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The S-Matrix in the Non-local Field Theory of Arnous and Heitler

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Summary. The general non-local field theory of ARNOUS and HEITLER is considered for a specially chosen c-number form factor. A technique for evaluating the S-matrix is developed and with its help it is proved that the theory is convergent throughout in all orders of perturbation. However, with this form factor strict Lorentz invariance is not maintained but is confined to the first order of perturbation, as some explicit calculations (e. g. the self mass of the electron) show.

Introduction

In recent years a lot of evidence for a finite size of the elementary particles has been accumulating. The evidence is fairly well-known and we shall not discuss it in detail except to say that (1) the evidence suggests a universal cut-off or 'least length' (the same for all particles, or rather all interactions) of the order of magnitude $1/M$, where M is the mass of the nucleon and that (2) the evidence concerns, among other things, the observability (and therefore finiteness) of certain quantities (e.g. self-masses) which hitherto were often regarded as unobservable, and, because they were infinite according to the usual theory, were renormalized away.

So far, the concept of a finite-sized particle is restricted to, and has been used in field theory, only in the extreme non-relativistic region. The evidence for the finiteness of the self-masses etc., however, if accepted, makes it imperative for this concept to be embodied somehow in the *relativistic* field theories also, which, of course implies a profound modification of these theories.

In other words, the observability of the self-masses, for example, requires that the usual relativistic theory be modified so that it becomes convergent**), and in such a way that the 'extended source' of the particles arises in the non-relativistic limit.

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**) Of course, there are other reasons too, for demanding convergence, such as the existence of 'ghost'-states.

The question is whether such a convergent theory exists at all in the relativistic region. This question is not trivial. Already a non-local relativistic theory has been proposed (KRISTENSEN, MØLLER 1952[1])) which in spite of being nonlocal is divergent. In two recent papers (ARNOUS, HEITLER *et al.*, in press[2])[3])) which will be referred to below as I and II an alternative approach to this problem has been suggested. In contrast to the non-local theory of KRISTENSEN and MØLLER, where relativistic invariance is assumed from the outset, it is proposed in I and II that the non-local theory be constructed in such a way that the convergence of the theory should be the *primary demand*. In view of the difficulty of reconciling convergence and invariance the relativistic invariance is considered only in the second place. In other words, it is proposed to generalize the idea of the finite source so that it should extend into the relativistic region, retaining the convergence, but not necessarily retaining strict relativistic invariance. The problems to be investigated would then be (a) whether within the framwork of the convergent theory strict relativistic invariance could be attained as well, or (b) whether strict relativistic invariance could be sacrificed *without* coming into contradiction with experiment. If any violation of exact relativistic invariance should occur it would of course be restricted to the inside of the source, about which we have no direct knowledge. We may perhaps visualize a situation where the violation of Lorentz-invariance is mild enough not to exhibit itself in the experiments up to the present. Thus this second question must be considered in spite of the fact that in the local theory, the success of the renormalization procedure seems to depend ultimately on an appeal to strict relativistic invariance.

A general framework for discussing these questions has been developed in I and II. There, the nonlocality is introduced into the theory by means of a form-factor in that part of the Hamiltonian which describes the interaction of two fields. This form-factor may be either a c-number or a q-number. Now as we have said, the primary demand should be for convergence. The purpose of the present paper is to establish that, for c-number form-factors at any rate, the form-factor can always be chosen so as to ensure convergence in all orders of the perturbation expansion (i. e. so as to ensure convergence of the S-matrix). It is shown further that this convergence is compatible with relativistic invariance in the *first* order of perturbation. Thus the theory proposed in I and II is actually the first convergent theory extending into the relativistic region.

But for this success, so far, a certain high price has had to be paid. It turns out, in fact, that (still for c-number form-factors) simultaneously with convergence, relativistic invariance in the higher orders of perturbation cannot be attained. This is shown below in detail for a particular

example (self-mass of the electron). However, the deviation from relativistic invariance in this case (i.e. the deviation from the Einstein mass-velocity relation) lies just outside the present range of experiment[4]). It will be of great interest to see whether more accurate experiments will maintain the *exact* validity of the Einstein relation or not. Whether for the more accurately measured effects of quantum-electrodynamics such as the Lamb-shift and the anomalous magnetic moment of the electron, the deviation from relativistic invariance is still outside the experimental range is not yet known. We have actually calculated the anomalous magnetic moment below (also the photon self-energy) but on account of some theoretical ambiguity our results for these are not yet final.

Thus the question (b) above whether strict relativistic invariance can be sacrificed or not, is still open. The question (a) whether strict relativistic invariance can be achieved simultaneously with convergence, is decided in the negative if we restrict ourselves to c-number form-factors, but is still very open if q-numbers factors are considered. The possibilities for q-number factors have hardly been investigated at all as yet, but it is fairly obvious that it will be possible to choose these, also, so as to ensure convergence. Whether, as for the c-number factors, choosing them in this way will exclude strict relativistic invariance remains to be seen.

For definiteness we have considered in this paper only non-local quantum-electrodynamics, but the convergence proof can easily be extended to cover all the known field-theories, including those which involve derivative coupling—as is pointed out in the text at the appropriate places. In addition to the convergence proof we have also developed in this paper a technique for calculating matrix elements of the S-matrix. It is the non-local analogue of the Feynman-Dyson method. It turns out to be more complicated than in the local case, but it can be used to advantage for the convergence proof and for a number of important and non-trivial calculations—self-mass of the electron, anomalous magnetic moment of the electron, proton-neutron mass-difference, etc.

The paper consists of three sections altogether. In the first section the calculating technique just mentioned is developed. In the second, the proof of convergence is given. In the last section the self-mass of the electron and the magnetic moment are discussed.

§ 1. Form of the S-matrix

In I, $S(t)$ was defined (in interaction representation) by

$$\left.\begin{aligned} i\,\frac{\partial S(t)}{\partial t} &= H(t)\,S(t) \\ S(-\infty) &= 1. \end{aligned}\right\} \tag{1.1}$$

H_0, the free-field Hamiltonian, was the same as in the local theory and the non-locality entered through $H(t)$ i.e.

$$H(t) = \int d^3x \int d^4(x'\, x''\, x''')\, \bar{\psi}(x')\, \gamma^\mu \psi(x'')\, A_\mu(x''')\, F(x-x', x-x'', x-x''') \tag{1.2}$$

$$= \int d^3x \int d^4(k'k''k''')\, \bar{\psi}(k')\, \gamma^\mu \psi(k'') A_\mu(k''')\, f(k', k'', k''')\, e^{-i(k''-k'-k)x} \tag{1.3}$$

where f, the form-factor, and $\bar{\psi}(k')$ etc. are the Fourier transforms of $F(x-x', x-x'', x-x''')$, $\bar{\psi}(x')$ etc. F is the non-local kernel and satisfies, in general, only the necessary normalization condition (I, (4)

$$f(k'', k'', o) = 1 \tag{1.4}$$

and the Hermiticity condition

$$f(k'\, k''\, k''') = f^*(k''\, k'\, k''') \tag{1.5}$$

A general F satisfying only (1.4), (1.5) does not produce Lorentz invariance even in the *first* order of perturbation theory. The condition for first order Lorentz-invariance is that the arguments of f be invariants i.e.

$$f(k', k'', k''') = f(\boldsymbol{k'}\cdot\boldsymbol{k''}, \boldsymbol{k''}\cdot\boldsymbol{k'''}, \boldsymbol{k'''}\cdot\boldsymbol{k'})^*). \tag{1.6}$$

Throughout this paper we shall assume that f satisfies (1.6). A particularly simple F (still satisfying (1.6), of course) is

$$\left.\begin{aligned} F(x-x', x-x'', x-x''') &= G(x-x', x-x'')\, \delta^4(x-x''') \\ f(\boldsymbol{k'}\cdot\boldsymbol{k''}, \boldsymbol{k''}\cdot\boldsymbol{k'''}, \boldsymbol{k'''}\cdot\boldsymbol{k'}) &= g(\boldsymbol{k'}\cdot\boldsymbol{k''}) \end{aligned}\right\} \tag{1.7}$$

In g only Fermion momentum variables occur (cf. (1.3)).

Solving (1.1), by iteration, we have as usual for $t = \infty$

$$S \equiv S(\infty) = 1 + \sum_{m=1}^{\infty} (-i)^m \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{m-1}} dt_m\, H(t_1)\, H(t_2) \ldots H(t_m). \tag{1.8}$$

In this section we shall formulate a method of calculating S analogous to the Feynman-Dyson method. In the next section we shall discuss the convergence of S. From (1.8)

$$S_{jk}^{(n)} = \langle j \mid (-i)^n \int_{-\infty}^{\infty} dt_1 \ldots dt_n\, H(t_1) \ldots H(t_n) \mid k \rangle \tag{1.9}$$
$$t_1 > t_2 > \ldots > t_n$$

Let us try to mould (1.9) into the Feynman-Dyson form. First

$$S_{jk}^{(n)} = \langle j \mid \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \ldots d^4 y_\alpha \ldots P\{H(y_1) \ldots H(y_\alpha) \ldots H(y_n)\} \mid k \rangle \tag{1.10}$$

*) $\boldsymbol{A}$ means 4-vector, $\vec{A}$ three-vector. The invariants $\boldsymbol{k'^2}$, $\boldsymbol{k''^2}$, $\boldsymbol{k'''^2}$ are trivial.

where (from (1.3))

$$H(y_\alpha) = \int d^4(k'_\alpha k''_\alpha k'''_\alpha)\, \bar{\psi}(k'_\alpha)\, y^\mu \psi(k''_\alpha)\, A_\mu(k'''_\alpha)\, f_\alpha(k'_\alpha, k''_\alpha, k'''_\alpha) e^{-i(k''_\alpha - k'_\alpha - k'''_\alpha)y_\alpha} \quad (1.11)$$

We are assuming that f_α satisfies (1.6).

P is the Dyson chronological operator. In contrast to the local case P is not necessarily Lorentz invariant. This is because here P operates on the y_α, not on y'_α, y''_α, y'''_α, the arguments of $\bar{\psi}, \psi, A$. The condition for the Lorentz-invariance of P is the usual

$$[H(x), H(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \quad (1.12)$$

Now all the momentum-space integrals occurring in the calculation of (1.10) come from *contractions*. These occur in the following way. Some $A_\mu(y'''_\beta)$ (or some $\bar{\psi}(y'_\beta)$, or $\psi(y''_\beta)$) creates a virtual particle which is then destroyed by another A (or $\bar{\psi}$ or ψ), $A_\mu(y'''_\alpha)$ say, standing to the left of $A(y'''_\beta)$ i.e. $y^0_\alpha > y^0_\beta$. (If it is destroyed in $\langle j|$ there is no contraction, and no integral). Then for $y^0_\alpha < y^0_\beta$ we get the reverse process and the effect of both processes on $S^{(n)}_{jk}$ may be written

$$y^0_\alpha > y^0_\beta\ S^{(n)}_{jk} = \langle j \int \ldots P \ldots \langle 0 \text{ photon} \mid A_\mu(k'''_\alpha)\, e^{iy_\alpha k'''_\alpha} f_\alpha(\ldots k'''_\alpha)$$
$$\ldots A_\mu(k'''_\beta) e^{iy_\beta k'''_\beta} f_\beta(\ldots k'''_\beta) \mid 0 \text{ photon} \rangle \ldots \mid k \rangle$$

$$y^0_\alpha < y^0_\beta\ S^{(n)}_{jk} = \langle j \int \ldots P \ldots \langle 0 \text{ photon} \mid A_\mu(k'''_\beta)\, e^{iy_\beta k'''_\beta} f_\beta(\ldots k'''_\beta)$$
$$\ldots A_\mu(k'''_\alpha) e^{iy_\alpha k'''_\alpha} f_\alpha(\ldots k'''_\alpha) \mid 0 \text{ photon} \rangle \ldots \mid k \rangle \quad (1.13)$$

P is now superfluous for $y^0_\alpha - y^0_\beta$, of course. Calculating these expressions in the usual way we get

$$y^0_\alpha \lessgtr y^0_\beta\ S^{(n)}_{jk} = \langle j \mid \int \ldots P$$
$$\ldots \frac{d^3 k'''_{\alpha\beta}}{\omega'''_{\alpha\beta}} e^{\pm i\omega'''_{\alpha\beta}(y^0_\alpha - y^0_\beta) - i\vec{k}'''_{\alpha\beta}(\vec{y}_\alpha - \vec{y}_\beta)} f_\alpha\left(\ldots \begin{matrix} \pm\omega'''_{\alpha\beta} \\ \vec{k}'''_{\alpha\beta} \end{matrix}\right) f_\beta\left(\ldots \begin{matrix} \mp\omega'''_{\alpha\beta} \\ \vec{k}'''_{\alpha\beta} \end{matrix}\right) \ldots \mid k \rangle \quad (1.14)$$

where $\omega'''^2_{\alpha\beta} = \vec{k}'''^2_{\alpha\beta} + m^2_{\alpha\beta}$, $m_{\alpha\beta}$ being the relevant mass (zero in this case, actually, but we leave it standing).

The step which leads to the Feynman-Dyson method is made at this point. In the local theory ($f_\alpha = f_\beta = 1$) the two expressions of (1.14), one for $y^0_\alpha > y^0_\beta$, one for $y^0_\alpha < y^0_\beta$, can be combined into a *single* expression, valid for *all* $y^0_\alpha - y^0_\beta$, i.e.

$$S^{(n)}_{jk} = \langle j \int \ldots P \ldots \frac{d^4 k'''_{\alpha\beta}}{k'''^2_{\alpha\beta} - m^2_{\alpha\beta} - i\varepsilon} e^{ik'''_{\alpha\beta}(y_\alpha - y_\beta)} \ldots \mid k \rangle \quad (1.15)$$

Combining in this way for all contracted variables (Wicks theorem) one arrives at the Feynman-Dyson matrix elements, which can be written down from the corresponding Feynman graphs and which are so much simpler than the 'elementary perturbation theory' expressions, obtained essentially by *not* combining the two parts of (1.14).

We try to carry out the same procedure in the non-local case. We write as Ansatz, in analogy to (1.15)

$$S_{jk}^{(n)} = \langle j \mid \int \ldots P \ldots \frac{d^4 k_{\alpha\beta}'''}{\boldsymbol{k}_{\alpha\beta}^2 - m_{\alpha\beta}^2 - i\varepsilon} e^{i\boldsymbol{k}_{\alpha\beta}'''(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)} G\,(k_{\alpha\beta}'''^0,\ \omega_{\alpha\beta}''',\ \vec{k}_{\alpha\beta}''' \ldots) \ldots \mid k \rangle \tag{1.16}$$

and determine G by multiplying (1.16) and (1.14) each by $e^{-i\boldsymbol{k}_{\alpha\beta}(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)}$ and integrating over *all* $(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)$. We find

$$G(k_{\beta\alpha}^0\ \omega_{\alpha\beta}\ \vec{k}_{\alpha\beta} \ldots) = \tfrac{1}{2}\left\{ f_\alpha\left(\ldots \begin{matrix}\omega_{\alpha\beta} \\ \vec{k}_{\alpha\beta}\end{matrix}\right) f_\beta \left(\ldots \begin{matrix}-\omega_{\alpha\beta} \\ \vec{k}_{\alpha\beta}\end{matrix}\right) + f_\alpha \left(\ldots \begin{matrix}-\omega_{\alpha\beta} \\ \vec{k}_{\alpha\beta}\end{matrix}\right) f_\beta \left(\ldots \begin{matrix}+\omega_{\alpha\beta} \\ \vec{k}_{\alpha\beta}\end{matrix}\right)\right\}$$

$$+ \frac{k_{\alpha\beta}^0}{2\,\omega_{\alpha\beta}} \left\{ f_\alpha \left(\ldots \begin{matrix}\omega_{\alpha\beta} \\ \vec{k}_{\alpha\beta}\end{matrix}\right) f_\beta \left(\ldots \begin{matrix}-\omega \\ \vec{k}\end{matrix}\right) - f_\alpha \left(\ldots \begin{matrix}-\omega \\ \vec{k}\end{matrix}\right) f_\beta \left(\ldots \begin{matrix}\omega \\ \vec{k}\end{matrix}\right) \right\} \tag{1.17}$$

Thus, in an obvious notation,

$$S_{jk}^{(n)} = \langle j \mid \int \ldots P \ldots \frac{d^4 k_{\alpha\beta}'''}{\boldsymbol{k}_{\alpha\beta}'''^2 - m_{\alpha\beta}^2 - i\varepsilon} e^{i\boldsymbol{k}_{\alpha\beta}'''(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)}$$

$$\left[\tfrac{1}{2}\{f_\alpha(+)\, f_\beta(-) + f_\alpha(-)\, f_\beta(+)\} + \frac{k_{\alpha\beta}'''^0}{2\,\omega_{\alpha\beta}} \{f_\alpha(+)\, f_\beta(-) - f_\alpha(-)\, f_\beta(+)\} \right] \tag{1.18}$$

is the non-local analogue of (1.15) and reduces again to (1.15) in the local limit $f = 1$. We can regard $G \mid \boldsymbol{k}^2 - \mu^2 - i\,\varepsilon$ as a modified propagator replacing the local $1 \mid \boldsymbol{k}^2 - \mu^2 - i\,\varepsilon$. Note that in (1.18) the spacetime integration has not yet been carried out—we have not as yet applied the δ-functions which occur for the momentum variables.

It might have been expected that instead of (1.18) we would get

$$S_{jk}^{(n)} = \langle j \mid \int \ldots P \ldots \frac{d^4 k_{\alpha\beta}'''}{\boldsymbol{k}_{\alpha\beta}'''^2 - m_{\alpha\beta}^2 - i\varepsilon} e^{i\boldsymbol{k}_{\alpha\beta}'''(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)} f_\alpha \left(\begin{matrix}k_{\alpha\beta}^{0\,'''} \\ \vec{k}_{\alpha\beta}'''\end{matrix}\right) f_\beta \left(\begin{matrix}-k_{\alpha\beta}'''^0 \\ \vec{k}_{\alpha\beta}'''\end{matrix}\right) \ldots \mid k \rangle \tag{1.19}$$

which is Lorentz-invariant for f satisfying (1.6). Instead, we get (1.18), which is not, generally, invariant for $n > 1$ (cf. § 3). In a sense it is an advantage not to arrive at (1.19) for $S_{jk}^{(n)}$ because it is already known that an $S_{jk}^{(n)}$ of the form (1.19) is not, generally, convergent[5)][6)]. Although (1.19) is not itself the correct expression, it will be useful for formulating a rule for writing down the correct $S_{jk}^{(n)}$.

So far we have discussed only a single contraction, and indeed only a Boson contraction. (It is shown below, however, that Fermion contractions lead to similar expressions.) What we want now is the analogue of (1.18) for a number of contractions carried out simultaneously (Wicks theorem). With the help of (1.19) we shall formulate a method for writing it down straight from the corresponding Feynman graph.

We remark first that in (1.19) only *one* factor $f_\alpha f_\beta$ occurs (in (1.18) there are four) so that the analogue of (1.19) for a number of simultaneous contractions is obtained by writing down the local theory matrix element, multiplying it by the *total* form-factor $\Phi \equiv \prod_{y=1}^{y=n} f_y$, (a product of n f_y's, each with three momentum variables) and then changing every pair of *contracted* variables k'''_{y_1}, k'''_{y_2} occurung in Φ into $(\mathrm{k}'''^0_{y_1 y_2}\ \vec{k}'''_{y_1 y_2})$ and $(-\ k'''^0_{y_1 y_2}\ \vec{k}'''_{y_1 y_2})$ respectively.

Next we remark that for the contraction $\alpha\,\beta$ above we can pass from (1.19) to (1.18) by making the transition

$$f_\alpha(\ldots{}^{k_0}_{\vec{k}})\, f_\beta(\ldots{}^{-k_0}_{\vec{k}}) \to \tfrac{1}{2}[f_\alpha(\ldots{}^{\omega}_{\vec{k}})\, f_\beta(\ldots{}^{-\omega}_{\vec{k}}) + f_\alpha(\ldots{}^{-\omega}_{\vec{k}})\, f_\beta(\ldots{}^{\omega}_{\vec{k}})]$$
$$+ \frac{k_0}{2\,\omega}\,[f_\alpha(\ldots{}^{\omega}_{\vec{k}})\, f_\beta(\ldots{}^{-\omega}_{\vec{k}}) - f_\alpha(\ldots{}^{-\omega}_{\vec{k}})\, f_\beta(\ldots{}^{\omega}_{\vec{k}})] \quad (1.20)$$

The process of making the transition (1,20) we call 'linearizing' $f_\alpha f_\beta$ with respect to $k^0_{\alpha\beta}$, because the right hand side of (1.20) contains $k^0_{\alpha\beta}$ only linearly. Further, consider the total form-factor $\Phi = \prod_{y=1}^{n} f_y$. As yet we have not integrated over space and time and have therefore not used the δ-functions in the momentum-space variables, so that at the present stage f_α and f_β are the *only* f_y's containing $k'''_{\alpha\beta}$. Thus the transition (1.19) $\to$ (1.18) may be regarded not only as a linearization of $f_\alpha f_\beta$ with respect to $k'''^0_{\alpha\beta}$ but as a linearization of the *total* form-factor with respect to $k'''^0_{\alpha\beta}$. This is true for every contraction, only we must note that it is true only *before* the application of the δ-functions.

The general method for writing down the analogue of (1.18) for a number of contractions is then evident: 'write down the local theory matrix element (in momentum space) from the corresponding Feynman graph (but before the use of the δ-functions at the corners), multiply *as in (1.19)* by the 'wrong' (non-linearized) total form-factor and then linearize this with respect to *all* contracted variables.' After linearization the δ-functions may be used. (The rule must be modified a little for Fermion propagators, see below. An explicit example of the use of this method is given in § 3).

In this way the Feynman-Dyson formalism *can* be retained in the non-local case. But it is only in a somewhat formal sense. The main advantage

of the formalism in the local case is the ease and simplicity it produces in the calculations. Here, a great deal of this advantage is lost, on account of the number of terms obtained after linearization according to the above method ($4\,k$ terms, *in general* (not always) where $k =$ number of contracted variables). So the rule can be applied to advantage only in certain cases (cf. § 3). The number of these cases is, of course, greater when $F(x-x', x-x'', x-x''') = G(x-x', x-x'')\,\delta^4(x-x''')$.

The alternative to using the rule above is to use the two terms of (1.14) *separately*, which is nothing other than using 'elementary perturbation theory'. This is also an unwieldy procedure and in each particular calculation, one has simply the coice of deciding which of the two methods is the less cumbersome.

We complete this section by considering Fermion contractions. Instead of (1.14) we then have

$$\gamma^0_\alpha > y^0_\beta\ S^{(n)}_{jk} = \langle j \,|\int \ldots P \ldots \langle 0\,\text{Fermion} \,|\, \psi(k''_\alpha)\ e^{-i\boldsymbol{y}_\alpha \boldsymbol{k}''_\alpha} f_\alpha(.\ k''_\alpha\ .)$$
$$\ldots \overline{\psi}\,(k'_\beta)\ e^{i\boldsymbol{y}_\beta \boldsymbol{k}'_\beta}\ f_\beta\,(k'_\beta \ldots) \,|\, 0\ \text{Fermion}) \ldots \,|\, k\rangle \qquad (1.21)$$

with a corresponding term for $y^0_\alpha < y^0_\beta$. These become[7])

$$\gamma^0_\alpha \lessgtr y^0_\beta\ S^{(n)}_{jk} = \langle j \,|\int \ldots P \ldots \frac{d^3k'_{\alpha\beta}}{\omega'_{\alpha\beta}}\ e^{\pm i\omega'_{\alpha\beta}\,(y^0_\alpha - y^0_\beta) - i\vec{k}'_{\alpha\beta}\,(\vec{y}_\alpha - \vec{y}_\beta)}$$
$$f_\alpha\left(\begin{matrix}\pm\omega'_{\alpha\beta}\\ \vec{k}'_{\alpha\beta}\end{matrix}\right) f_\beta\left(\begin{matrix}\pm\omega'_{\alpha\beta}\\ \vec{k}'_{\alpha\beta}\ldots\end{matrix}\right) [\pm\gamma^0\,\omega'_{\alpha\beta} - \vec{k}\cdot\vec{\gamma}'_{\alpha\beta} + m_{\alpha\beta}] \ldots \,|\, k\rangle \qquad (1.22)$$

In the local case these terms combine into

$$S^{(n)}_{jk} = \langle j \,|\int \ldots P \ldots \frac{d^4k'_{\alpha\beta}}{\boldsymbol{k}'^2_{\alpha\beta} - m^2_{\alpha\beta} - i\varepsilon}\ e^{i\boldsymbol{k}'_{\alpha\beta}\,(\boldsymbol{y}_\alpha - \boldsymbol{y}_\beta)}\ [\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{\gamma} + m_{\alpha\beta}] \ldots \,|\, k\rangle \qquad (1.23)$$

In the non-local case we make the Ansatz (1.16) as before, and for this contraction it turns out that

$$G(k^0, w, \vec{k}\ldots) = \tfrac{1}{2}\{f_\alpha(\tfrac{\omega}{\vec{k}})\ f_\beta(\tfrac{\omega}{\vec{k}}\ldots)\ [\gamma^0\omega - \vec{\gamma}\cdot k$$
$$+ m] + f_\alpha(\tfrac{-\omega}{\vec{k}})\ f_\beta\,(\tfrac{-\omega}{\vec{k}}\ldots)\ [-\gamma^0\omega - \vec{\gamma}\cdot\vec{k} + m]\}$$
$$+ \frac{k^0}{2\,\omega}\{f_\alpha(\tfrac{\omega}{\vec{k}})\ f_\beta(\tfrac{\omega}{\vec{k}}\ldots)\ [\gamma^0\omega - \vec{\gamma}\cdot\vec{k} + m] - f_\alpha(\tfrac{-\omega}{\vec{k}})\ f_\beta(\tfrac{-\omega}{\vec{k}}\ldots)\ [-\gamma^0\omega - \vec{\gamma}\cdot\vec{k} + m]\} \qquad (1.24)$$

in other words, what is linearized in this case is not the form factor alone, but the form-factor times $[\gamma^0\,k_0 - \vec{\gamma}\cdot\vec{k} + m]$. Note that for Fermion propagators the products $f_\alpha(^+)\ f_\beta(^+)$ and $f_\alpha(^-)\ f_\beta(^-)$ occur. For Bosons (cf. 1.20) we had $f_\alpha(^+)\ f_\beta(^-)$ and $f_\alpha(^-)\ f_\beta(^+)$.

The general method for writing down matrix elements formulated above must therefore be modified to writing the local propagators in the form $\gamma \cdot k + m \mid k^2 - m^2$ (not $1 \mid \gamma \cdot k - m$) and then linearizing, not the total form-factor, but the total form-factor times the local *numerator* (a product of terms of the type $\gamma \cdot k + m$, here).

This general method of calculation has already been used explicitly in a non-trivial case—the calculation of the neutron-proton mass-difference[8]). Incidentally, the rule 'linearize the total form-factor times the numerator' is valid also for derivative couplings.

§ 2. Convergence of the S-Matrix

In this section it will be shown that with a suitable choice of form-factor $S_{jk}^{(n)}$ can be made finite i.e. the ultra-violet divergences can be removed from the theory. (We shall not be concerned, of course, with the convergence of the expansion $\sum_1^\infty e^n S_{jk}^{(n)}$).

The general method of proof is as follows: As a preliminary it is shown that the integration over the *fourth* component of all the integration variables occurring in $S_{jk}^{(n)}$ is convergent, independently of the form-factor. Then, in the proof proper, it is shown that the form-factor can be chosen to make the remaining (threedimensional) integrations convergent. This is shown first for the general form-factor $F(x - x', x - x'', x - x''')$ and then for the case $F = G(x - x', x - x'')\, \delta^4(x - x''')$. In the proof, no claim is made to mathematical rigour, and for simplicity a small finite mass will be assigned to the photon.

We proceed now to the preliminary stage of the proof, i.e. the proof that the fourth component integration converges. According to § 1

$$S_{jk}^{(n)} = \int \text{'local theory expression'} \tag{2.1}$$

$\times$ 'linearized (form factor $\times$ numerator)' where the 'linearized (form factor $\times$ numerator)' is a finite sum of terms of the form*)

$$W^{(n)} = \prod_{y=1}^{n} f_\gamma \begin{pmatrix} \pm\omega_y' & \pm\omega_y'' & \pm\omega_y''' \\ \vec{k}_y' & \vec{k}_y'' & \vec{k}_y''' \end{pmatrix} \times [\pm\gamma^0\omega_y' - \vec{\gamma}\cdot\vec{k}_y' + m]\,[\pm\gamma^0\omega_y'' - \vec{\gamma}\cdot\vec{k}_y'' + m] \times \left(\frac{k_y'^0}{\omega_y'}\right)^{\delta_y'} \left(\frac{k_y''^0}{\omega_y''}\right)^{\delta_y''} \left(\frac{k_y'''^0}{\omega_y'''}\right)^{\delta_y'''} \tag{2.2}$$

with every $\delta = 0$ or 1. Assuming that f satisfies (1.6) (first order Lorentz-invariance) f_γ will be

$$f_\gamma\,(\omega_y'\,\omega_y'' \pm \vec{k}_y'\cdot\vec{k}_y'',\ \omega_y''\,\omega_y''' \pm \vec{k}_y''\cdot\vec{k}_y''',\ \omega_y'''\,\omega_y' \pm \vec{k}_y'''\cdot\vec{k}_y') \tag{2.3}$$

*) Apart from external variables.

We need discuss only the general term (2.2) of the sum. The contribution to $S_{jk}^{(n)}$ of (2.2) is essentially*)

$$S_{jk}^{(n)} = \prod_{t=1}^{t=Fi+Bi} \int d^4x \int \frac{d^4 k_t}{\boldsymbol{k}_t^2 - m_t^2 - i\varepsilon} \; e^{i\boldsymbol{k}_t(x_\alpha^t - x_\beta^t)} \; W^{(n)}(k_t^0, \omega_t, \vec{k}_t \ldots) \tag{2.4}$$

where F_i and B_i are the numbers of internal Fermion and Boson lines respectively, $\boldsymbol{k}_t$ is the variable of any internal contracted line (contracted between the corners x_l^α and x_β^t), and m_t is the mass of the corresponding particle.

The time integrations in (2.4) (represented by dx_0) lead to one 'overall δ-function' (energy conservation) and $n-1$ internal δ-functions, which reduce the $B_i + F_i$ independent variables k_t^0 to $B_i + F_i - n + 1$ variables (k_s^0 say). The remaining $n-1$ variables (k_l^0 say) become linear functions of the k_s^0 and of the external energies (k_e^0 say). It is shown in appendix A that *for each k_s^0, at least one k_l^0 is a linear function of it.*

Now in (2.4) on account of (2.2) the fourth components of the variables appear only in the form

$$K = \prod_{s=1}^{Fi+Bi-n+1} \frac{(k_s^0|\omega_s)^{\delta_s}}{k_s^{0^2} - \omega_s^2 - i\,\varepsilon_s} \prod_{l=1}^{n-1} \frac{(k_l^0|\omega_l)^{\delta_s}}{k_l^{0^2} - \omega_l^2 - i\varepsilon_l} \tag{2.5}$$

$$\omega^2 = \boldsymbol{k}^2 + m^2$$

On account of what we have just said, any k_s^0 ($k_{s_1}^0$ say) occurs linearly in at least one of the k_l^0. As a result, for $|k_{s_1}^0| \to \infty$, K behaves like $1 \,/\, |k_{s_1}^{0^m}|$ $m \geqslant 2$ and it is easy to see from (2.5) that this is true no matter how the other k_s^0 vary (even if they, too, tend to ∞). In other words,

$$K \to 1 \,/\, |k_s^{0^m}| \quad m \geqslant 2 \quad |k_s^0| \to \infty \quad \textit{uniformly} \tag{2.6}$$

But (2.6) is a sufficient condition for the convergence of the fourth component ponent integration, so that the preliminary proof is now complete. Note that no use has been made of the explicit form of the form-factor in deriving (2.6).

We proceed now to the proof proper, the proof of the convergence of the remaining (three dimensional) integrations for suitable choices of the form-factor. Let us consider the fourth component integration in (2.4) as already carried out (by the method of residues). The remaining integrand is a sum of terms consisting of

(a) an algebraic function of the variables $\vec{\boldsymbol{k}}_t$ and ω_t coming from (2.5) and the $[\pm\gamma^0\omega' - \vec{\gamma}\cdot\vec{k}' + m]\,[\pm\gamma^0\omega'' - \vec{\gamma}\cdot\vec{k}'' + m]$ part of $W^{(n)}$ in (2.2);

*) Apart from external variables.

(b) $n-1$ δ^3-functions in the $\vec{k}_t$ (and external momenta) coming from the d^3x integration;

(c) the $\prod_{\gamma=1}^{n} f_\gamma$ part of $W^{(n)}$ in (2.2).

This means that, apart from $\prod_{\gamma=1}^{n} f_\gamma$, the integrals can diverge at worst algebraically. Our task is to show that f_γ can be chosen so as to prevent this algebraic divergence.

Consider first the case $F = F(x-x', x-x'', x-x''')$. In this case we shall retain all the $B_i + F_i$ variables $\vec{k}_t$ i.e. we shall not *use* the $n-1$ δ^3-functions in the integrand. If now for each of the variables we can show

(1) that it occurs in at least one of the form factors f_γ;
(2) that one of the three arguments of this f_γ (which are in any case positive indefinite, $\omega'\omega'' \pm \vec{k}'\vec{k}'' \geqslant 0$) behaves like $\vec{k}_t^m$ $m > 0$ for $\vec{k}_t$ large i.e. increases monotonically with some power of $\vec{k}_t$, then it is easy to see that f can be chosen to ensure convergence. We need only choose $f = f(a, b, c)$ to be a function of its arguments a, b, c such that it decreases faster than any power of a, b, c for large a, b, c (e. g. $f = e^{-\frac{a^2+b^2+c^2}{\lambda^2}}$) and then $\prod_{\gamma=1}^{n} f_\gamma$ will decrease faster than any power of any $\vec{k}_t$ for large $\vec{k}_t$. Our task reduces therefore to showing that the conditions (1) and (2) hold for each $\vec{k}_t$.

Consider first all the variables, $\boldsymbol{k}_t^\alpha$ say, whose lines in a Feynman graph meet *external* lines at one corner (at least). The form-factor at this corner contains the argument

$$x = a, b \text{ or } c = \sqrt{\vec{k}_t^{\alpha 2} + m_t^2}\sqrt{\vec{k}_e^2 + m_e^2} - |\vec{k}_t^\alpha||\vec{k}_e| + |\vec{k}_t^\alpha||\vec{k}_e|(1 \mp \cos \hat{k_t^\alpha k_e}) \quad (2.7)$$

where $\vec{k}_e$ is the external momentum. Hence for the $\vec{k}_t^\alpha$ condition (1) is satisfied. Assuming, now, that the photon mass is finite ($m_e \neq 0$), (27) will be a monotonically increasing function of $\vec{k}_t^\alpha$ for large $\vec{k}_t^\alpha$ (since even when $\cos \hat{k_t^\alpha k_e} = \pm 1$, $\sqrt{\vec{k}_e^2 + m_e^2} \neq |k_e|$). So condition (2) is satisfied. Hence for the k_t^α the integration can certainly be made convergent.

Now comes the crucial point. Because the integrations over the $\vec{k}_t^\alpha$ are convergent, we can regard the $\vec{k}_t^\alpha$ as *finite* (cut-off at some finite value determined by the cut-off parameter in f, λ say, or better still, we can make a small error by summarily cutting them off at some finite value higher than that determined by λ). Then the $\vec{k}_t^\alpha$ will also satisfy $\sqrt{\vec{k}_t^{\alpha 2} + m_t^2} \neq |\vec{k}_t^\alpha|$ and if we consider the variables, $\vec{k}_t^\beta$ say, whose lines meet the $\vec{k}_t^\alpha$

lines at one corner (at least) then, in exactly the same way we can deduce that the integral over the $\vec{k}_t^{\beta}$ can be made convergent. Repetition of this process shows that the integration over all the variables $\vec{k}_t$ can be made convergent, for if after a certain number of steps, there remain only variables whose lines do not meet at a corner either the lines already considered, or external lines, then the remaining lines form the graph of a vacuum effect, which should be neglected anyway. This completes the proof for $F = F(x - x', x - x'', x - x''')$. The method of producing convergence just outlined has already been used explicitly in a non-trivial case, the graphs of Figure I b, e, c, which occur in the proton-neutron mass-difference calculation[8]).

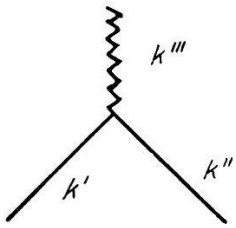

Fig. 1

We now consider the case $F = G(x - x', x - x'')\,\delta^4(x - \,''')$. This case is more difficult because of the $F_i + B_i$ variables only the F_i Fermion variables appear in the form-factors, e.g. at the corner shown in Figure 1, f has only the single argument

$$a_{\pm} = \sqrt{\vec{k}''^2 + m^2}\,\sqrt{\vec{k}'^2 + m^2} \pm \vec{k}' \cdot \vec{k}'' \tag{2.8}$$

However if we use the δ^3-function at this corner, we have the extra condition

$$\vec{k}' + \vec{k}''' - \vec{k}'' = 0 \tag{2.8}$$

which either eliminates $\vec{k}'''$ as a variable, or introduces it into the form-factor at the expense of $\vec{k}'$ or $\vec{k}''$, e.g.

$$a_{\pm} = \sqrt{\vec{k}'^2 + m^2}\,\sqrt{(\vec{k}' + \vec{k}''')^2 + m^2} \pm \vec{k}' \cdot (\vec{k}' + \vec{k}''') \tag{2.9}$$

In this way all the *independent* variables occur in at least one form-factor. From (2.8) and (2.9) we can see that

(a) if $\vec{k}'$ is finite, $a_{\pm}$ is monotonic in $\vec{k}''$ or $\vec{k}'''$ for large $\vec{k}''$, $\vec{k}'''$, (whichever is the independent variable)

(b) if $\vec{k}''$ is finite, $a_{\pm}$ is monotonic in $\vec{k}'$ or $\vec{k}'''$ for large $\vec{k}'$, $\vec{k}'''$ (whichever is the independent variable) *but*

(c) if $\vec{k}'''$ is finite, a_- is *not* monotonic in $\vec{k}'$ or $\vec{k}''$ (whichever is the independent variable), although a_+ is.

Our task is, therefore, to establish the convergence making use of only (a) and (b).

Consider first the Fermion lines of a graph. These occur only in (1) continuous open chains with external variables at either end, or (2) closed loops. For any one of these open chains consider one of the external variables at the end. The momentum of this line is certainly finite (constant). Therefore at the corner where this line splits into two internal lines (Boson + Fermion) (a) and (b) tell us that the integration over whichever of these we regard as independent can be made convergent, just as discussed above for the case $F(x - x', x - x'', x - x''')$.

Hence this independent variable (and therefore on account of (2.8) the other dependent one too) can be regarded as finite. Thus the second variable in the Fermion chain is finite. Proceeding to the next corner we can deduce similarly that the third Fermion variable is finite and its integration can be made convergent, and so on. The result is that all the Fermion variables of a chain can be regarded as finite (and the integration over any of them which are to be integrated will be convergent).

For closed loops, on the other hand, it is well known that all the Fermion variables can be expressed in terms of one of them (+ Boson variables). It is shown in appendix B that the integration over this variable can always be made convergent, so that this variable anyway can be regarded as finite. And so, starting with this one we can then proceed around the loop just as we did along the open Fermion line and establish that all the other Fermion variables are finite too.

Thus the integrations over *all* the independent Fermion variables in a graph can be made finite, and *all* the Fermion variables may be regarded as finite. From (a) and (b) it then follows that the same is true for the Boson variables. This completes the proof for the case $F = G(x - x', x - x'')\,\delta^4(x - x''')$.

The primary purpose of this paper has thus been achieved. The non-local theories of I and II, embody within them the possibility of convergence. As considered above, the form-factor f is not algebraic, but a closer, though much more tedious, examination shows that algebraic functions will also be sufficient to produce convergence. In practice (§ 3) we have found that even a function which tends quite slowly to zero for large values of its arguments will produce convergence. This is hardly surprising in view of the mild nature of the divergence in quantum-electrodynamics. Incidentally, the above proof (including the preliminary proof) for *non*-algebraic f holds also for derivative couplings.

It might be asked why the present theory produces convergence and the Christensen-Møller theory does not. In terms of form-factors, the answer is that in the present theory only time-like momentum vectors

occur in the form-factor in all orders of perturbation, wheres in the Christensen-Møller theory space-like vectors appear in the higher orders. The divergence of that theory can be traced to these space-like vectors. The space-like vectors do not occur in the present theory on account of the way in which the form-factor is introduced – via the Interaction Hamiltonian in the interaction picture.

§ 3. Some Calculations

In this section we carry out some explicit calculations. The most important of these is the calculation of the self-energy of the electron which we discuss in detail. We discuss also the calculation of the induction tensor and of the magnetic moment of the electron but, on account of the theoretical ambiguity still associated with these, we do not discuss them in detail.

For definiteness we use the (c-number) form-factor already proposed by ARNOUS and HEITLER[9])

$$f^2 = \frac{\lambda^4}{\lambda^4 + (p_0 q_0 - \vec{p}\cdot\vec{q})^2 - (p_0^2 - \vec{p}^2)\,(q_0^2 - \vec{q}^2)} \tag{3.1}$$

where p and q are the momenta of the two electron lines at a corner of a Feynman graph (corresponding to $\boldsymbol{k}'$, $\boldsymbol{k}''$ of Fig. 1) and $\lambda^4 = m^2 K^2$ where m is the electron mass and K is an assumed universal cut-off of the order of magnitude of the *nucleon* mass. (3.1) is one of the simplest form-factors satisfying (1.4) (1.5) and (1.6). It is sufficiently strong to make the electron self-energy (second order in e) convergent. A stronger form-factor may be necessary for general convergence, but, in principle, for the self-energy the effect of (3.1) should be the same as the stronger one.

a) *Electron Self-energy*

As mentioned in II, in the non-local theory it is not quite clear whether the expectation value of the energy is

$$\langle P_4 \rangle \equiv \langle H_0 + H \rangle \text{ or } \langle P_4^\theta \rangle = \langle P_4 + \theta \rangle$$

where P_4^θ is a gauge-invariant tensor. (In the local case these expectation values are identical). *Here we shall assume that it is* $\langle P_4 \rangle$. The self-energy of an electron is then (with 'switch-on')

$$\langle t \mid P_4 \mid t \rangle - \langle -\infty \mid H_0 \mid -\infty \rangle \tag{3.2}$$

for a one-electron state $|-\infty\rangle$, when $t\rangle$ is the state at a time t with the interaction fully switched on. Using (II (60))

$$\left.\begin{aligned}\text{Self-energy} &= \langle t \mid P_4 \mid t\rangle - \langle -\infty \mid H_0 \mid -\infty\rangle \\ &= \langle -\infty | S^{-1}(t)(H_0+H)\, S(t) - H_0 | -\infty\rangle \\ &= \underset{T=\infty}{\overset{Lt}{}} \frac{i}{2T} \langle -\infty \mid S(T) - S^*(T) | -\infty\rangle \\ &= \underset{T=\infty}{\overset{Lt}{}} \frac{i}{T} \langle -\infty | S(T) - 1 | -\infty\rangle\end{aligned}\right\} \quad (3.3)$$

Here $S(T)$ is the usual S-matrix, and so we can calculate the self-energy directly from S. We consider only the order e^2. The corresponding Feynman diagram is as shown in Figure 2.

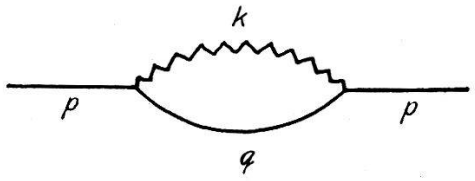

Fig. 2

According to § 1, the S-matrix is obtained by writing down the local theory integrand in momentum space (before the use of the δ-functions at corners 1, 2)[10])

$$\frac{e^2}{(2\pi)^3} \frac{m}{E_p}\, i(-i)\,(-i)^2\, \bar{u}(p)\, \gamma^\mu \frac{1}{\boldsymbol{k}^2-\mu^2} \frac{\boldsymbol{\gamma}\cdot\boldsymbol{q}+m}{\boldsymbol{q}^2-m^2}\, \gamma_\mu\, u(p) \quad (3.4)$$

multiplying it by the 'wrong' (non-linearized) form-factor (3.1)

$$-\frac{e^2}{(2\pi)^3} \frac{m}{E_p}\, \bar{u}(p)\, \gamma^\mu \frac{1}{\boldsymbol{k}^2-\mu^2} \frac{\boldsymbol{\gamma}\cdot\boldsymbol{q}+m}{\boldsymbol{q}^2-m^2}\, \gamma_\mu\, u(p) \cdot \frac{\lambda^4}{\lambda^4 + (p_0 q_0 - \vec{p}\cdot\vec{q})^2 - (p_0^2-\vec{p}^2)(q_0^2-\vec{q}^2)} \quad (3.5)$$

and then linearizing with respect to k_0 and q_0 (only q_0 actually, in this case). The result is

$$\begin{aligned}-\frac{e^2}{(2\pi)^3} \frac{m}{E_p} \frac{1}{\boldsymbol{k}^2-\mu^2} \frac{1}{\boldsymbol{q}^2-m^2}\, \bar{u}(p) \Big\{ &(\gamma^0 q_0 - \vec{\gamma}\cdot q + m)\, \frac{f(+)+f(-)}{2} \\ &+ \Big[\gamma^0 E + \frac{q_0}{E}(-\vec{\gamma}\cdot q + m)\Big] \frac{f(+)-f(-)}{2} \Big\}\, u(p), \\ E = \sqrt{\vec{q}^2 + m^2} \quad f(\pm) &= \frac{\lambda^4}{\lambda^4 + (p_0 E \pm \vec{p}\cdot\vec{q})^2 - m^4}\end{aligned} \quad (3.6)$$

The δ-functions $\delta^4(p-k-q)$ then reduce $\int d^4k \int d^4q$ to $\int d^4k$, leaving one overall δ-function $\delta^4(0)$. The integration with respect to k_0 and the properties of γ_s' then lead to*)

*) The $\bar{u}$, u are normalized so that $m\, \bar{u}\, \gamma_\alpha\, u = p_\alpha$.

$$-\frac{e^2}{(2\pi)^3}\frac{m}{E_p}\frac{2\pi i}{m}\delta^4(0)\int\frac{d^3k}{2\omega E[(E+\omega)^2-p_0^2]}$$
$$\times\left\{\begin{array}{l}[-p_0^2E+(\vec{p}^2-\vec{p}\cdot\vec{k}+2m^2)(E+\omega)][f(+)+f(-)]\\ +[-p_0E(E+\omega)+(\vec{p}^2-\vec{p}\cdot\vec{k}+2m^2)p_0][f(+)-f(-)]\end{array}\right\}E=\sqrt{(\vec{p}-\vec{k})^2+m^2}\,\omega=|\vec{k}| \tag{3.7}$$

This expression can also be obtained by an 'elementary perturbation theory' calculation (CH. TERREAUX, unpublished). Using (3.7) in (3.3) we first calculate the self-energy to the order $\vec{p}^2\,|\,m^2$ for $\vec{p}^2 \ll m^2$. The $\delta^4(0)$ disappears due to the normalization of the kets and the interpretation of T in (3.3) as $2\pi\,\delta(0)$, and the result of a straightforward but very lengthy calculation is then

electron self-energy $(\delta E_p) =$

$$=\frac{1}{2\pi\cdot 137}\frac{3m^2}{E_p}\left[\left(\log\frac{2K}{m}-\frac{1}{6}\right)-\frac{2}{3}\frac{\vec{p}^2}{m^2}+0\,\frac{\vec{p}^4}{m^4}+0\,\frac{m^2}{\Lambda^2}\right] \tag{3.8}$$

From the general theory of II, it can be seen that there is no radiative correction to the 3-momentum of an electron so that the self-*mass*, δm, defined by $E_p+\delta E_p=\sqrt{\vec{p}^2+(m+\delta m)^2}$, is given by

$$\delta m_{e^2}\simeq\frac{E_p}{m}\,\delta E_p\simeq\frac{3m}{2\pi\cdot 137}\left[\left(\log\left(\frac{2K}{m}\right)-\frac{1}{6}\right)-\frac{2}{3}\frac{\vec{p}^2}{m^2}\right] \tag{3.9}$$

In a Lorentz-invariant theory (3.9) would be independent of $\vec{p}$. Thus (3.9) already shows that Lorentz-invariance is violated.

More important, one can see that this will be true for *any* c-number form-factor (not just (3.1)) as follows. In (3.7) the explicit form of f has not yet been used, so that (3.7) applies to *any* c-number form-factor. Now any acceptable choice of form-factor must

(a) make (3.7) convergent (and must in particular tend to zero for large $\vec{k}$),
(b) behave in the same way for $\vec{p}$ and $\vec{q}$ (from (1.5), the Hermiticity condition).

(a) means that

$$\underset{|p|=\infty}{Lt}\int d^3k\times(\text{Integrand})=\int d^3k\times\underset{|p|=\infty}{Lt}(\text{Integrand})$$

and (b) means that

$$\underset{|\vec{p}|=\infty}{Lt}(\text{Integrand})\to 0,\ \text{since}\ \underset{|\vec{q}|=\infty}{Lt}(\text{Integrand})=\underset{|\vec{k}|=\infty}{Lt}(\text{Integrand})\to 0.$$

Hence *the self-mass will tend to zero with increasing* $|\vec{p}|$. At high energies the 'bare' mass will be observable. Since the self-mass is not zero for $\vec{p}=0$ (rest system of the electron) it will, therefore, be $\vec{p}$-dependent, and so Lorentz-invariance will be violated.

That $\delta m_{e^2} \to 0$ for $|\vec{p}| \to \infty$ is satisfactory in that δm is therefore always $\lesssim 3\,m/100$ for all $\vec{p}$ (the correction for $\vec{p} = 0$ is approximately $3\,m/100$). Hence the deviation from the Einstein mass-velocity relation due to δm will be $\lesssim 3\%$ (though not necessarily much less for some values of $\vec{p}$). The expansion in (3.9) is valid for $|\vec{p}|\,|\,m \lesssim 1/2$ approximately, and for these values of $|\vec{p}|\,|\,m$ the deviation from the Einstein law predicted by (3.9) is $\sim 0{,}05\%$. At present experimental results are available for $|\vec{p}|\,|\,\mathrm{m} \lesssim 1$ $(\mathrm{v}|\mathrm{c} \sim 0{,}7)$ and these are accurate to about $0{,}1\%$[4]). The above theoretical results are therefore just outside the present range of experiment. We think however that the deviation found with the particular form-factor chosen is too large and that with a more suitable choice of form-factor the deviation might be made considerably smaller. It should be emphasized again, perhaps, that these theoretical results are valid only for *c*-number form-factors, and only under the assumption that $\langle P_4 \rangle$ is the energy.

Note in (3.9) that even when the cut-off K tends to infinity, $\delta m_e{}^2$ does not become invariant, i.e. we do not get the local limit, if by the latter we understand the usual local integrals which tend invariantly to infinity. This is a little surprising, but one must remember that the local integrals are, in fact, ambiguous and their values depend on the method of calculation[11]). That we do not get the local limit in (3.9) means that the *c*-number form-factors considered prescribe a method of calculation different from the usual invariant prescriptions (which is already clear from § 1).

We regard the fact that $\delta m_e{}^2$ does not become invariant even for large K as an indication that *c*-number form-factors are insufficient, and if any satisfactory form factors exist at all *q*-number form-factors will ultimately have to be used to modify the local theory in the proper way.

(b) *The induction tensor and the magnetic moment of the electron:*

Using the methods given in § 1 we have also calculated the induction tensor (the matrix element corresponding to the graph shown in Fig. 3) and the magnetic moment of the electron (given by the usual local theory graphs). The induction tensor is important for calculating the self-energy of the photon[12]) and also because it plays a part in the magnetic moment calculation (Fig. 4). Using a *c*-number form-factor, and again assuming that $\langle P_4 \rangle$ is the energy, we have, so far, not been able to obtain the experimental results for either the photon self-energy (zero) or the magnetic moment. On account of this we refrain from giving the calculation in detail here. It is by no means certain, however, that we shall not in the future be able to obtain the correct (experimental) result, because a number of questions in this connection are still very open.

One question, for example, is the definition of the energy ($\langle P_4 \rangle$ or $\langle P_4 + \theta \rangle$?). Another is the question of renormalization (mass and charge). A third is whether matters could be improved by using q-number form-factors. Finally, even among c-number form-factors, the

Fig. 3

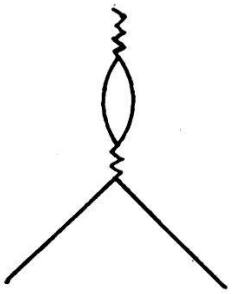

Fig. 4

results (unlike the conclusion drawn for the self-energy) depend on the particular form-factor chosen. It may yet be possible to choose a suitable one. For the two calculations just considered, therefore, our results are by no means conclusive.

Acknowledgements

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Appendix A

It is required to show that on using the n δ-functions at the n corners of a Feynman graph, so that $n-1$ of the variables k_l^0 are expressed in terms of the other $B_i + F_i - n + 1$ variables k_s^0 (and external momenta k_e^0), then for each k_s^0, at least one k_l^0 is a linear function of it.

The n δ-functions at the corners of a graph correspond directly to a set of n linear equations in the k_s^0, k_l^0 and k_e^0 with three terms in each equation

(from the three lines meeting at the relevant corner). Every contracted variable k_s^0 and k_l^0 appears in just two of these equations (the ones corresponding to the corners at each end of the contracted line). Expressing the $n-1$ k_l^0 in terms of the k_s^0, means forming linear combinations of these n equations so that, in one resultant equation only the k_e^0 appear (corresponding to the 'overall' δ-function) and in each of the other $n-1$ resultant equations one k_l^0 is expressed in terms of the k_s^0 and k_e^0. Now no k_s^0 appears (by definition) in the one equation for the k_e^0. Hence each k_s^0 must appear in at least one of the remaining $n-1$ equations, for, if not, it could not be determined at all (in terms of all the other variables) from the new set of n equations, which is impossible, since these are equivalent to the original set, from two of which it *could* be determined. Hence each k_s^0 appears in at least one of the $n-1$ equations for the k_l^0 i.e. at least one k_l^0 is a linear function of it.

Appendix B

It is required to show that if in a closed loop all the Fermion variables are expressed in terms of one of them, the integral over this one can be made convergent.

If c, the number of corners in the loop, is greater than four ($c > 4$) the integral in question is convergent even in the local theory (power of denominator $>$ power of numerator in the integrand). In the non-local case this should be, a fortiori, true since any reasonable form-factor will be $\leqslant 1$ for all values of the variables in it, even if it does not tend monotonically to zero for large values. Hence we need discuss only $c \leqslant 4$ and, indeed, only $c = 4$ and $c = 2$, since $c = 0,1$ are not loops at all and $c = 3$ is taken care of by Furry's theorem (if we invoke C invariance!).

For $c = 4$ the integral without the form-factor is almost convergent—it is logarithmically divergent (in the variable $\vec{k}$ say). And although the arguments of the four form-factors in which $\vec{k}$ appears do not increase monotonically for large $\vec{k}$, this is true only for very special values ($\cos \widehat{k k_i} = \pm 1$) of the angles $\widehat{k k_i}$, where $\vec{k}_i$ are the other momenta occurring in these form-factors cf. (2.9)). Now in addition to the integration over $|\vec{k}|$, there is an *angle* integration to be carried out, and one can see that the angle integration (which may extend over the special values mentioned) will produce one more power of $|\vec{k}|$ (at least) in the denominator. On account of the logarithmic nature of the divergence without a form-factor, this extra power of $|\vec{k}|$ in the denominator will be sufficient to make the integral over $|\vec{k}|$ convergent. This completes the proof for the case $c = 4$.

Admittedly, the argument used is not as rigorous as one might wish, but it seems to us to be correct. The convergent integral

$$\int_0^\infty \frac{dk}{\sqrt{k^2+\mu^2}} \int_{\theta=0}^{\theta=\pi} \frac{d(\cos\theta)}{[1+(k^2+\mu^2)\cos^2\theta]} \tag{B.1}$$

furnishes a simple example of the idea involved.

We consider finally the case $c = 2$. Since in this case there *are* only two corners, at each corner there is either *pair* creation or *pair* destruction. On account of this, in (2.9) only the *positive* sign (i.e. a_+) will appear. For finite Boson momentum external to the loop, a_+ is monotonic in the Fermion variable and the integral (which is quadratically divergent in the local case) can be made convergent. (This can easily be verified explicitly.) That we have assumed the Boson momentum external to these loops finite is no restriction because in any given graph for at least one of the $c=2$ loops the external Boson momenta must come from some other type of Fermion line (already discussed) or else must be external to the whole graph, for otherwise these $c = 2$ loops would form a vacuum effect. For this one loop the Boson momenta will be finite. For the remaining $c = 2$ loops it can then be established in exactly the same way that at least one must have finite external Boson momenta. And so on for all the $c = 2$ loops. This completes the proof for the case $c = 2$, and so for all c.

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