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# Analyticity of Wightman functions at completely space-like points

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## Introduction

The Wightman function<sup>1)</sup>

$$\mathfrak{B}(z_0, z_1, \dots, z_n) \equiv W(\zeta_1, \dots, \zeta_n) \text{ with } \begin{aligned} \zeta_k &= z_k - z_{k-1} \\ z_k &= x_k + i y_k \end{aligned} \quad \zeta_k = \xi_k + i \eta_k \quad (1)$$

is defined by analytic continuation of  $\langle A(x_0)A(x_1)\dots A(x_n) \rangle_0$  where  $A$  is a scalar field operator.

It is known to be analytic in  $UPR_n'$ , i.e. in the union of the extended tubes after permutation of the variables. The real points in  $R_n'$  are the  $J$  points<sup>2)</sup> defined by the condition

$$(\sum_i \lambda_i \xi_i)^2 < 0 \quad \text{for} \quad \sum_i \lambda_i = 1, \lambda_i \geq 0 \quad (2)$$

We define the  $S$  points as the real points such that

$$1. \quad (x_i - x_k)^2 < 0 \quad \text{for} \quad i \neq k \quad (3)$$

2. For no permutation

$$(i_0, \dots, i_n) \text{ of } (0, \dots, n) \text{ is } (x_{i_1} - x_{i_0}, \dots, x_{i_n} - x_{i_{n-1}})$$

a  $J$  point (otherwise stated:  $(\xi_1, \dots, \xi_n) \notin UPR_n'$ ).

For  $n = 2$ ,  $S$  points do not exist, which means that a completely space-like point (3) is a permuted  $J$  point.

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For  $n = 3$ , the following is a  $S$  point:

$$\begin{aligned} x_0 &= (1 - \varepsilon, 1, 1, 0), \quad x_1 = (1 - \varepsilon, -1, -1, 0), \quad x_2 = (\varepsilon - 1, 1, -1, 0), \\ x_3 &= (\varepsilon - 1, -1, 1, 0) \end{aligned}$$

where  $\varepsilon$  is a small positive number\*\*).

The purpose of this note is to prove the following

### Theorem \*\*\*)

The real completely space-like points are contained in the envelope of holomorphy of  $UPR_n'$ .

In fact, the analyticity at completely space-like points follows from local commutativity expressed as a condition on the boundary values of  $\mathfrak{M}^3$ ). It also follows from the existence of boundary values only<sup>4</sup>). It is however interesting, in the frame of the Wightman Programme, that this existence already follows from the assumed analyticity of  $\mathfrak{M}$  in  $UPR_n'$ .

### Proof

$\zeta_k$  is chosen of the following form

1.  $(\xi_k)$  belongs to a completely space-like point.
2.  $(\eta_k)$  is along the time axis:  $\eta_k = (t_k, \vec{0})$ .

Thus, we consider  $W$  as a function of  $3n$  real (space) variables and of  $n$  complex (time) variables.

Let  $(t_k)$  have any fixed value, then  $(y_k)$  is determined up to a common additive constant. We order  $(y_k)$  in  $(y_{i_k})$  according to increasing time coordinates. We have thus

$$W(\zeta_1, \dots, \zeta_n) = \mathfrak{M}(z_0, z_1, \dots, z_n) = \mathfrak{M}(z_{i_0}, z_{i_1}, \dots, z_{i_n}) = W(\zeta_1', \dots, \zeta_n')$$

with  $t_i' \geq 0$ . If for all  $i$ ,  $t_i' > 0$ , then,  $W$  is analytic in  $(\zeta_i)$ .

If, for exactly  $m$  coordinates  $t_i'$ ,  $t_i' = 0$ , we say that  $W$  has an  $m$ -singularity at  $(\zeta_i)$ . In the space  $(t_i)$ , the singularities appear thus on a finite number of linear subspaces,  $m$ -singularities being located on subspaces with dimension  $n-m$ . From this it follows that an  $m$ -singularity is never limiting point of  $m'$ -singularities for  $m' > m$ . We now use induction to remove the  $m$ -singularities for increasing  $m$ . When  $m = 1$ , there is no singularity because the  $\xi'$  corresponding to  $t' = 0$

\*\*) This result is due to O. STEINMANN. I thank him for his permission to quote it.

\*\*\*) This theorem was proved originally by DYSON, but remained unpublished (F. J. DYSON, private communication).

is space-like and  $\zeta'$  can therefore be sent into the imaginary upper half cone by an infinitesimal complex Lorentz transformation<sup>2)</sup>. For  $m > 1$ , let  $t_1'', \dots, t_m''$  be the vanishing arguments in  $(t_i'')$ , then,  $W$  is analytic if  $(t_i')$  is held fixed except for  $t_1''$ , and  $t_2''$  which vary over some independent neighbourhoods of their original positions and

$$t_1'' \text{ and } t_2'' \text{ are not both zero} \quad (4)$$

We call  $x$  and  $y$  respectively the complex variables from which  $t_1''$  and  $t_2''$  originate. Applying then the Kantensatz<sup>5)</sup> to the intersection of the analytic hypersurfaces:

$$\operatorname{Im} x = 0, \quad \operatorname{Im} y = 0$$

we see that there are no singularities on this intersection. This completes the proof of the theorem.

We finally remark that it simply follows from the proof that  $\mathfrak{W}(z_0, z_1, \dots, z_n)$  is analytic when the vectors  $z_i$  have real space components and purely imaginary time components except when two of them coincide<sup>6)</sup>.

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- 6) Cf. J. SCHWINGER, Annual International Conference on High Energy Physics at CERN (1958), p. 134.