A Time-keeping Problem Connected with the Gravitational Red-shift

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Abstract. It is shown in accordance with general relativity theory that a clock carried on a particle which describes a circular orbit in a central gravitational field appears to go slow as compared with a clock carried on a particle at a greater orbital distance. The factor appearing in the comparison can be interpreted by a combination of the effects concerned in Einstein's gravitational red-shift and in Einstein's clock-paradox, being an instructive illustration of these effects.

1. In calculations of the gravitational red-shift it is usually assumed that the light-source and the observer are at rest in the frame of reference employed. This assumption can be justified in the context. Nevertheless, bodies cannot remain at rest in this sense in a purely gravitational field. Therefore it is of interest to study a case where the bodies concerned are explicitly taken to be in possible states of motion in the field. Such a study helps to elucidate the problem of time-keeping which is involved in the interpretation of the red-shift.

The example we use appears to be the simplest available. We consider two particles (with observers) describing circular orbits in the gravitational field of a massive body, conveniently called the 'star'. We take the case where the orbital periods are commensurable in the sense that any particular configuration of the whole system is repeated periodically. Our object is to calculate the times between successive repetitions in the reckonings of the two observers and to interpret the results. The treatment is to be in accordance with the standard theory of general relativity.

2. The gravitational field of a 'star' of mass M is given in a standard notation by the Schwarzschild metric

$$ds^2 = \left(1 - 2\,\frac{m}{r}\right) c^2\, dt^2 - \left(1 - 2\,\frac{m}{r}\right)^{-1} dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2 \vartheta\, d\varphi^2 \quad (1)$$

where $m = \gamma M/c^2$ and γ is the gravitational constant. We consider motion only in a particular plane $\varphi =$ constant, as we may do without loss of generality.

One of the equations for a geodesic in the space-time is

$$\frac{d}{ds}\left[\left(1-2\frac{m}{r}\right)^{-1}\frac{dr}{ds}\right]+\left(1-2\frac{m}{r}\right)^{-2}\frac{m}{r^2}\left(\frac{dr}{ds}\right)^2-r\left(\frac{d\vartheta}{ds}\right)^2+\frac{m\,c^2}{r^2}\left(\frac{dt}{ds}\right)^2=0\,.$$

If $r=a$, where a is a constant, this reduces to

$$m\,c^2\left(\frac{dt}{ds}\right)^2-a^3\left(\frac{d\vartheta}{ds}\right)^2=0\,. \tag{2}$$

With $r=a,\ dr=0,\ d\varphi=0$ we have from (1)

$$\left(1-2\frac{m}{a}\right)c^2\left(\frac{dt}{ds}\right)^2-a^2\left(\frac{d\vartheta}{ds}\right)^2=1\,. \tag{3}$$

If (2), (3) are satisfied, it is easily verified that all the conditions for a geodesic are fulfilled. Thus (2), (3) are necessary and sufficient conditions for a test-particle to describe a circular orbit $r=a$ in the given field.

From (2) we obtain $c\,dt/d\vartheta=(a^3/m)^{1/2}$ so that if ϑ increases by $2\,\pi$ then t increases by the amount T where

$$c\,T=2\,\pi\left(\frac{a^3}{m}\right)^{1/2} \tag{4}$$

We call T the *coordinate period* of the orbit.

From (2), (3) we obtain also $ds/d\vartheta=(a^3/m)^{1/2}\,(1-3\,m/a)^{1/2}$ and (see below) we assume $a>3\,m$. Hence if ϑ increases by $2\,\pi$ then s increases by the amount $c\,P$ where

$$c\,P=2\,\pi\left(\frac{a^3}{m}\right)^{1/2}\left(1-3\,\frac{m}{a}\right)^{1/2}=\left(1-3\,\frac{m}{a}\right)^{1/2}c\,T\,. \tag{5}$$

We call P the *proper-period* of the orbit; according to the postulates of relativity theory, this is the period assigned by an observer attached to the particle.

3. Consider now two particles (with observers) describing circular orbits of the sort obtained above. Quantities associated with them will be distinguished by suffixes 1, 2 respectively. Suppose the orbits to be such that

$$T_2=p\,T_1 \tag{6}$$

where p is an integer and $p>1$, so that (4) gives $a_2=p^{2/3}\,a_1$. Suppose also that for $t=0$ we have $\vartheta_1=0,\ \vartheta_2=0$.

Then for $t = T_2$ we have $\vartheta_1 = 2\,p\,\pi$, $\vartheta_2 = 2\,\pi$ which gives for the whole system the same geometrical configuration as that for $t = 0$. In other words, a change of origin of the t-coordinate by the amount T_2 makes no change in the geometrical specification of the system.

Thus the picture of the whole system formed by either observer is repeated at intervals T_2 of t. From (5), (6) the proper-time intervals between such repetitions are for the two observers, respectively,

$$p\,P_1 = \left(1 - 3\frac{m}{a_1}\right)^{1/2} T_2, \qquad P_2 = \left(1 - 3\frac{m}{a_2}\right)^{1/2} T_2. \tag{7}$$

Since $a_2 > a_1$, we have $P_2 > p\,P_1$.

We have, therefore, *a system which, including the observers, returns to a state precisely the same as a previous one and is such that different observers assign different times for the cycle.*

4. The further analysis is sufficiently illustrated by assuming $m/a_2 \ll m/a_1 \ll 1$. This is the case of a particle 2 remote from the star and a particle 1 much closer to the star but far enough for m/a_1 to be a small quantity. (It is well-known that for any actual body the ratio m/r is small compared with unity at any exterior point.)

In this case (7) yields approximately

$$\frac{P_2}{p\,P_1} = 1 + \frac{3}{2}\frac{m}{a_1}. \tag{8}$$

This means, in particular, that if observer 2 sees a clock carried by observer 1 to register the passage of proper-time $p\,P_1$, then observer 2 using a similar clock carried by himself will assign a time-interval P_2 to this occurrence, where $P_2 > p\,P_1$ in accordance with (8).

Neglecting terms in m/a_2, observer 2 will regard himself as being in a Minkowski space-time. To a first approximation, he will regard observer 1 as having a constant speed $a_1\,d\vartheta_1/dt = c\,(m/a_1)^{1/2} = V$, say, relative to himself. Consequently, he will expect a clock carried by observer 1 to appear to go slow in accordance with the *time-dilatation* factor

$$(1 - V^2/c^2)^{-1/2} \doteqdot 1 + \frac{1}{2}\,m/a_1.$$

Further, the usual derivation of the *gravitational red-shift* means precisely that, if an observer sees a clock in a region where the gravitational potential is less by an amount ψ than that at his own position, then the clock will appear to him to go slow by a factor $1 + \psi/c^2$, assuming $\psi/c^2 \ll 1$. In the present case, observer 2 regards observer 1 as being always at a distance from the star where, to a first approximation, the potential is less than at himself by the amount $\psi = \gamma\,M/a_1$. So, from the definition of m, we have $1 + \psi/c^2 = 1 + m/a_1$.

Thus observer 2 can interpret the term $3/2\ m/a_1$ in (8) as comprising $1/2\ m/a_1$ from the usual time-dilatation and m/a_1 from the same effect as the gravitational red-shift. Thus we have an example in which *the gravitational red-shift is exhibited by a system whose total behaviour in the gravitational field is taken into account.*

5. *Discussion.* (a) It is sometimes asked, Does a clock 'really' go slow if it is placed in a gravitational field? There is, of course, no meaning to the question if it be restricted to what can be observed at the position of the clock itself. But the gravitational red-shift does imply, for example that the time-interval between two events on the Sun as measured by a clock at the Sun's surface is less than the time-interval between the same events measured by a similar clock on the Earth. This is in agreement with the interpretation of our example.

(b) In the example, the behaviour of a clock carried by one observer as actually seen by the other observer *at any particular epoch* will be complicated by the first-order DOPPLER effect. But this does not affect the results as they have been stated. Naturally, it would be possible to work in terms of what is observed at each instant and then to integrate over the cycle. In the example there is no fundamental significance in taking the coordinate-periods to be commensurable; this assumption merely simplifies the exposition, in effect, by yielding the integrated result directly.

(c) The result (8) means also that the observer 1 will see a clock carried by 2 to go *fast* (on the average) by the factor calculated. The part m/a_1 of the term $3/2\ m/a_1$ now comes from a gravitational 'blue-shift'. The part $1/2\ m/a_1$ still comes from the time-dilatation exactly as in the standard result of EINSTEIN'S *clock-paradox*, of which the present example affords an instructive confirmation. It should be noted that the approximation employed does *not* place observer 1 in a MINKOWSKI space-time, so there is no question of obtaining the effect of the time-dilatation directly from the motion of 2 relative to 1. It is the erroneous attempt to do this that would produce the paradox. The difference between the two observers, and in particular the 'discrepancy' $1/2\ m/a_1$ in their time-keeping contributed by the time-dilatation, is absolute. It should, in fact, be further noted that a relation of the type (8) between time-reckonings of two observers does not arise in special relativity since that theory cannot treat such cyclic processes involving, as they must, observers in accelerated relative motion.

(d) It may not have been remarked before that, in accordance with (5), the distance $3\ m = a_0$, say, is a critical distance for the SCHWARZSCHILD space-time. It is the distance for which a circular orbit demands an orbital speed equal to the speed of light.