Zeitschrift: Helvetica Physica Acta

Band: 29 (1956)

Heft: [4]: Supplementum 4. Fünfzig Jahre Relativitätstheorie = Cinquantenaire de la Théorie de la Relativité = Jubilee of Relativity Theory

Artikel: On the definition of inertial systems in general relativity

Autor: Pirani, F.A.E.

DOI: https://doi.org/10.5169/seals-112743

On the Definition of Inertial Systems in General Relativity

by F. A. E. Pirani (Dublin)

Since general relativity theory is covariant under general coordinate transformations, it does not exhibit any immediately obvious family of preferred coordinate systems similar to the inertial systems of special relativity theory or Newtonian mechanics. Some people are inclined to the view that no such coordinate systems can be or indeed should be defined, and hold that to attempt such a thing is against the spirit of the theory, while others (e. g. Fock) make definitions whose physical significance is obscure. At any event, since absolute space and time do not appear in general relativity as primary concepts, the concepts *uniform motion* and *non-rotation* cannot appear either, until it has made clear relative to what the motion is supposed to be uniform and non-rotating.

This can be done in two ways: (a) in terms of a single observer who refers observations to a clock measuring proper time and to a local coordinate system defined by a triad of unit vectors in each of his instantaneous 3-spaces; (b) in terms of an extended family of observers whose world lines are used as a system of reference. These two methods cannot be put into any general correspondence because the coordinate systems defined by individual observers will not in general form a holonomic system, and because the coordinate directions defined by sets of neighbouring observers will not in general subtend constant angles at one another. The first method – the description of observations relative to a single observer – seems closer to practical methods, and will form the main concern of this note.

In either formulation the concept of non-rotation can be defined either in terms of local dynamical phenomena or in terms of astronomical observations. But (essentially because of the principle of equivalence) uniform motion cannot be defined purely locally in an adequate way.

The relations between locally and extendedly defined reference frames on the one hand, and the two kinds of observations – dynamical and astronomical – on the other, are the subject of various statements of uncertain status which have appeared in the literature under the name

'MACH's principle of the relativity of inertia'. Without being concerned about which, if any, of these can properly be attributed to MACH, I shall mention several of them in order to make clear which of them is relevant here:

(1) Kinematically equivalent motions are dynamically equivalent.

(2) The gravitational field (metric tensor) is *determined* by the material content of space-time (energy-momentum tensor).

(3) In the absence of matter, space-time should necessarily be MINKOWSKIAN.

(3*) In the absence of matter, the field equations should have no solutions at all.

(4) The local reference frames in which NEWTON's laws are approximately valid (without the introduction of Coriolis or centrifugal forces) are those frames which are approximately non-rotating relative to the distant stars.

It seems to me that (1) is a statement which might be true and might be false in Newtonian mechanics, and is in fact false, while in a generally covariant theory with a two-sheeted null cone determined by the metric it is almost trivially true.

The difficulty facing (2) is that the field equations are differential equations, so that the problem of choosing boundary conditions arises. To reject boundary conditions altogether as conceptually undesirable would seem to place a very strict interpretation on the word 'determined'. The extent to which boundary conditions are required is now well understood, since the work of LICHNEROWICZ and his school. If, however, one wishes to avoid them altogether, it would appear that either one must replace the field equations by integral equations, which is hardly a practical proposition, or else introduce some kind of statistical postulate of a cosmological type.

Statement (3) has no operational basis; there cannot be any empirical content to a statement about the necessary metrical structure of an empty space-time. It may have some aesthetic appeal, but this has no empirical basis, because the fact that a region of the dimensions of the solar system is observed to be approximately MINKOWSKIAN is evidence only for the relative smallness of the gravitational constant. In a Riemannian space-time, every sufficiently small region is MINKOWSKIAN to any given approximation.

The aesthetic appeal of (3*) is more readily understood than that of (3), but neither of these statements is true of the field equations of general relativity theory.

The idea which is relevant here is (4). This is an empirical result which is felt by some to have the status of a basic principle – to constitute a test of the validity of mechanical theories. In Newtonian mechanics it is effectively a postulate. The conclusion to be reached in this note is that as far as general relativity is concerned, (4) is an accident, not a fundamental law – an empirical result which is only approximately confirmed by theory, and this only when the gravitational field is slowly varying in space and time.

In order to arrive at this conclusion it is necessary to have an acceptable definition of 'local reference frames in which NEWTON's laws are approximately valid'. It is not hard to see that for an individual observer such frames may be defined by parallel propagation along his world line of the triad of space-like vectors which constitute his local reference frame, at least if the observer is moving along a geodesic. Consider a freely falling observer P, with geodesic world line, and consider further a cloud of freely falling test particles near P with velocities which are small relative to P. The motion of these particles will be described by the equation of geodesic deviation[1]). If now a vierbein of orthonormal vectors is introduced, whose timelike member is P's 4-velocity, the spacelike members defining his local reference frame, and if the vierbein is propagated parallelly along P's world line, then the motion of nearby test particles, written in terms of the vierbein, is like that of a continuous fluid in which the circulation is a constant. If further it is supposed that the world lines of all the other particles intersect the world line of P at some instant (as if P threw out a cloud of dust particles in all directions at that instant), then the circulation is always zero, which is to say that the motion of a cloud of free test particles out from a point, referred to parallelly propagated axes, is *irrotational*. In this sense, then, parallel propagation defines a system of axes in which NEWTON's laws are approximately valid, at least for a freely falling observer. If, for example, this definition is employed for an observer with a 'circular' geodesic orbit in SCHWARZSCHILD space-time, then the axes exhibit that secular rotation known as the DE SITTER-SCHOUTEN effect.

If the orbit of P is not a geodesic, because he is subject to non-gravitational forces, then parallel propagation is unsuitable, because the space axes do not remain permanently orthogonal to the observer's world line. FERMI propagation has been proposed [1] as an alternative (which reduces to parallel propagation if the orbit becomes geodesic). PAPAPETROU's spinning test particles [2] give an interesting illustration of the fitness of this. A spinning test particle is described by its 4-velocity v^μ and by a

1) The following argument is based on J. L. SYNGE, Duke Mathematical J. *1*, 527 (1935).

skew tensor $S^{\mu\nu}$ representing the spin. PAPAPETROU's equations are not determinate, but if they are supplemented by the condition $S^{\mu\nu}\, v_\nu = 0$, whose physical meaning is just that the spin angular momentum be conserved, then $S^{\mu\nu}$ can be replaced by a spin vector $H^\mu = \varepsilon^{\mu\nu\varrho\sigma}\, S_{\nu\varrho}\, v_\sigma$ lying in the instantaneous 3-space of the spinning particle, and then it follows from PAPAPETROU's equations that H^μ is of constant magnitude and satisfies the equations

$$(\delta^\mu_\nu - v^\mu\, v_\nu)\,\frac{\delta H^\nu}{\delta s} = 0\,,$$

which are exactly the equations of FERMI propagation. This is to say that the axis of a spinning test particle is fixed relative to FERMI-propagated axes.

One can investigate local dynamical behaviour in a different way by trying to define something like a FOUCAULT pendulum. One would not want to be too realistic about this, because it would involve solving the two-body problem, and introducing the constraint imposed by the suspending wire, and so on, but one can reach a plausible sort of 3-dimensional oscillator by introducing a non-gravitational force to replace the earth's gravitational field as the restoring force for the pendulum bob. Thus the suspension of the apparatus – the earth, plus a CARDAN'S suspension and some springs, for example – is supposed to have a *given* path, which will be assumed geodesic, and to produce no gravitational field. Then the bob of the pendulum would move freely, were it not for the action of the springs, which, say, restore the pendulum with a force proportional to its displacement from the given geodesic. Not surprisingly, these assumptions lead to the equation

$$\frac{\delta^2 \eta^\mu}{\delta s^2} + R^\mu_{\nu\varrho\sigma}\, v^\nu\, \eta^\varrho\, v^\sigma + k^2\, \eta^\mu = 0$$

for the infinitesimal displacement η^μ of the pendulum bob from the given geodesic (and orthogonal to it). This is just the equation of geodesic deviation, with the last term added on. If these equations are solved in a SCHWARZSCHILD space-time, then the model behaves very much like a FOUCAULT pendulum, except for a small gravitational couple (analogous to the DE SITTER-SCHOUTEN effect) exerted by the central mass. This couple, although small for parameter values taken from the solar system case, can become large in strong and rapidly varying fields.

Thus there is a variety of ways of defining non-rotation by dynamical experiments. To define non-rotation relative to the stars is not quite so straightforward, because one has to be careful not to introduce irrelevant abberation effects, and also one has to decide how to weight stars at different distances.

As far as aberration is concerned, one can always check definitions by going back to the SCHWARZSCHILD case, where of course the MINKOWSKIAN boundary condition represents distant stars at rest.

A measure of rotation relative to the stars can be formulated like this: Consider an observer P with velocity v^μ, and suppose that his local reference frame is defined by three orthonormal vectors $\lambda_\mu^{(a)}$, which are propagated along the world line of P in a given way. Then to a photon of 4-momentum p^μ, he will assign energy $E = p^\mu v_\mu$ and direction defined by direction cosines $\nu^{(a)} = E^{-1} \lambda_\mu^{(a)} p^\mu$. If he receives light from a particular star, then he can plot the motion of that star's projection on a unit sphere fixed to his reference frame by measuring the direction of arrival of successive photons. If he does this for a continuous distribution of stars over the sky, he can construct expressions like $\mu_{(a)} = \int \varepsilon_{abc} \nu^{(b)} \delta\nu^{(c)}/\delta s \, f \, d\omega$. Here $d\omega$ is solid angle, suitably defined, f is a weighting function, and $\delta\nu^{(c)}/\delta s$ is the rate of change of direction with respect to proper time, of the stars in a given element of solid angle. In general, such expressions will not vanish. In particular, they will not as a rule vanish even if the axes are FERMI propagated. That is to say that in a general Riemannian space-time, there is no choice of local reference frame which can be made by an observer so that the projections of all the distant stars on his unit sphere are at rest. Furthermore, if he adopts a reference frame in which NEWTON's laws are approximately valid, then he will in general find that the positions of the distant stars, referred to that frame, change secularly.

There are of course some cosmological models, such as the ROBERTSON-WALKER models, in which all the stars appear to fundamental observers to have fixed directions. In world models which are homogeneous but not isotropic, this need not be the case. It is possible, therefore, that consideration of such models would yield conclusions about the large scale structure of the universe, by showing that in non-isotropic world-models statement (4) above would not be even approximately true, while consideration of world-models like the SCHWARZSCHILD solution shows that when irregularities are admitted, (4) cannot be an exact principle, but only an approximate statement.

Diskussion – Discussion

A. D. FOKKER: 1. The geodesic precession was referred to as the DE SITTER-SCHOUTEN effect. As a historical fact, SCHOUTEN only found 2/3 of the right amount. I myself have been able to give the correct theory [3].

2. The geodesic precession means that a gyroscope carried along by the earth after a year will *not* point to the same fixed star as before.

3. Given a cloud of free falling particles: how many of these free motions do we need to adjust a geodesic frame of reference in such a way that *all* the other motions (in first approximation) would appear as straight and uniform? That would be the ultimate fact contained in GALILEI's principle of inertia.

F. A. E. PIRANI: 1. I apologize. The fault is really EDDINGTON'S, for he gives the reference to SCHOUTEN, and not to yourself.

2. I quite agree. My remark was intented to refer to the *approximate* empirical situation.

3. I don't know, but I should guess that three would be enough.

H. P. ROBERTSON: I should like to ask Dr. PIRANI whether according to his criterion the matter involved in the GÖDEL solution is rotational or not?

F. A. E. PIRANI: I have not stated the criterion completely, but in any event I do not think anyone has worked out the answer to your question using it.

References

[1] WALKER, A. G., Proc. Edinburgh Math. Soc. [2] *4,* 170 (1935).
[2] PAPAPETROU, A., Proc. Roy. Soc. [A] *209,* 248 (1951).
[3] FOKKER, A. D., Proc. Roy. Acad. Amsterdam, *23,* 729 (1920).