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Covariant Hyperquantization*)

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(29. III. 56.)

Abstract. The differential equations for the vacuum expectation values of chronological products of field operators are transcribed into an operator formalism which is covariant under the Lorentz group. This is accomplished by introducing a new set of field operators with the same transformation properties as the ordinary fields to which they correspond. They operate in a certain linear vectorspace Ω and they satisfy very simple commutation rules. The vectors in Ω have no relations to the statevectors of the system. They represent instead the solutions of the Heisenberg field equations provided they are suitably restricted by certain subsidiary conditions, subsequently called the *field-conditions*. The vectors which satisfy these conditions are constructed in closed form.

It is shown that the scalar product in Ω cannot be the positive definite product characteristic for the Hilbert space. In fact the scalar product in Ω must be symmetrical in the two factors for a covariant formalism.

Introduction.

The present paper has a two-fold purpose. On the one hand we shall base the formalism of hyperquantization [1][2][3][4] on the well-known mathematical notions of multilinear algebra. On the other hand we develop this theory in a relativistically covariant form.

The two points are not unrelated. Indeed as we shall see the emphasis of the purely algebraic aspects of the formal manipulations involved here show clearly that the scalar products of the hyperquantization space subsequently called Ω play an entirely different role from the scalar products of the state vectors in Hilbert space. Whereas the latter are directly related to observable quantities (viz., expectation values and matrix elements), the former are not. In fact it is possible to develop the formalism without specifying the scalar product. The choice of the scalar product is essentially determined by the requirement of the relativistic covariance of the formalism. The decisive point is the transformation properties of

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a spinor field. Because the spinor components transform with the complex coefficients of a non-unitary matrix, the only covariant scalar product of the hyperquantization space Ω is a symmetrical scalar product. This space is therefore not a Hilbert space in contrast to the space of the state vectors.

In the previous work [1][2][3][4] the assumption has always been made that the scalar product in Ω is the unitary product characteristic of a Hilbert space. The formalism which arises from this assumption has very awkward transformation properties under the Lorentz group. For instance the scalar product (Equation (26)) of reference[4] is not invariant under Lorentz transformations and the vectors introduced by Coester (reference[1]) are sums of vectors, each of which satisfies a different transformation law. Since all these theories are merely formal transcriptions of manifestly covariant field theories it should be possible to develop hyperquantization with covariant equations. This is accomplished in the present paper.

The theory is closely related to the method of generating functionals introduced by Schwinger[5] and subsequently used by many authors[6][7][8][9][10]. In fact there is a one-to-one correspondence between such functionals and the vectors in Ω. We prefer the formalism of hyperquantization primarily because it is possible to avoid the use of classical external currents which are assumed as anticommuting c-numbers. In the hyperquantization theory it is not necessary to introduce such questionable mathematical objects. It is therefore preferable to the method of functionals.

In the present paper we shall develop the theory for a scalar and a spinor field with self-interactions, these are meant to serve as examples. In a subsequent paper we shall extend it to quantum electrodynamics. The paper is divided into three parts. In part I we give a brief review of some of the basic mathematical tools involved. In part II we treat the example of the scalar field and in part III we discuss the spinor field.

It must be stressed that nothing that is presented in this paper contains any new physical ideas. This is equally true for all the other works quoted above. In spite of this we believe that such reformulations of existing theories can be quite useful. Such a formulation may serve as a framework for new theories. Thus for instance we may take the point of view that the equations for the vectors in the space Ω are the basic equations of the theory and forget the origin from which they arise. This means that we have replaced the field operators of the ordinary theory which satisfy the standard commutation rules and a set of field equations by new field operators which satisfy much simpler commutation rules and no field equations. The physical content of the theory is then entirely derivable from the vectors in the space Ω which are subjected to certain subsidiary conditions. (See Equations (66), (112) and (113) below.) We shall call these the *field conditions*. These vectors in Ω are then the new mathematical objects which replace the solutions of the Heisenberg equations. It will become apparent that to each solution of these equations belongs a certain vector in Ω.

One of the most important advantages of the hyperquantization formalism is the fact that with it we are able to derive general relations independent of the perturbation theory. In fact the existence of explicit solutions in closed form, although only of a formal character, enables us to read off such general relations with great ease. There are many such relations known today; the symmetry properties of the S-matrix, Ward's identity, and the low-energy limits of S-matrix elements are examples. The possibility of extending these results is the main interest of this approach.

Part I-Multilinear Products of Vectorspaces[11]).

(1) *Linear vectorspaces and their Kronecker products.*

We shall operate in an n-dimensional linear vectorspace $R = R_n$ over the field of complex numbers. A linearly independent set of n vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ forms a base in R. Any other vector $\boldsymbol{a} \in R$ may be represented as a linear combination

$$\boldsymbol{a} = a^1 \boldsymbol{e}_1 + a^2 \boldsymbol{e}_2 + \cdots + a^n \boldsymbol{e}_n .$$

The uniquely defined set of n complex numbers $\{a^1, a^2, \ldots, a^n\}$ are the contravariant components of the vector $\boldsymbol{a}$ in the base $\boldsymbol{e}_1, \ldots \boldsymbol{e}_n$.

The Kronecker product [denoted by $\boldsymbol{a} \times \boldsymbol{b}$] is a mapping of ordered pairs of vectors $\boldsymbol{a} \in R$ and $\boldsymbol{b} \in R$ into a linear vectorspace $R \times R$ of dimension n^2, which satisfies the following conditions:

$$\left.\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} \\ (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} \\ (\lambda\, \boldsymbol{a}) \times (\boldsymbol{b}) &= \lambda(\boldsymbol{a} \times \boldsymbol{b}) \\ (\boldsymbol{a}) \times (\mu\, \boldsymbol{b}) &= \mu(\boldsymbol{a} \times \boldsymbol{b}) \end{aligned}\right\} \qquad (2)$$

(λ and μ any complex numbers).

The null-vector in $R \times R$ is represented by 0×0. The Kronecker products $\boldsymbol{e}_i \times \boldsymbol{e}_k$ $(i, k = 1, 2, \ldots n)$ of a base $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots \boldsymbol{e}_n$ in R

define a base in $R \times R$. Referred to this base the Kronecker product $\boldsymbol{a} \times \boldsymbol{b}$ of two vectors

$$\boldsymbol{a} = \{a^1, a^2, \ldots, a^n\}$$

$$\text{and} \quad \boldsymbol{b} = \{b^1, b^2, \ldots, b^n\}$$

is $\boldsymbol{a} \times \boldsymbol{b} = \{a^1 b^1, a^1 b^2, \ldots, a^1 b^n, a^2 b^1, \ldots a^2 b^n, \ldots, a^n b^1, \ldots a^n b^n\}$. (3)

The Kronecker product can be generalized to an ordered set of f vector $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \boldsymbol{a}_f$ in R. The vectors $\boldsymbol{a}_1 \times \boldsymbol{a}_2 \times \cdots \times \boldsymbol{a}_f$ form a linear vectorspace of dimension n^f which we shall denote by $R \times R \times \cdots \times R$ or $\underset{f}{\times} R$.

(2) *The symmetrical product.*

The symmetrical product or S-product of two vector $\boldsymbol{a} \in R$ and $\boldsymbol{b} \in R$ is defined by

$$\boldsymbol{a} \square \boldsymbol{b} \equiv \tfrac{1}{2} (\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{a}) . \tag{4}$$

The terminology refers to the symmetry property

$$\boldsymbol{a} \square \boldsymbol{b} = \boldsymbol{b} \square \boldsymbol{a} . \tag{5}$$

The set of vectors of the form $\boldsymbol{a} \square \boldsymbol{b}$ are a linear subspace in the space $R \times R$ of dimension $\frac{1}{2} n (n + 1)$. We denote this space by $R \square R \equiv \square R$.

More general we can define the S-product of a set of f vector $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_f$ in R by setting

$$\boldsymbol{a}_1 \square \boldsymbol{a}_2 \square \cdots \square \boldsymbol{a}_f \equiv \frac{1}{f!} \sum_P \boldsymbol{a}_{i_1} \times \boldsymbol{a}_{i_2} \times \cdots \times \boldsymbol{a}_{i_f} . \tag{6}$$

The sum is extended over all $f!$ permutations

$$P = \begin{pmatrix} 1 \; 2 \ldots f \\ i_1 \, i_2 \ldots i_f \end{pmatrix}$$

of the f indices.

The S-products (6) of f vectors in R_n are a linear subspace of $\underset{f}{\times} R$ of dimension $\binom{n+f-1}{f}$. We denote this space by $\underset{f}{\square} R$. The $\binom{n+f-1}{f}$ different products

$$\omega_{r_1 r_2 \ldots r_f} \equiv \boldsymbol{e}_{r_1} \square \boldsymbol{e}_{r_2} \square \cdots \square \boldsymbol{e}_{r_f}, \quad (r_1 \leqslant r_2 \leqslant \cdots \leqslant r_f) \tag{7}$$

are a base vector system in $\underset{f}{\square} R$. A general vector $\omega_f \in \underset{f}{\square} R$ may be written as a linear combination of the base vectors (7)

$$\omega_f = \frac{1}{f!} \sum_{r_1 \ldots r_f} \lambda^{r_1 \cdots r_f} \omega_{r_1 \ldots r_f} . \tag{8}$$

The $\lambda^{r_1 r_2 \dots r_f}$ are called the contravariant components of the vector ω_f in the base (7). They are symmetrical functions of the f indices $r_1\, r_2 \dots r_f$. Under coordinate transformation they transform like symmetrical contravariant tensors of rank f. The summation in (8) is not restricted by the inequalities of (7). Therefore each component occurs exactly $f!$ times.

(3) *The antisymmetrical product.*

The antisymmetrical product or A-product (also called alternate or Grassman product) of two vectors $a \in R$ and $b \in R$ is defined by

$$\boldsymbol{a} \circ \boldsymbol{b} \equiv \tfrac{1}{2}(\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a}) . \tag{9}$$

It has the properties

$$\left.\begin{aligned} \boldsymbol{a} \circ \boldsymbol{b} &= -\boldsymbol{b} \circ \boldsymbol{a} \\ \text{and} \quad \boldsymbol{a} \circ \boldsymbol{a} &= 0 . \end{aligned}\right\} \tag{10}$$

The set of all products of the form (9) is a linear subspace of $R \times R$ of dimension $\frac{1}{2}\, n\,(n-1)$.

The A-product of a set of f vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_f$ in R is defined by

$$\boldsymbol{a}_1 \circ \boldsymbol{a}_2 \circ \dots \circ \boldsymbol{a}_f \equiv \sum_P \epsilon(P)\, \boldsymbol{a}_{i_1} \times \boldsymbol{a}_{i_2} \times \dots \times a_{i_f} . \tag{11}$$

The summation is extended over all permutations and $\epsilon(P)$ is the signature of the permutation P, i.e.

$$\epsilon(P) = \begin{cases} +1 \text{ for } P \text{ even} \\ -1 \text{ for } P \text{ odd}. \end{cases} \tag{12}$$

The A-products of f vectors form a linear subspace of $\underset{f}{\times} R$ of dimension $\binom{n}{f}$. We shall denote this space by

$$R \circ R \circ \dots \circ R \equiv \underset{f}{\circ} R .$$

The A-product of f vectors is zero if and only if the vectors are linearly dependent. The equation $\boldsymbol{a} \circ \boldsymbol{a} = 0$ is a special case of this. The vectors

$$\omega_{r_1 r_2 \dots r_f} \equiv \boldsymbol{e}_{r_1} \circ \boldsymbol{e}_{r_2} \circ \dots \circ \boldsymbol{e}_{r_f}, \qquad r_1 < r_2 < \dots < r_f . \tag{13}$$

are linearly independent and form a complete set; they are therefore a base in $\underset{f}{\circ} R$.

Every vector $\omega_f \in \underset{f}{\circ} R$ can be represented as a linear combination

$$\omega_f = \frac{1}{f!} \sum \mu^{r_1 r_2 \dots r_f}\, \omega_{r_1 r_2 \dots r_f} \tag{14}$$

with the uniquely determined antisymmetrical components $\mu^{r_1 r_2 \dots r_f}$

which have the transformation properties of a contravariant antisymmetrical tensor of rank f.

When $f = n$ there exists exactly one A-product. It is the determinant of the n vectors in R.

(4) *Creation and annihilation operators.*

The definitions for $\underset{f}{\square} R$ and $\underset{f}{\circ} R$ given above applies for all positive integers $f > 0$. We shall need an extension of this definition to the case $f = 0$. In that case we define both spaces as a one-dimensional vectorspace represented by one single vector ω_0 identical with the set of complex numbers. This vector will play an important role in the following. We denote it by the „vacuum"-vector. It has nothing to do with the physical vacuum state but shares many of the properties of the state vector for the vacuum.

We can now consider the *set* of vectors $\omega_f \in \underset{f}{\square} R$ with $f = 0, 1, 2, \ldots$ Such sets form a new vectorspace which will be denoted by Ω_s. It is the union of all the spaces $\underset{f}{\square} R$ for all values of f. We write a general vector of Ω_s in the form

$$\omega = \{\omega_f\} \qquad (f = 0, 1, \ldots). \tag{15}$$

The space Ω_s is a linear vectorspace if we define the addition and multiplication with complex numbers λ according to the rules

$$\left.\begin{array}{ll} & \omega + \omega' = \{\omega_f + \omega_f'\} \\ \text{when} & \omega = \{\omega_f\} \text{ and } \omega' = \{\omega_f'\} \\ \text{and} & \lambda\,\omega = \{\lambda\,\omega_f\}. \end{array}\right\} \tag{16}$$

In a similar way we define a linear vectorspace Ω_a which is the space of *sets* of vectors $\omega_f \in \underset{f}{\circ} R$

$$\omega = \{\,\omega_f\,\} \qquad (f = 0, 1, \ldots, n)$$

with the algebraic operations defined also according to (16). Since f in this case is restricted to values of $f \leqslant n$ the space Ω_a is of finite dimension equal to

$$\sum_{f=0}^{n} \binom{n}{f} = 2^n.$$

It follows that the vectors in this space are closed with respect to the alternating product, i. e. they form an algebra, the so called Grassman algebra. The space Ω_s on the other hand is of infinite dimensions.

In the following we shall write $\omega_{r_1 \ldots r_f}$ for a vector in Ω which has the only non-vanishing component $\omega_f \equiv \omega_{r_1 \ldots r_f}$.

$$\omega_{r_1 \ldots r_f} = \{0, 0, \ldots, \omega_f, 0, \ldots\}. \tag{17}$$

Such a vector is called homogeneous of rank f. The homogeneous base vectors of rank $f = 0, 1, \ldots$ form a base vector system[12] in Ω. A general vector $\omega \in \Omega$ may be represented as a linear combination

$$\omega = \sum_{f=0}^{\infty} \frac{1}{f!} \sum_{r_1 \ldots r_f} \lambda^{r_1 \ldots r_f} \omega_{r_1 \ldots r_f} \tag{18}$$

with arbitrary complex numbers $\lambda^{r_1 \ldots r_f}$ $(f = 0, 1, \ldots)$.

We shall now define the creation and annihilation operations for the two cases separately.

(a) *The symmetrical case.*

The *creation operators* $\zeta_r^\dagger$ are a set of n linear operators in Ω_s, defined by the relations

$$\zeta_r^\dagger \colon \omega_{r_1 \ldots r_f} \to \zeta_r^\dagger \omega_{r_1 \ldots r_f} = \omega_{r r_1 \ldots r_f} \equiv \boldsymbol{e}_r \,\square\, \omega_{r_1 \ldots r_f}. \tag{19}$$

The annihilation operators ζ_r are defined by

$$\zeta_r \colon \omega_{r_1 \ldots r_f} \to \zeta_r \omega_{r_1 \ldots r_f} = \sum_{\mu=1}^{f} \delta_{r r_\mu} \omega_{r_1 \ldots r_{\mu-1} r_{\mu+1} \ldots r_f}. \tag{20}$$

For ω_0 we define $\zeta_r \omega_0 = 0$. For any other vector the operators are defined by the linearity condition. Thus for instance a vector

$$\omega = \sum_{f=0}^{\infty} \frac{1}{f!} \sum_{r_1 \ldots r_f} \lambda^{r_1 \ldots r_f} \omega_{r_1 \ldots r_f} \tag{21}$$

in Ω_s is transformed into

$$\zeta_r \omega \equiv \sum_{f} \frac{1}{f!} \sum_{r_1 \ldots r_f} \lambda^{r_1 \ldots r_f} \zeta_r \omega_{r_1 \ldots r_f} \tag{22}$$

These linear operators satisfy the commutator relations

$$\left.\begin{array}{c} [\zeta_r, \zeta_s] = [\zeta_r^\dagger, \zeta_s^\dagger] = 0 \\ [\zeta_r, \zeta_s^\dagger] = \delta_{rs} \end{array}\right\} \tag{23}$$

only the last of these is not immediately obvious. It can be verified as follows

$$\zeta_r \zeta_s^\dagger \omega_{r_1 \ldots r_f} = \zeta_r \omega_{s r_1 \ldots r_f} = \delta_{rs} \omega_{r_1 \ldots r_f} + \sum_{\mu=1}^{f} \delta_{r r_\mu} \omega_{s r_1 \ldots r_{\mu-1} r_{\mu+1} \ldots r}$$

$$\zeta_s^\dagger \zeta_r \omega_{r_1 \ldots r_f} = \zeta_s^\dagger \sum_{\mu=1}^{f} \delta_{r r_\mu} \omega_{r_1 \ldots r_{\mu-1} r_{\mu+1} \ldots r_f} = \sum_{\mu=1}^{f} \delta_{r r_\mu} \omega_{s r_1 \ldots r_{\mu-1} r_{\mu+1} \ldots r_f}.$$

Hence
$$[\zeta_r, \zeta_s^\dagger]\, \omega_{r_1 \ldots r_f} = \delta_{rs}\, \omega_{r_1 \ldots r_f}.$$

The operators are linear, hence for any vector $\omega \in \Omega_s$ Equation (21) and (24) give
$$[\zeta_r, \zeta_s^\dagger]\, \omega = \delta_{rs}\, \omega \quad \text{or} \quad [\zeta_r, \zeta_s^\dagger] = \delta_{rs}.$$

Because of their property (19) the creation operators can be used for the representation of the general vector in Ω_s. Repeated use of (19) gives
$$\omega_{r_1 \ldots r_f} = \zeta_{r_1}^\dagger \ldots \zeta_{r_f}^\dagger\, \omega_0. \tag{24}$$
Thus the general base vector can be obtained from the "vacuum" vector by repeated application of creation operators.

We observe here that the operators ζ and $\zeta^\dagger$ are completely independent. In particular they are not the Hermitian conjugate of one another, since we have as yet no scalar product in the space Ω and therefore no definition of Hermitian conjugation. We shall defer the definition of the scalar product to the next section after defining corresponding operators for the antisymmetrical case.

(b) *The antisymmetrical case.*

The creation operators $\eta_r^\dagger$ are a set of n linear operators in Ω_a defined by
$$\eta_r^\dagger:\ \omega_{r_1 \ldots r_f} \to \eta_r^\dagger\, \omega_{r_1 \ldots r_f} = \boldsymbol{e}_r \circ \omega_{r_1 \ldots r_f}. \tag{25}$$
The annihilation operators η_r have the defining property
$$\eta_r:\ \omega_{r_1 \ldots r_f} \to \eta_r\, \omega_{r_1 \ldots r_f} = \sum_{\mu=1}^{f} (-1)^{\mu-1}\, \delta_{r r_\mu}\, \omega_{r_1 \ldots r_{\mu-1} r_{\mu+1} \ldots r_f}. \tag{26}$$
When operating on arbitrary vectors they are determined by the linearity condition. These operators satisfy the commutator relations
$$\left.\begin{aligned} \{\eta_r, \eta_s\} &= 0, \\ \{\eta_r^\dagger, \eta_s^\dagger\} &= 0, \\ \{\eta_r, \eta_s^\dagger\} &= \delta_{rs}. \end{aligned}\right\} \tag{27}$$
These may be verified in complete analogy to the symmetrical case giving due attention to the different sign factors. The η_r operating on the "vacuum" vector are defined again by
$$\eta_r\, \omega_0 = 0. \tag{28}$$
Any of the base vectors can be represented by repeated creation operators operating on the "vacuum" vector. For instance
$$\omega_{r_1 \ldots r_f} = \eta_{r_1}^\dagger \ldots \eta_{r_f}^\dagger\, \omega_0. \tag{29}$$

(5) *The scalar product in Ω.*

A scalar product in the space $R = R_n$ will induce a product in the spaces $\underset{f}{\square} R$ and $\underset{f}{\circ} R$. For instance the scalar product of two vectors $\boldsymbol{a}_1 \times \boldsymbol{b}_1$ and $\boldsymbol{a}_2 \times \boldsymbol{b}_2$ is defined by

$$(\boldsymbol{a}_1 \times \boldsymbol{b}_1) \cdot (\boldsymbol{a}_2 \times \boldsymbol{b}_2) \equiv (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)\,(\boldsymbol{b}_1 \cdot \boldsymbol{b}_2) \tag{30}$$

where $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$ and $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2$ are the scalar products in R. The generalization to linear combinations of Kronecker products as they occur in $\underset{f}{\square} R$ and $\underset{f}{\circ} R$ is then accomplished by the use of the distributive property of the scalar product.

Since our vector space is constructed over the field of complex numbers the scalar product in R is defined as a function of ordered pairs of vectors with values in the complex number field. It shall further have the property of linearity with respect to the second factor

$$\left.\begin{aligned} \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \\ \boldsymbol{a} \cdot \lambda \boldsymbol{b} &= \lambda\,(\boldsymbol{a} \cdot \boldsymbol{b}) \end{aligned}\right\} \tag{31}$$

(λ any complex number).

We also require a symmetry property. The following two possibilities present themselves

(a) orthogonal metric $\quad \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (32a)

(b) unitary metric $\quad \boldsymbol{a} \cdot \boldsymbol{b} = (\boldsymbol{b} \cdot \boldsymbol{a})^*$ (32b)

The second of these is the scalar product which is commonly employed for the definition of a metric in a complex vector space. It leads to a positive definite expression for $\boldsymbol{a} \cdot \boldsymbol{a}$. In case (a) this quantity is generally complex.

The value for the scalar product can be fixed by assigning arbitrary values for the scalar products of the base vectors. For instance

$$\boldsymbol{e}_r \cdot \boldsymbol{e}_s = \delta_{rs}, \qquad (r, s = 1, \ldots n). \tag{33}$$

We shall now discuss the scalar product which is induced in the two vectorspaces Ω_s and Ω_a by the product in R.

(a) *Symmetrical case.*

The scalar product between any two base vectors in Ω_s is obtained from (33), the definition of the $\omega_{r_1 \ldots r_f}$, and the generalization of (30) for arbitrary number of factors

$$(\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{f'}}) = \left\{\begin{matrix} \sum\limits_P \delta_{r_1 r'_{i_1}} \ldots \delta_{r_f r'_{i_f}} & \text{for } f = f' \\ 0 & \text{for } f \neq f'. \end{matrix}\right\} \tag{34}$$

The sum on the right hand side is extended over all permutations of the f indices $r_1 r_2 \ldots r_f$. An immediate consequence of (34) is the relation

$$(\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r_f'}) = \sum_{\mu=1}^{f} \delta_{r_1 r'_\mu} (\omega_{r_2 \ldots r_f}, \omega_{r_1' \ldots r'_{\mu-1} r'_{\mu+1} \ldots r'_f}) \quad (35)$$

which will be used later.

We can now prove that the operators ζ and $\zeta^\dagger$ are the adjoint of each other in the sense that for any arbitrary two vectors ω_1 and ω_2 in Ω_s

$$(\omega_1, \zeta\, \omega_2) = (\zeta^\dagger\, \omega_1, \omega_2). \quad (36)$$

Because the operators ζ and $\zeta^\dagger$ are linear it is sufficient to verify this property for two base vectors in Ω_s and for such pairs only for which (36) is not zero.

$$\begin{aligned}(\omega_{r_1 \ldots r_f}, \zeta_r\, \omega_{r_1' \ldots r'_{f+1}}) &= \Big(\omega_{r_1 \ldots r_f}, \sum_{\mu=1}^{f+1} \delta_{r r'_\mu}\, \omega_{r_1' \ldots r'_{\mu-1} r'_{\mu+1} \ldots r'_{f+1}}\Big) \\ &= \sum_{\mu=1}^{f+1} \delta_{r r'_\mu} (\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{\mu-1} r'_{\mu+1} \ldots r'_{f+1}})\end{aligned}$$

and

$$\begin{aligned}(\zeta_r^\dagger\, \omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{f+1}}) &= (\omega_{r r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{f+1}}) \\ &= \sum_{\mu=1}^{f+1} \delta_{r r'_\mu} (\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{\mu-1} r'_{\mu+1} \ldots r'_{+1}}).\end{aligned}$$

In the last step we have used relation (35). The two expressions are thus equal and (36) is established.

We note that (36) is valid for either of the two cases (*a*) or (*b*). This is so because the representation of the operators ζ and $\zeta^\dagger$ is real in the base vector system which we have chosen. The two kinds of adjoint operations are the transposition and Hermitian conjugation which are identical for a real matrix.

(b) *Antisymmetrical case.*

In this case the scalar product between any two base vectors has the value

$$(\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r'_{f'}}) = \begin{cases} \sum_P \epsilon(P)\, \delta_{r_1 r'_{i_1}} \ldots \delta_{r_f r'_{i_f}} & \text{for } f' = f \\ 0 & \text{for } f' \neq f. \end{cases} \quad (37)$$

The summation is extended over all permutations P of the indices and $\epsilon(P)$ is the signature of the permutation. From (37) follows the relation

$$(\omega_{r_1 \ldots r_f}, \omega_{r_1' \ldots r_{f'}}) = \sum_{\mu=1}^{f} (-1)^{\mu-1} \delta_{r_1 r_\mu'} (\omega_{r_2 \ldots r_f}, \omega_{r_1' \ldots r'_{\mu-1} r'_{\mu+1} \ldots r_{f'}}) \quad (38)$$

which can be used with advantage for proving the property

$$(\omega_1, \eta_r \omega_2) = (\eta_r^\dagger \omega_1, \omega_2) \tag{39}$$

in complete analogy for the derivation of (36).

(6) *Extension to continuous indices.*

For the applications we need a generalization of this formalism to the case of continuous variables. We shall include the possibility of mixed discrete and continuous variables. The summation will be indicated by a generalized integral which is written without differential with the summation variables indicated under the integral sign when necessary.

In this case we start with an infinite-dimensional vector space $R = R_\infty$. There exist an infinite set of linearly independent vectors $\boldsymbol{a}(x)$ which are labelled by one or several parameters and possibly one or several discrete indices. This variable x plays the role of the index r in the preceding sections.

The symmetrical and antisymmetrical products are defined in analogy to (6) and (11)

$$\boldsymbol{a}(x_1) \square \boldsymbol{a}(x_2) \square \cdots \square \boldsymbol{a}(x_f) = \frac{1}{f!} \sum_P \boldsymbol{a}(x_{i_1}) \times \boldsymbol{a}(x_{i_2}) \times \cdots \times \boldsymbol{a}(x_{i_f}) \tag{40}$$

$$\boldsymbol{a}(x_1) \circ \boldsymbol{a}(x_2) \circ \cdots \circ \boldsymbol{a}(x_f) = \frac{1}{f!} \sum_P \epsilon(P)\, \boldsymbol{a}(x_{i_1}) \times \cdots \times \boldsymbol{a}(x_{i_f}). \tag{41}$$

The base vectors in Ω_s and Ω_a are obtained from a base vector system $\boldsymbol{e}(x_1), \boldsymbol{e}(x_2), \ldots$ in R by

$$\omega(x_1 \ldots x_f) = \boldsymbol{e}(x_1) \square \cdots \square \boldsymbol{e}(x_f) \quad \text{for the space } \Omega_s \tag{42}$$

$$\omega(x_1 \ldots x_f) = \boldsymbol{e}(x_1) \circ \cdots \circ \boldsymbol{e}(x_f) \quad \text{for the space } \Omega_a. \tag{43}$$

In either case we can write for a general vector

$$\omega = \sum_{f=0}^{\infty} \frac{1}{f!} \int_{x_1 \ldots x_f} \lambda(x_1, \ldots x_f)\; \omega(x_1, \ldots x_f) \tag{44}$$

where the components $\lambda(x_1 \ldots x_f)$, $(f = 0, 1, \ldots)$ are either symmetrical or antisymmetrical functions of the $x_1\, x_2 \ldots x_f$. The base vectors $\omega(x_1 \ldots x_f)$ are the respective products of the base vectors in R.

$$\left.\begin{aligned} \omega(x_1\, x_2 \ldots x\ ) &= \boldsymbol{e}(x_1) \square \boldsymbol{e}(x_2) \square \cdots \square \boldsymbol{e}(x_f) && \text{for } \Omega_s \\ \omega(x_1\, x_2 \ldots x_f) &= \boldsymbol{e}(x_1) \circ \boldsymbol{e}(x_2) \circ \cdots \circ \boldsymbol{e}(x_f) && \text{for } \Omega_a. \end{aligned}\right\} \tag{45}$$

In the symmetrical case the operators $\zeta^\dagger(x)$ and $\zeta(x)$ are defined in complete analogy to (19) and (20)

$$\begin{aligned}\zeta^\dagger(x):\ \omega(x_1\ldots x_f) \rightarrow \zeta^\dagger(x)\,\omega(x_1\ldots x_f) &= \omega(x\,x_1\ldots x_f)\\ &\equiv \boldsymbol{e}(x)\,\Box\,\omega(x_1\ldots x_f)\end{aligned} \tag{46}$$

$$\begin{aligned}\zeta(x):\ \omega(x_1\ldots x_f) &\rightarrow \zeta(x)\,\omega(x_1\ldots x_f)\\ &= \sum_{\mu=1}^{f} \delta(x_1, x_\mu)\,\omega(x_1\ldots x_{\mu-1}\,x_{\mu+1}\ldots x_f).\end{aligned} \tag{47}$$

These operators satisfy the commutation rules

$$\left.\begin{aligned}[\zeta(x), \zeta(x')] = [\zeta^\dagger(x), \zeta^\dagger(x')] = 0\\ [\zeta(x), \zeta^\dagger(x')] = \delta(x, x').\end{aligned}\right\} \tag{48}$$

The δ-function which occurs in these formulas must be visualized as a generalized δ-function referring to the continuous parameters and discrete indices which are denoted by the variables x.

If we define a metric in Ω_s in analogy to (34) we have

$$\big(\omega(x_1\ldots x_f), \omega(x_1'\ldots x_{f'}')\big) = \begin{cases}\sum\limits_P \delta(x_1, x_{i_1}')\ldots\delta(x_f, x_{i_f}') & \text{for } f = f'\\ 0 & \text{for } f \neq f'\end{cases} \tag{49}$$

and $\zeta^\dagger(x)$ is the adjoint of $\zeta(x)$:

$$(\omega_1, \zeta(x)\,\omega_2) = (\zeta^\dagger(x)\,\omega_1, \omega_2). \tag{50}$$

In the antisymmetrical case the operators $\eta^\dagger(x)$ and $\eta(x)$ are defined by

$$\begin{aligned}\eta^\dagger(x):\ \omega(x_1\ldots x_f) \rightarrow \eta^\dagger(x)\omega(x_1\ldots x_f) &= \omega(x\,x_1\ldots x_f)\\ &\equiv \boldsymbol{e}(x)\circ\omega(x_1\ldots x_f)\end{aligned} \tag{51}$$

$$\begin{aligned}\eta(x):\ \omega(x_1\ldots x_f) &\rightarrow \eta(x)\,\omega(x_1\ldots x_f)\\ &= \sum_{\mu=1}^{f}(-1)^{\mu-1}\,\omega(x_1\ldots x_{\mu-1}\,x_{\mu+1}\ldots x_f)\end{aligned} \tag{52}$$

and they satisfy

$$\left.\begin{aligned}\{\eta(x), \eta(x')\} = \{\eta^\dagger(x), \eta^\dagger(x')\} = 0\\ \{\eta(x), \eta^\dagger(x')\} = \delta(x, x')\end{aligned}\right\} \tag{53}$$

In the metric with the scalar product

$$\big(\omega(x_1\ldots x_f), \omega(x_1'\ldots x_{f'}')\big) = \begin{cases}\sum\limits_P \epsilon(P)\delta(x_1, x_{i_1}')\ldots\delta(x_f, x_{i_f}') & \text{for } f=f'\\ 0 & \text{for } f \neq f'\end{cases} \tag{54}$$

the operators $\eta(x)$ and $\eta^\dagger(x)$ are again adjoint to each other

$$(\omega_1, \eta(x)\,\omega_2) = (\eta^\dagger(x)\omega_1, \omega_2). \tag{55}$$

Relations (55) and (50) are valid for either an orthogonal or unitary scalar product.

Part II-Hyperquantization of a Scalar Field.

(1) *The field equations and the commutation rules.*

We consider a neutral scalar field, represented by a scalar field variable $\Phi(x)$ in the Heisenberg picture[13]). It satisfies a field equation of the form

$$\Box\,\Phi(x) = J(x) \tag{56}$$

where $\Box \equiv \partial_\mu\,\partial^\mu - M^2$, M is the (unrenormalized) mass and $J(x)$ is the interaction term which for most of the following considerations need not be specified. In general $J(x)$ is a functional of the field variables. If we need to indicate this functional dependence we shall write $J = J\{\Phi(x)\}$. The simplest example of an interaction is a term of the form

$$J\,\{\Phi(x)\} = \lambda\,\Phi^2(x) \tag{57}$$

where λ is a coupling constant.

The Heisenberg variables satisfy commutation rules which are simple only on space-like surfaces. They may be written in the form

$$\left.\begin{aligned} &[\Phi(x),\,\Phi(x')]\,\delta(x^0-x'^0) = 0\\ &[\partial_0\,\Phi(x),\,\Phi(x')]\,\delta(x^0-x'^0) = -i\,\delta(x-x')\,. \end{aligned}\right\} \tag{58}$$

In this form the relations are valid for all values of x and x'.

(2) *The correlation functions.*

The difficulty of constructing explicit solutions of (56) is well-known. In recent work it has been emphasized that it is sufficient for the applications of the theory to know expectation values of certain ordered products of field operators (subsequently called correlation functions). In most cases it will be sufficient to know only the expectation values for a "true" vacuum state which we visualize as the state of lowest energy (assuming that such a state exists). Of particular importance is this expectation value for the chronological or T-product, defined as

$$T\{\Phi(x_1)\ldots\Phi(x_f)\} = \Phi(x_{i_1})\ldots\Phi(x_{i_f})$$

such that $x^0_{i_1} \geqslant x^0_{i_2} \geqslant \cdots \geqslant x^0_{i_f}$.

The particular set of correlation functions thereby obtained are called the τ-functions

$$\tau(x_1\ldots x_f) \equiv \langle T\{\Phi(x_1)\ldots\Phi(x_f)\}\rangle\,. \tag{60}$$

All of the following considerations will be valid for general matrix elements or expectation values. There is no need at this stage of specifying the states for which the expectation value is calculated.

Another type of correlation function is obtained with the concept of the ordered or S-product of field operators. The S-product can only be defined for a free field ($J = 0$). In that case one can prove that the two types of products are related by the ordering theorem of WICK[14]). One can with SALAM and MATHEWS[14]) use the relation between the S- and T-products as the defining equation for the S-products. These relations involve the „contraction-symbol'' between the two kinds of products which in this case is simply $i\Delta(x_1 - x_2)$. We denote these functions by σ and shall refer to them as σ-functions[16]).

Thus we have a set of equations, beginning as follows

$$\begin{aligned}
\sigma(x_1) &= \tau(x_1)\\
\sigma(x_1 x_2) &= \tau(x_1\, x_2) + i\,\Delta(x_1 - x_2)\\
\sigma(x_1\, x_2\, x_3) &= \tau(x_1\, x_2\, x_3) + i\,\Delta(x_1 - x_2)\,\tau(x_3)\\
&\quad + i\Delta(x_1 - x_3)\,\tau(x_2) + i\,\Delta(x_2 - x_3)\,\tau(x_1)\,. \qquad (61)\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Here $\Delta(x)$ is the inhomogeneous Δ-function denoted by $\Delta_c(x)$ in reference[13]). We shall omit the index, since no other functions will be used in this paper.

One could introduce another type of σ-functions which are related to the τ-functions in a corresponding way except that the Δ-functions are replaced by Δ'-functions[8]). These σ'-functions seem to be of advantage when renormalization questions are considered. We shall not need them here but merely mention that the formalism can be applied equally well to either case.

It can be shown[17]) that the S-matrix element for the scattering process involving n free particles is directly expressible in terms of the σ-functions in the form

$$S(x_1 \ldots x_n) = \Box_1 \ldots \Box_n\, \sigma(x_1 \ldots x_n)\,. \qquad (62)$$

Thus when these functions are known, the S-matrix is essentially known too.

(3) *The differential equations for the τ-functions.*

The τ- and σ-functions satisfy certain differential equations as a result of the Heisenberg field equations and the commutation rules.

They were given by MATHEWS and SALAM[15]). For the scalar case under consideration these equations are

$$\Box \langle T\{\Phi(x)\Phi(x_1)\ldots\Phi(x_f)\}\rangle = \langle T\{J(x)\,\Phi(x_1)\ldots\Phi(x_f)\}\rangle$$
$$+\, i \sum_{\mu=1}^{f} \delta(x - x_\mu) \langle T\{\Phi(x_1)\ldots\Phi(x_{\mu-1})\,\Phi(x_{\mu+1})\ldots\Phi(x_f)\}\rangle$$
$$(f = 0, 1, \ldots) \tag{63}$$

Each solution of the system (56) furnishes a solution of (63). It seems plausible that the solutions of (63) also determine the solution of (56), but a complete proof of this is not known to us[18]). The remarkable feature of the system (63) is that whereas the solutions of (56) are not known, the system (63) can be solved in closed form. This is accomplished with the formalism of hyperquantization[19]).

In order to transcribe equation (63) into this formalism we introduce the vector Ω_τ defined by

$$\Omega_\tau = \sum_{f=0}^{\infty} \frac{1}{f!} \int_{x_1\ldots x_f} \tau(x_1\ldots x_f)\,\omega(x_1\ldots x_f) \tag{64}$$

where the base vectors are given by equation (42) or by the equivalent expression

$$\omega(x_1\ldots x_f) = \zeta^\dagger(x_1)\ldots\zeta^\dagger(x_f)\,\omega_0\,. \tag{65}$$

The system (63) is then easily recognized as the $(x_1\ldots x_f)$ component of the *field-condition*

$$[\Box\,\zeta(x) - i\,\zeta^\dagger(x) - J\{\zeta(x)\}]\,\Omega_\tau = 0\,. \tag{66}$$

The equivalent equation in Schwinger's method of functionals was given by K. SYMANZIK[6]).

Explicit solutions of (66) are now constructed as follows. We consider first the free-field case with the corresponding vector ω_τ satisfying

$$[\Box\,\zeta(x) - i\,\zeta^\dagger(x)]\,\omega_\tau = 0\,. \tag{67}$$

It has the solution

$$\omega_\tau = e^{\Sigma}\,\omega_0\,, \tag{68}$$

$$\Sigma = -\frac{i}{2} \int_{xx'} \zeta^\dagger(x)\,\Delta(x - x')\,\zeta^\dagger(x')\,. \tag{69}$$

This may be verified using the following relations

$$\Box\,\Delta(x - x') = -\,\delta(x - x') \tag{70}$$

$$e^{\Sigma}\,\Box\,\zeta(x)\,e^{-\Sigma} = \Box\,\zeta(x) - i\,\zeta^\dagger(x) \tag{71}$$

and

$$\Box\,\zeta(x)\,\omega_0 = 0\,. \tag{72}$$

Equation (71) follows directly from the general operator identity

$$e^{\Sigma} 0\, e^{-\Sigma} = 0 + [\Sigma, 0] + \frac{1}{2!}[\Sigma, [\Sigma, 0]] + \cdots \tag{73}$$

and the commutation rules (48).

Returning now to equation (66) we can express its solution in terms of the functional K related to $J(x)$ by the functional derivative

$$J(x) = \frac{\delta K}{\delta \zeta(x)} \tag{74}$$

or the equivalent form

$$J(x) = [K, \zeta^{\dagger}(x)]. \tag{75}$$

For the special example (57) K would be given simply by

$$K = \int_x K(x) \tag{76}$$

$$\text{with} \quad J(x) = \frac{\partial K(x)}{\partial \zeta(x)} \tag{77}$$

$$\text{or} \quad K(x) = \frac{1}{3} \lambda\, \zeta^3(x). \tag{78}$$

In any case it is now easy to verify that the solution of the field-condition (66) can be written in the form

$$\left.\begin{aligned} \Omega_\tau &= e^{\Lambda}\, \omega_\tau \\ \text{with} \quad \Lambda &= -iK + \text{constant.} \end{aligned}\right\} \tag{79}$$

This is based on the identity:

$$e^{\Lambda}\big(\Box\, \zeta(x) - i\, \zeta^{\dagger}(x)\big)\, e^{-\Lambda} = \Box\, \zeta(x) - i\, \zeta^{\dagger}(x) - J(x) \tag{80}$$

which follows from (73) and (75).

The complete solution of (66) appears now in the closed form

$$\Omega_\tau = e^{\Lambda} e^{\Sigma}\, \omega_0. \tag{81}$$

It should be pointed out that there are many other solutions of equation (66), depending on the boundary conditions adopted. This ambiguity is reflected in the ambiguity of the solutions of (70). In fact we could obtain the whole class of solutions for (66) by choosing an arbitrary inhomogeneous Δ-function in (69). The particular choice we have made corresponds to the solution which is needed for the calculation of the S-matrix element (62).

A further ambiguity results from the normalization of the vectors Ω_τ. It is clear that Ω_τ remains a solution when it is multiplied with an aribtrary constant. This constant may be chosen in such a way that

$$|(\omega_0, \Omega_\tau)| = 1.$$

This condition fixes the constant in equation (79) and normalizes the vacuum to vacuum transition probability to unity, which is physically reasonable. The vector Ω_τ is then determined up to an arbitrary phase factor. We can choose the latter so that

$$(\omega_0, \Omega_\tau) = 1. \tag{82}$$

(4) *The equation for the σ-functions.*

We shall now derive the corresponding equations for the σ-functions. This is very simple in the hyperquantization formalism. We define the vector Ω_σ by

$$\Omega_\sigma = \sum_{f=0}^{\infty} \frac{1}{f!} \int_{x_1 \ldots\ldots x_f} \cdots \int \sigma(x_1 \ldots x_f)\, \omega(x_1 \ldots x_f). \tag{83}$$

There is a one-to-one correspondence between the vectors Ω_σ and Ω_τ and on account of the linearity of the system (61) this is a linear correspondence. Thus we see: *The reordering process corresponds to a linear transformation of the space Ω.*

The linear operator which effects this transformation is

$$\left.\begin{aligned} &\Omega_\sigma = e^{-\Sigma}\,\Omega_\tau, \\ \text{with}\quad &\Sigma = -\frac{i}{2}\int_{xx'} \zeta^\dagger(x)\, \Delta(x-x')\, \zeta^\dagger(x'). \end{aligned}\right\} \tag{84}$$

For the proof we express the τ- and σ-functions as scalar products

$$\left.\begin{aligned} \tau(x_1 \ldots x_f) &= \big(\omega(x_1 \ldots x_f), \Omega_\tau\big) = (\omega_0, \zeta(x_f) \ldots \zeta(x_1)\, \Omega_\tau) \\ \sigma(x_1 \ldots x_f) &= \big(\omega_0, \zeta(x_f) \ldots \zeta(x_1)\, \Omega_\sigma\big) = (\omega_0, \zeta(x_f) \ldots \zeta(x_1)\, e^{-\Sigma}\,\Omega_\tau). \end{aligned}\right\} \tag{85}$$

The latter can be transformed on account of

$$e^{-\Sigma^\dagger}\, \omega_0 = \omega_0 \tag{86}$$

and

$$e^{\Sigma}\, \zeta(x)\, e^{-\Sigma} = \zeta(x) + i \int_{x'} \Delta(x-x')\, \zeta^\dagger(x') \tag{87}$$

hence

$$\sigma(x_1 \ldots x_f) = (\omega_0, e^{\Sigma}\, \zeta(x_f)\, e^{-\Sigma} \ldots e^{\Sigma}\, \zeta(x_1)\, e^{-\Sigma}\, \Omega_\tau). \tag{88}$$

Multiplying out the binomial expressions (87) in (88) and using (85) we obtain just the system (61).

From (84) and (81) follows

$$\Omega_\sigma = e^{-\Sigma}\, e^{\Lambda}\, e^{\Sigma}\, \omega_0. \tag{89}$$

*

This vector satisfies the new field-condition

$$\left(\Box\,\zeta(x) - J\{\zeta'(x)\}\right)\Omega_\sigma = 0 \tag{90}$$

$$\text{with}\quad \zeta'(x) = e^{-\Sigma}\,\zeta(x)\,e^{\Sigma} = \zeta(x) - i\int_{x'} \Delta(x-x')\,\zeta^\dagger(x') \tag{91}$$

which corresponds to (66). When equation (90) is written out in components, we obtain the set of equations for the functions which corresponds to (63).

For free fields ($\Lambda = 0$) we find with (89)

$$\omega_\sigma = \omega_0. \tag{92}$$

Thus in the absence of interaction the σ-functions reduce to a scalar quantity. We note a certain analogy of the vector Ω_σ to the interaction picture[13]) in ordinary quantum mechanics. The operator Σ plays the role of the free Hamiltonian and Λ that of the interaction operator. We can define a transformed Λ' by

$$e^{\Lambda'} = e^{-\Sigma}\,e^{\Lambda}\,e^{\Sigma} \tag{93}$$

which is obtained from Λ by replacing everywhere ζ by ζ' (Equation (91)). In the analogy just mentioned it corresponds to the interaction operator in the interaction picture.

We note, however, that in order that this analogy exists it is essential that the Δ-function satisfies Equation (70). The σ'-functions, defined with the Δ'-function do not exhibit such a simple analogy to the interaction picture.

Part III-Hyperquantization of a Spinor Field.

(1) *Field equations and commutation rules.*

The spinor field in the Heisenberg picture is assumed to satisfy the Heisenberg field equation[13])

$$(\boldsymbol{\partial} + m)\,\Psi(x) = I(x). \tag{94}$$

Here $I(x)$ is a spinor function or functional of Ψ, representing the interaction term. When it is necessary to indicate its dependence on the field variables we shall write $I\{\Psi(x)\}$. The simplest possible case is an interaction of the form

$$I = \lambda\,\Psi(\overline{\Psi}\,\Psi) \tag{95}$$

where $\overline{\Psi} = \Psi^\dagger A$ is the adjoint spinor and A is the metrix which transforms the γ_μ according to [13])

$$A\,\gamma_\mu\,A^{-1} = -\,\gamma_\mu^\dagger.$$

In the following we shall not make use of the explicit form (95) of the interaction. It is only mentioned here as an example of a typical self interaction.

By taking the adjoint operation on (94) we obtain also the equation

$$\overline{\Psi}(x)\,(\overleftarrow{\partial} - m) = -\,\overline{I}(x) \tag{96}$$

where the inverted differential operator differentiates the function which stands *before* it [13]). The commutation rules for the field may be written for our purpose in the form

$$\gamma^0 \{\Psi(x),\, \overline{\Psi}(x')\}\,\delta(x^0 - x'^0) = -\,i\,\delta(x - x'). \tag{97}$$

(2) *The correlation functions.*

We define the τ-functions depending on the two sets of variables $x_1 \ldots x_r,\, y_1 \ldots y_s$ by

$$\tau(x_1 \ldots x_r,\, y_1 \ldots y_s) = \langle T\{\Psi(x_1) \ldots \Psi(x_r)\,\overline{\Psi}(y_1) \ldots \overline{\Psi}(y_s)\}\rangle. \tag{98}$$

They are antisymmetrical functions in both sets of variables. The set (x) which refers to the spinor Ψ is always written *before* the set (y) which refers to the spinor $\overline{\Psi}$. The two sets are not interchangeable.

We can also define the σ-function by using the Wick identity in analogy to (61). For the first few of these functions we have for instance

$$\sigma(x, y) = \tau(x, y) - i\,S(x - y)$$

$$\sigma(x_1 x_2, y_1 y_2) = \tau(x_1 x_2, y_1 y_2) + iS(x_1 - y_1)\,\tau(x_2, y_2) - iS(x_1 - y_2)\,\tau(x_2, y_1)$$
$$- \, iS(x_2 - y_1)\,\tau(x_1, y_2) + iS(x_2 - y_2)\,\tau(x_1, y_1). \tag{99}$$

. . . .

The S-functions which appear in these relations are the functions S_c. Since no other functions appear we shall omit the index c. For the definition see reference [13]). The S-matrix element referring to $r + s$ free particles can be expressed in terms of the σ-functions by the relation

$$S(x_1 \ldots x_r, y_1 \ldots y_s) = (\partial_1 + m) \ldots (\partial_r + m)\, \sigma(x_1 \ldots x_r,\, y_1 \ldots y_s) \times$$
$$\times\, (\overleftarrow{\partial}_1 - m) \ldots (\overleftarrow{\partial}_s - m) \tag{100}$$

which is the analogue of equation (62) [20]). In equation (100) the differential operators on the left operate on the variables x while those on the right operate on the variables y.

The τ-functions satisfy a system of differential equations which were given by MATHEWS and SALAM[15]) for a scalar field interacting with a spinor field. Specialized to our case they appear as

$$(\partial + m)\, \tau(x\, x_1 \ldots x_r, y_1 \ldots y_s) = \langle T\{ I(x)\, \Psi(x_1) \ldots \Psi(x_r), \bar{\Psi}(y_1) \ldots \bar{\Psi}(y_s) \}\rangle$$
$$- i \sum_{\sigma=1}^{s} (-1)^{\sigma+r-1}\, \delta(x - y_s)\, \tau(x_1 \ldots x_r, y_1 \ldots y_{\sigma-1}\, y_{\sigma+1} \ldots y_s) \quad (101)$$

and

$$\tau(x_1 \ldots x_r, y\, y_1 \ldots y_s)\, (\rho - m) = -\langle T\{ \Psi(x_1) \ldots \Psi(x_r), \bar{I}(y)\, \bar{\Psi}(y_1) \ldots \bar{\Psi}(y_s) \}\rangle$$
$$- i \sum_{\varrho=1}^{r} (-1)^{\varrho-1}\, \delta(y - x_\varrho)\, \tau(x_1 \ldots x_{\varrho-1}\, x_{\varrho+1} \ldots x_r, y_1 \ldots y_s). \quad (102)$$

We transcribe these equations into the hyperquantization formalism by defining in complete analogy to the symmetrical case a vector Ω_τ for each set of τ-functions

$$\Omega_\tau = \sum \frac{1}{r!\, s!} \int_x \int_y \tau(x_1 \ldots x_r, y_1 \ldots y_s)\, \omega(x_1 \ldots x_r, y_1 \ldots y_s). \quad (103)$$

The base vectorsystem introduced here depends on the pair of sets of variables $x_1 \ldots x_r$ and $y_1 \ldots y_s$. It is antisymmetrical with respect to permutations of the x and y separately. The vectors of the system are constructed as multiple GRASSMAN products of two sets of vectors in the dual spaces R and R^*. The two sets of variables are therefore analogue to the contra- and co-variant indices of a tensor.

In conformity with this interpretation we define the scalar product of two base vectors according to

$$\big(\omega(x_1 \ldots x_r, y_1 \ldots y_s), \omega(x_1' \ldots x_{r'}', y_1' \ldots y_{s'}')\big) = \begin{cases} 0 \text{ for } r' \neq s \text{ or } s' \neq r \\ (-1)^{rs} \sum\limits_P \epsilon(P)\, \delta(x_1 - y_{i_1}') \ldots \end{cases}$$
$$\ldots \delta(x_r - y_{i_r}')\, \delta(y_1 - x_{j_1}') \ldots \delta(y_s - x_{j_s}') \text{ for } r' = s \text{ and } r = s'. \quad (104)$$

The summation is extended over all pairs of permutations of the form

$$P = \begin{pmatrix} 1 & 2 & \ldots & r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & s \\ j_1 & j_2 & \ldots & j_s \end{pmatrix}$$

and $\epsilon(P)$ is the product of the signature of these permutations.

We can now define creation and annihilation operators with respect to this base vector system as follows

$$\left.\begin{aligned}
&\eta^{\dagger}(z)\,\omega(x_1\ldots x_r,\,y_1\ldots y_s) = (-1)^r\,\omega(x_1\ldots x_r,\,z\,y_1\ldots y_s)\\
&\bar{\eta}^{\dagger}(z)\,\omega(x_1\ldots x_r,\,y_1\ldots y_s) = \omega(z\,x_1\ldots x_r,\,y_1\ldots y_s)\\
&\eta(z)\,\omega(x_1\ldots x_r,\,y_1\ldots y_s)\\
&\quad=\sum_{\varrho=1}^{r}\delta(z-x_\varrho)\,(-1)^{\varrho-1}\,\omega(x_1\ldots x_{\varrho-1}\,x_{\varrho+1}\ldots x_r,\,y_1\ldots y_s)\\
&\bar{\eta}(z)\,\omega(x_1\ldots x_r,\,y_1\ldots y_s)\\
&\quad=\sum_{\sigma=1}^{s}\delta(z-y_\sigma)\,(-1)^{r+\sigma-1}\,\omega(x_1\ldots x_r,\,y_1\ldots y_{\sigma-1}\,y_{\sigma+1}\ldots y_s).
\end{aligned}\right\}\quad(105)$$

The operators $\eta(z)$ and $\eta^{\dagger}(z)$ on the one hand and $\bar{\eta}(z)$ and $\bar{\eta}^{\dagger}(z)$ on the other hand are adjoint to one another with respect to the scalar product (104). It is sufficient to verify this for scalar products which do not vanish. For instance

$$\begin{aligned}
&\big(\omega(x_1\ldots x_r,\,y_1\ldots y_s)\,,\,\eta^{\dagger}(z)\,\omega(x_1'\ldots x_s',\,y_1'\ldots y_{r-1}')\big)\\
&\qquad=(-1)^s\big(\omega(x_1\ldots x_r,\,y_1\ldots y_s),\,\omega(x_1'\ldots x_s',\,z\,y_1'\ldots y_{r-1}')\big).
\end{aligned}$$

According to (104) this becomes

$$\begin{aligned}
&(-1)^{(r+1)s}\sum_{\varrho=1}^{r}(-1)^{\varrho-1}\,\delta(z-x_\varrho)\sum_{P}\epsilon(P)\,\delta(x_1-y_{i}')\ldots\delta(x_{\varrho-1}-y_{i_{\varrho-1}}')\times\\
&\quad\times\delta(x_{\varrho+1}-y_{i_{\varrho+1}}')\ldots\delta(x_r-y_{i_r}')\,\delta(y_1-x_{j_1}')\ldots\delta(y_s-x_{j_s}')\\
&\quad=\big(\eta(z)\,\omega(x_1\ldots x_r,\,y_1\ldots y_s),\,\omega(x_1'\ldots x_s',\,y_1'\ldots y_{r-1}')\big)\,.
\end{aligned}$$

Hence for any two base vectors ω_1 and ω_2

$$(\omega_1,\,\eta^{\dagger}(z)\,\omega_2) = (\eta(z)\,\omega_1,\,\omega_2). \qquad (106)$$

In a similar way one verifies that for any two base vectors

$$(\omega_1,\,\bar{\eta}^{\dagger}(z)\,\omega_2) = (\bar{\eta}(z)\,\omega_1,\,\omega_2). \qquad (107)$$

The operators satisfy the following set of commutation rules

$$\begin{aligned}
&\{\eta^{\dagger}(z_1),\,\bar{\eta}^{\dagger}(z_2)\} = \{\eta(z_1),\,\bar{\eta}(z_2)\} = \{\eta(z_1),\,\eta(z_2)\}\\
&=\{\eta^{\dagger}(z_1),\,\eta^{\dagger}(z_2)\} = \{\bar{\eta}(z_1),\,\bar{\eta}(z_2)\} = \{\bar{\eta}^{\dagger}(z_1),\,\bar{\eta}^{\dagger}(z_2)\} = 0
\end{aligned}\qquad(108)$$

and

$$\{\eta(z_1),\,\bar{\eta}^{\dagger}(z_2)\} = \{\eta^{\dagger}(z_1),\,\bar{\eta}(z_2)\} = \delta(z_1-z_2). \qquad (109)$$

As an example we verify the last one

$$\eta(z_1)\,\bar{\eta}^\dagger(z_2)\,\omega(x_1\ldots x_r, y_1\ldots y_s) = \eta(z_1)\,\omega(z_2\,x_1\ldots x_r, y_1\ldots y_s)$$
$$= \sum_{\varrho=1}^{r} \delta(z_1-x_\varrho)\,(-1)^{\varrho-1}\,\omega(z_2\,x_1\ldots x_{\varrho-1}\,x_{\varrho+1}\ldots x_r, y_1\ldots y_s)$$
$$+\,\delta(z_1-z_2)\,\omega(x_1\ldots x_r, y_1\ldots y_s)$$

and

$$\bar{\eta}^\dagger(z_2)\,\eta(z_1)\,\omega(x_1\ldots x_r, y_1\ldots y_s)$$
$$= \bar{\eta}^\dagger(z_2)\sum_{\varrho=1}^{r} \delta(z_1-x_\varrho)\,(-1)^{\varrho-1}\,\omega(x_1\ldots x_{\varrho-1}\,x_{\varrho+1}\ldots x_r, y_1\ldots y_s)$$
$$= \sum_{\varrho=1}^{r} \delta(z_1-x_\varrho)\,(-1)^{\varrho-1}\,\omega(z_2\,x_1\ldots x_{\varrho-1}\,x_{\varrho+1}\ldots x_r, y_1\ldots y_s).$$

Since the base vector is arbitrary, we have established the first of the equations (109). In a similar way one can prove all the other commutator relations.

We can construct the complete set of base vectors by operating with creation operators on the „vacuum" vector defined by

$$\eta(z)\,\omega_0 = \bar{\eta}(z)\,\omega_0 = 0\,. \qquad (110)$$

We find

$$\omega(x_1\ldots x_r, y_1\ldots y_s) = \bar{\eta}^\dagger(x_1)\ldots\bar{\eta}^\dagger(x_r)\,\eta^\dagger(y_1)\ldots\eta^\dagger(y_s)\,\omega_0\,. \qquad (111)$$

The orthogonality relation may now be verified directly by using the commutation rules and equation (110).

It is now easy to see that the equation (101) is the $(x_1\ldots x_r, y_1\ldots y_s)$ component of the field condition

$$[(\boldsymbol{\partial}+m)\,\eta(x)+i\,\eta^\dagger(x)-I\,\{\eta(x)\}]\,\Omega_\tau = 0. \qquad (112)$$

Similarly the equation (102) is obtained from

$$[\bar{\eta}(x)\,(\boldsymbol{\varrho}-m)+i\,\bar{\eta}^\dagger(x)+\bar{I}\{\eta(x)\}]\,\Omega_\tau = 0\,. \qquad (113)$$

In these equations $I\{\eta(x)\}$ is the interaction term with the field variable $\Psi(x)$ replaced by $\eta(x)$ and $\bar{\Psi}(x)$ replaced by $\bar{\eta}(x)$. The two equations (112) and (113) are completely equivalent to the two systems (101) and (102).

(3) *Transformation properties and relativistic invariance.*

We shall now pay special attention to the transformation properties of the operators η, $\eta^\dagger$, $\overline{\eta}$ and $\overline{\eta}^\dagger$. The natural assumption is that they are spinors, that is that they transform under Lorentz transformations L exactly the same way as the operators Ψ, $\overline{\Psi}$ respectively.

$$L: \quad x \to x' = Lx, \Psi \to \Psi'$$

where

and

$$\left.\begin{aligned} \Psi'(x') &= S\,\Psi(x) \\ \overline{\Psi}'(x') &= \overline{\Psi}(x)\,S^{-1}. \end{aligned}\right\} \tag{114}$$

The coefficients of the 4×4 spinor matrix are complex but S is not a unitary matrix (except for space-rotations). It follows that the correlation functions transform according to a formula such as

$$\tau'(x_1' \ldots x_r', y_1' \ldots y_s') = U_r\,\tau(x_1 \ldots x_r, y_1 \ldots y_s)\,V_s \tag{115}$$

where u and v are Kronecker products

$$\left.\begin{aligned} U_r &= \underbrace{S \times S \times \cdots \times S}_{r} \\ V_s &= \underbrace{S^{-1} \times S^{-1} \times \cdots \times S^{-1}}_{s}. \end{aligned}\right\} \tag{116}$$

The assumption that the η's transform like spinors or that

$$\left.\begin{aligned} \eta'(x') &= S\,\eta(x) \\ \overline{\eta}'(x') &= \eta\,(x) S^{-1} \end{aligned}\right\} \tag{117}$$

and a similar set for the operators $\eta^\dagger\ \overline{\eta}^\dagger$, has the advantage that the commutation rules for the η's are invariant*)

$$\{\eta'(x_1), \overline{\eta}'^\dagger(x_2)\} = \delta(x_1 - x_2), \text{ etc.} \tag{118}$$

Furthermore the representation (111) for the base vectors ω shows that these vectors transform according to $\omega \to \omega'$

$$\omega'(x_1' \ldots x_r', y_1' \ldots y_s') = U_s\,\omega(x_1 \ldots x_r, y_1 \ldots y_s)\,V_r. \tag{119}$$

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*) It may be not superfluous to remind the reader that the symbol x in all these formulas stands not only for the space-time variables but also for the spinor components. The δ-functions in (118) and the summations in (119) must be interpreted correspondingly.

Hence the expression

$$\int_x \int_y \tau(x_1 \ldots x_r, y_1 \ldots y_s)\, \omega(x_1 \ldots x_r, y_1 \ldots y_s)$$
$$= \int_{x'} \int_{y'} \tau'(x_1' \ldots x_r', y_1' \ldots y_s')\, \omega'(x_1' \ldots x_r', y_1' \ldots y_s')$$

is an invariant under Lorentz transformations. It follows that the vectors Ω_τ defined by (103) are also invariant. The field conditions (112) and (113) which are the basic equations in the theory are now *covariant equations*, and the relativistic invariance of the formalism is evident.

It will now become apparent that the transformation property of the Ω_τ is only compatible with a symmetrical scalar product of the form (32a). The crucial condition is that the operators η and $\eta^\dagger$ are adjoint to one another and that they both transform identically under Lorentz transformations. Hence the two expressions

$$(\omega_1, \eta(z)\,\omega_2) = (\eta^\dagger(z)\,\omega_1, \omega_2)$$

must transform the same way. This is only the case if

$$(\eta'^\dagger(z')\,\omega_1, \omega_2) = (\omega_2, \eta'^\dagger(z')\,\omega_1)$$
$$= S(\eta^\dagger(z)\,\omega_1, \omega_2).$$

With the unitary scalar product (32b) the right-hand side would instead transform with the complex conjugate matrix S^*.

(4) *Solutions of the field conditions.*

The vector ω_τ representing the free field satisfies the free-field conditions

$$\left.\begin{aligned} [(\partial + m)\,\eta(x) + i\,\eta^\dagger(x)]\,\omega_\tau &= 0 \\ [\eta(x)\,(\rho - m) + i\,\eta^\dagger(x)]\,\omega_\tau &= 0. \end{aligned}\right\} \tag{120}$$

The solution may be expressed in the form

$$\omega_\tau = e^\Sigma \omega_0 \tag{121}$$

with

$$\Sigma = i \int_y \int_{y'} \bar{\eta}^\dagger(y)\, s(y - y')\, \eta^\dagger(y'). \tag{122}$$

The ,,vacuum'' vector ω_0 is defined by

$$\eta(x)\omega_0 = \bar{\eta}(x)\,\omega_0 = 0 \quad \text{for all } x. \tag{123}$$

Since

$$e^{\Sigma}\eta(x)e^{-\Sigma} = \eta(x) - i\int_y S(x-y)\,\eta^{\dagger}(y) \tag{124}$$

and

$$(\boldsymbol{\partial} + m)\,S(x) = -\,\delta(x) \tag{125}$$

we verify that $\eta(x)\omega_0 = 0$ implies

$$\begin{aligned} 0 &= e^{\Sigma}(\boldsymbol{\partial} + m)\,\eta(x)\,e^{-\Sigma}\,e^{\Sigma}\,\omega_0 \\ &= [(\boldsymbol{\partial} + m)\,\eta(x) + i\,\eta^{\dagger}(x)]\,\omega_T. \end{aligned}$$

In a similar way one can establish the second of equations (120).

Solutions of the field conditions with the interaction terms are obtained by a second transformation

$$\Omega_T = e^{\Lambda}\omega_T \tag{126}$$

with

$$\Lambda = i\int_y L\{\eta(y)\} + \text{constant} \tag{127}$$

where the functional $L\{\eta(y)\}$ is obtained from the interaction term by

$$I\{\eta(x)\} = \Big[\int_y L\{\eta(y)\},\, \eta^{\dagger}(x)\Big]. \tag{128}$$

Using the relation (73), we obtain

$$e^{\Lambda}[(\boldsymbol{\partial} + m)\,\eta(x) + i\,\eta^{\dagger}(x)]\,e^{-\Lambda} = (\boldsymbol{\partial} + m)\,\eta(x) + i\,\eta^{\dagger}(x) - I\{\eta(x)\}$$

which, together with (126) implies the equation (112). The equation (113) is then also satisfied with the expression (126) for Ω_T.

We obtain again as in the scalar case a whole family of solutions by choosing for S all the different S-functions which satisfy the equation (125). The particular solution which is obtained by choosing for S the S_c-function is the one needed for the expression (100) of the S-matrix element. We have also indicated the ambiguity arising from the normalization constant by adding an arbitrary constant in (127). This constant can be chosen in such a way that

$$|(\omega_0, \Omega_T)| = 1.$$

The vector Ω_T is then determined up to an arbitrary phase factor.

We can also obtain an equation for the Ω_σ vector and a corresponding set of equations for the σ-functions. Since this transformation is closely analogue to the scalar case and adds nothing significantly new to the results we shall not carry this out explicitly.

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