) M. GELL-MAN and F. LOW, Phys. Rev. **84**, 350 (1951).
[3]) W. PAULI, Meson Theory of Nuclear Forces, New York 1946.
[4]) M. M. LÉVY, Phys. Rev. **88**, 72 (1952).
[5]) G. WENTZEL, Phys. Rev. **89**, 684 (1953).
[6]) W. PAULI, Ausgewählte Kapitel aus der Feldquantisierung, Zürich 1951.
[7]) F. J. DYSON, Cornell Lectures (unpublished).
[8]) L. HULTHÉN, Arkiv Mat. Astron. Fysik, **28** A, n° 5 (1941).