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Analyticity of Wightman functions at completely space-like points

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Introduction

The Wightman function[1])

$$\mathfrak{W}(z_0, z_1, \ldots, z_n) \equiv W(\zeta_1, \ldots, \zeta_n) \text{ with } \quad \zeta_k = z_k - z_{k-1} \qquad z_k = x_k + i\, y_k \quad \zeta_k = \xi_k + i\, \eta_k \tag{1}$$

is defined by analytic continuation of $\langle A(x_0)A(x_1)\ldots A(x_n)\rangle_0$ where A is a scalar field operator.

It is known to be analytic in UPR_n', i.e. in the union of the extended tubes after permutation of the variables. The real points in R_n' are the J points[2]) defined by the condition

$$(\sum_i \lambda_i\, \xi_i)^2 < 0 \quad \text{for} \quad \sum_i \lambda_i = 1,\ \lambda_i \geqslant 0 \tag{2}$$

We define the S points as the real points such that

1. $$(x_i - x_k)^2 < 0 \quad \text{for} \quad i \neq k \tag{3}$$

2. For no permutation

$$(i_0, \ldots, i_n) \text{ of } (0, \ldots, n) \text{ is } (x_{i_1} - x_{i_0}, \ldots, x_{i_n} - x_{i_{n-1}})$$

a J point (otherwise stated: $(\xi_1, \ldots, \xi_n) \notin UPR_n'$).

For $n = 2$, S points do not exist, which means that a completely space-like point (3) is a permuted J point.

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For $n = 3$, the following is a S point:

$$x_0 = (1 - \varepsilon, 1, 1, 0), x_1 = (1 - \varepsilon, -1, -1, 0), x_2 = (\varepsilon - 1, 1, -1, 0), \\ x_3 = (\varepsilon - 1, -1, 1, 0)$$

where ε is a small positive number**).

The purpose of this note is to prove the following

Theorem***)

The real completely space-like points are contained in the envelope of holomorphy of UPR_n'.

In fact, the analyticity at completely space-like points follows from local commutativity expressed as a condition on the boundary values of $\mathfrak{W}$[3]). It also follows from the existence of boundary values only[4]). It is however interesting, in the frame of the Wightman Programme, that this existence already follows from the assumed analyticity of $\mathfrak{W}$ in UPR_n'.

Proof

ζ_k is chosen of the following form

1. (ξ_k) belongs to a completely space-like point.
2. (η_k) is along the time axis: $\eta_k = (t_k, \vec{0})$.

Thus, we consider W as a function of $3\,n$ real (space) variables and of n complex (time) variables.

Let (t_k) have any fixed value, then (y_k) is determined up to a common additive constant. We order (y_k) in (y_{i_k}) according to increasing time coordinates. We have thus

$$W(\zeta_1, \ldots, \zeta_n) = \mathfrak{W}(z_0, z_1, \ldots, z_n) = \mathfrak{W}(z_{i_0}, z_{i_1}, \ldots, z_{i_n}) = W(\zeta_1', \ldots, \zeta_n')$$

with $t_i' \geqslant 0$. If for all i, $t_i' > 0$, then, W is analytic in (ζ_i).

If, for exactly m coordinates t_i', $t_i' = 0$, we say that W has an m-singularity at (ζ_i). In the space (t_i), the singularities appear thus on a finite number of linear subspaces, m-singularities being located on subspaces with dimension n-m. From this it follows that an m-singularity is never limiting point of m'-singularities for $m' > m$. We now use induction to remove the m-singularities for increasing m. When $m = 1$, there is no singularity because the ξ' corresponding to $t' = 0$

**) This result is due to O. Steinmann. I thank him for his permission to quote it.

***) This theorem was proved originally by Dyson, but remained unpublished (F. J. Dyson, private communication).

is space-like and ζ' can therefore be sent into the imaginary upper half cone by an infinitesimal complex Lorentz transformation[2]). For $m > 1$, let $t_1'', \ldots, t_m''$ be the vanishing arguments in (t_i''), then, W is analytic if (t_i') is held fixed except for t_1'', and t_2'' which vary over some independent neighbourhoods of their original positions and

$$t_1'' \text{ and } t_2'' \text{ are not both zero} \tag{4}$$

We call x and y respectively the complex variables from which t_1'' and t_2'' originate. Applying then the Kantensatz[5]) to the intersection of the analytic hypersurfaces:

$$\operatorname{Im} x = 0, \quad \operatorname{Im} y = 0$$

we see that there are no singularities on this intersection. This completes the proof of the theorem.

We finally remark that is simply follows from the proof that $\mathfrak{W}(z_0, z_1, \ldots, z_n)$ is analytic when the vectors z_i have real space components and purely imaginary time components except when two of them coincide[6]).

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References

1) A. S. Wightman, Phys. Rev. *101*, 860 (1956). – D. Hall and A. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. *31*, No. 5 (1957).
2) R. Jost, Helv. Phys. Acta *30*, 409 (1957).
3) F. J. Dyson, Phys. Rev. 110, *579* (1958).
4) R. Jost, private communication.
5) H. Behnke & P. Thullen, Ergebn. d. Math. *3*, Nr. 3, Berlin (1934), S. 52, Satz 20.
6) Cf. J. Schwinger, Annual International Conference on High Energy Physics at CERN (1958), p. 134.