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Further Investigations on the Non-Local Convergent Field Theory

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Summary: The non-local field theory of HEITLER and ARNOUS is compared with experiment in a rather exhaustive manner. It turns out that the theory in its original form is not in agreement with *all* experimental results (in particular it does not predict a zero self-mass for the photon) but that a certain very ad hoc modification of the theory can be introduced so that all the discrepancies between the theory and the present day experimental data vanish. It is pointed out that the Heitler-Arnous theory is so far the only theory which provides a reasonable explanation of the various mass differences.

Introduction

In spite of the striking successes of renormalized field theory, it is becoming increasingly evident that such a theory is insufficient to account for a number of experimental results. The most obvious example of this state of affairs is provided by the various mass-differences ($\pi^+ - \pi^\circ$, $p - n$, $\Sigma^+ - \Sigma^-$ etc.) for which the renormalized theory offers no explanation whatever. To explain these mass-differences a different type of theory is necessary, and obviously one of the first requirements of such a theory is that it be convergent.

In two recent papers[1][2], which will be referred to below as I and II, one such theory has been proposed. We shall call this the Heitler-Arnous or HA theory. A detailed investigation of this theory carried out in I, II, a third paper[3], the present paper and some unpublished work, shows that the HA theory can certainly be made convergent in all orders of perturbation, that it offers an explanation of the mass-differences in good qualitative agreement, at least, with experiment[4][5][6][7] and that it is in excellent agreement with experiment for a large number of other effects, such as Compton-scattering (Klein-Nishina formula) but not for *all* known effects, at least in its original form, a situation which will be discussed presently.

It turns out, however, that in the HA theory it is virtually impossible to obtain strict relativistic and gauge-invariance simultaneously with convergence. (Of course, the theory is invariant under purely *spatial* rotations.) This difficulty of reconciling invariance with convergence (which is, of course, common to *all* known field theories and which has existed ever since field theory, classical or quantized, exists) suggests a critical attitude toward the exact fulfillment of the former, in spite of the fact that the success of the Lamb shift and other calculations of the local theory rest on a *postulated* relativistic and gauge-invariance.

The primary purpose of the present paper (which is carried out in Part II i.e. sections 4–8) is to investigate in detail the consequences of the lack of gauge and relativistic invariance in the HA theory, by comparing that theory with experiment. In view of the difficulties mentioned we shall here take the attitude that *only if the lack of invariance involves a contradiction with experiment shall we regard this lack of invariance as a weakness of the theory.*

The result of this investigation is already known for certain physical processes, namely, those which do not involve integration over virtual momenta in their theoretical calculation (e.g. Compton scattering) and those which involve only integrations which would be convergent in the local *un*renormalized theory. In these cases, there is a certain departure from relativistic and gauge invariance in second and higher orders of perturbation*, but it is much too small to be in contradiction with present day experiments (with energies less than 1 BeV). The important question is: what about the effects such as the Lamb shift which involve for their calculation integrations over virtual momenta which would diverge in the local *un*renormalized theory? In this paper the investigation of such effects will be carried out in a fairly exhaustive manner, all second order effects, as well as all the (experimentally) important higher order effects (anomalous magnetic moment of the electron, Lamb shift and radiative corrections to scattering) being taken into account. The result of the investigation is that the HA theory runs into a serious difficulty in connection with the photon self-energy (it turns out to be negative definite and not zero as required). Furthermore the same difficulty reappears, and in a more acute form, in some of the higher order effects, notably the anomalous magnetic moment effect. The theory also encounters other difficulties in connection with the magnetic moment, but these are not quite so serious.

The remarkable fact in connection with the photon self-energy is that its negative definite character follows, roughly speaking, *only* from the

*) In first order of perturbation, there is a departure from the results of the local theory, but with suitable formfactors (cf. section 1) this departure may be relativistically and gauge invariant. In any case, it is again much too small to be in contradiction with experiment.

fact that the Hamiltonian (in the interaction picture) is linear in the radiation field (i. e. of first order in e) and is Hermitian! A consequence of this is that the photon self-energy must be negative definite in *any* theory satisfying these simple conditions. In particular, it must be negative definite in the local theory also. But in that case, the situation is saved by the fact that the photon self-energy is also *infinite*, which allows certain manipulations (depending for their success on the very divergence of the integral in question) to be used in order to make it zero. The inconsistency of such a procedure is obvious.

Suggestions for circumventing the photon selfenergy difficulty in the HA theory (e.g. by means of a charge renormalization) will be given. There are six suggestions in all. These do not necessarily exhaust all the possibilities, of course, and further suggestions would be welcome. It turns out that only one of the six suggestions is of any use. It consists in adding to the first-order Hamiltonian certain terms of higher order in e (in particular a term of second order in e, i.e. a term bilinear in the radiation field. This second order term is of such a form that it can be regarded as the non-local analogue of the photon mass "renormalization" term of the local theory). This suggestion has the merit that it not only removes the photon self-energy difficulty, but can be so modified that it also *removes all further discrepancies* between the HA theory and experiment. It has the drawback, however, that the form of the new higher order terms in the Hamiltonian is ad hoc in the extreme, and so too much value should not be attached to this suggestion. Perhaps the necessity of adding such higher order terms to the Hamiltonian shows that the starting point (interaction picture) of the HA theory, though the most suitable for obtaining convergence, may not be the simplest. It could be that the higher order terms in the Hamiltonian emerge as natural consequences of a different starting point. It might be well to add that these extra terms in the Hamiltonian do not disturb the convergence of the HA theory in any way.

The result of all these considerations in that the HA theory in its original form (as in I and II) is not in full agreement with experiment, and can only be brought into full agreement by an ad hoc modification, as has just been described. Thus, it cannot be regarded as a satisfactory theory by any means. But it has certain merits. Its *does* account for all the usual effects of quantum electrodynamics even if it does so in an ad hoc way. It offers simultaneously a reasonable explanation of the various mass-differences. And once the need for a convergent theory is accepted, it furnishes a simple model of such a theory – in fact, it is the *only* existing model apart from the extreme non-relativistic extended source model (of which it is a natural extension). Perhaps the present situation can best be seen by contrasting the local renormalized theory, and the HA theory schematically, as follows:

	Local theory (divergent)	HA theory (convergent)
mass-differences	unobtainable	qualitative agreement with experiment
effects involving either no integration over virtual momenta or only integrations which would be convergent in the local *un*renormalized theory	excellent agreement with experiment	excellent agreement with experiment
effects involving integrations which would be divergent in the local *un*renormalized theory	excellent agreement with experiment using renormalisation and prescriptions which *enforce* invariance	agreement with experiment reached by adding an ad hoc second order term to the Hamiltonian

The main part of the present paper, which has just been discussed, is carried out in Part II i.e. sections 4–8. In Part I, i.e. sections 1–3, some theoretical questions which arose in I and II are settled. First, some specific types of form-factor are discussed. Then it is shown that, by the addition of a certain second order term H_2 to the Hamiltonian, the HA theory can be made gauge invariant (in second order of perturbation theory at any rate) but that it probably can *not* be made relativistically invariant. Only the results (just mentioned) of Part I, and not the details are necessary for Part II.

PART I

Type of form-factor

As mentioned in I, the non-local HA theory consists in replacing, in the interaction picture, the usual interaction Hamiltonian by a non-local interaction i.e.

$$H(x_0) = e \int d^3x\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\, A_\mu(x) \rightarrow e \int d^3x\, d^4(x'\, x''\, x''')\, \bar{\psi}(x') \times$$
$$\times\, \Gamma_\mu(x - x', x - x'', x - x''')\, \psi(x'')\, A_\mu(x''') \tag{1.1}$$

and leaving the *free* fields $\psi(x)$, $A_\mu(x)$ unchanged.

In I and II virtually no restrictions were placed on the form-factor $\Gamma_\mu(x - x', x - x'', x - x''')$. In fact, it had only to be translational invariant and to satisfy a hermiticity condition and a necessary normalization condition. In this section we want to examine the structure of $\Gamma_\mu (x - x', x - x'', x - x''')$ in more detail.

In the first place we assume that $\Gamma_\mu(x - x', x - x'', x - x''')$ contains no creation or destruction operators (i.e. it is a c-number in the Hilbert space of these operators) but we allow it to be a function of the sixteen linearly independent γ-matrices (i.e. it may be a q-number in spinor space. For short, we shall call it a q-number form-factor in this case). Secondly, for the sake of simplicity, we can restrict ourselves, except where explicitly mentioned, to the special case $\Gamma_\mu(x - x', x - x'', x - x''')$ equal to $\Gamma_\mu(x - x', x - x'')\, \delta^4(x - x''')$. It will be seen that the more general type of form-factor $\Gamma_\mu(x - x', x - x'', x - x''')$ has no great adavantage over the simpler $\Gamma_\mu(x - x', x - x'')\, \delta^4(x - x''')$, particularly in connection with the most serious difficulty of the HA theory, the negative definite photon self-energy. With $\Gamma_\mu(x - x', x - x'')$ we can write

$$H(x_0) = \int d^3x\, Q_\mu(x)\, A_\mu(x)$$

with

$$Q_\mu(x) = \int d^4(x'\, x'')\, \overline{\psi}(x')\, \Gamma_\mu(x - x', x - x'')\, \psi(x'') \,. \tag{1.2}$$

We now wish to see what conditions are imposed on $\Gamma_\mu(x - x', x - x'')$ by demanding *relativistic invariance in first order of perturbation theory* (this means in the terms proportional to e in a perturbation theory expansion, not the first non-vanishing terms, which may be of order e^2 or higher for particular effects.). Writing $\Gamma_\mu(p, q)$ for the Fourier transform of $\Gamma_\mu(x - x', x - x'')$, it is not difficult to see that first order relativistic invariance demands that $\Gamma_\mu(p, q)$ be a convariant function of its arguments i.e.

$$\Gamma_\mu(p, q) = \gamma_\mu\, [A(\boldsymbol{p} \cdot \boldsymbol{q}) + \gamma_5\, A'(\boldsymbol{p} \cdot \boldsymbol{q})] + p_\mu\, [B(\boldsymbol{p} \cdot \boldsymbol{q}) + \gamma_5\, B'(\boldsymbol{p} \cdot \boldsymbol{q})] + \\ + q_\mu\, [C\, (\boldsymbol{p} \cdot \boldsymbol{q}) + \gamma_5\, C'(\boldsymbol{p} \cdot \boldsymbol{q})] \tag{1.3}$$

where $\boldsymbol{p} \cdot \boldsymbol{q}$ means a four-vector product and A, A', ... C' are arbitrary functions. When the form-factor is an ordinary c-number factor, only A is not zero.

If, now, in addition to first order relativistic invariance, we demand *first order gauge invariance* we have

$$e \int d^4x\, Q_\mu(x)\, A_\mu(x) = e \int d^4x\, Q_\mu(x)\, [A_\mu(x) + \Lambda(x),_\mu] \tag{1.4}$$

$$\Lambda(x),_\mu = \frac{\partial}{\partial x_\mu}\, \Lambda(x) \,,$$

from which

$$e \int d^4x \, Q_\mu(x),_\mu \Lambda(x) = 0 \tag{1.5}$$

since $\Lambda(x)$ is arbitrary this means that

$$Q_\mu(x),_\mu = 0 \, . \tag{1.6}$$

The effect of (1.6) on the $\Gamma_\mu(p, q)$ of (1.3) is easily seen by taking the Fourier transform of $Q_\mu(x)$ in (1.2). We find (since (1.6) must be valid for all possible states)

$$\begin{gathered} B - C = 0 \, , \\ B' - C' = \frac{2 i \, m \, A'}{m^2 + \boldsymbol{p} \cdot \boldsymbol{q}} \\ (m = \text{electron mass}) \end{gathered} \tag{1.7}$$

which reduces the six independent functions of (1.3) to four. Note, however, that for a c-number factor (1.7) is automatically fulfilled $(A' = B = B' = C = C' = 0)$.

Equations (1.3) and (1.7) are therefore the required conditions for first order relativistic and gauge-invariance respectively. As was shown in I and II they are by no means sufficient, in general, to produce invariance in higher orders of perturbation. In the following sections, except where explicitly mentioned, we shall not assume that the form-factor satisfies either (1.3) or (1.7). On the one hand we shall at times allow it, for example, to be a much more complicated function of the γ's than that given in (1.3) and on the other hand, we shall sometimes find it convenient to use the simplest of all form-factors, the special c-number cut-off, given in terms of $\Gamma_\mu(p, q)$ by

$$\Gamma_\mu(p, q) = \gamma_\mu \, f(\vec{p}, \vec{q}, \lambda)$$

where λ is the cut-off parameter, $\vec{p}$ and $\vec{q}$ are three-vectors, and f (a c-number) is some monotonically decreasing function of $\vec{p}$ and $\vec{q}$ e.g. $\lambda^4/\lambda^4 + \vec{p}^4 + \vec{q}^4$.

2. Second-order gauge-invariance

It has been shown in I and II that in the HA theory, with interaction Hamiltonian as in (1.1) (we shall call this H_1 since it is of first order in e), neither the S-matrix $S(H_1)$ nor the expectation value of the energy $\langle t \mid P_4 \mid t \rangle$ is necessarily gauge-invariant in higher orders of perturbation theory – in particular in second order (order e^2). The conditions for gauge-invariance have been given in I and II. It appears, however, that these conditions are very difficult (perhaps impossible) to fulfill in a theory which is at the same time convergent. We shall assume therefore that the S-matrix, $S(H_1)$, of the ordinary HA theory is *not* gauge-invariant and we show now how this difficulty may be circumvented (at least in second order)*).

There are two ways of circumventing the difficulty. The first way e, to add to the HA Hamiltonian H_1 a Hamiltonian H_2 of second order in is so that for the total S-matrix we have (in second order)

$$S_{e^2}(H_1 + H_2) = S_{e^2}(H_1) + S_{e^2}(H_2)\,, \qquad S_{e^2}(H_2) = i \int_{-\infty}^{\infty} dx_0\, H_2(x_0)\,. \tag{2.1}$$

The problem then is to find H_2 such that $S_{e^2}(H_1 + H_2)$ is gauge-invariant and such that $S_{e^2}(H_2) \to 0$, in the local limit. The second way to circumvent the difficulty has already been suggested in II. There, it was suggested that since P_4 of

$$\langle t\,|P_4|\,t\rangle - \langle -\infty\,|H_0|-\infty\rangle = \lim_{T\to\infty} \frac{i}{T} \langle -\infty\,|\,S(T) - 1\,|-\infty\rangle \tag{2.2}$$

(cf. II(60)) is not gauge-invariant, a term P_4^θ should be added to P_4 so that $P_4 + P_4^\theta$ would be gauge-invariant. This is what is done in the local theory but, unlike the local theory, in the HA theory $\langle t\,|\,P_4^\theta\,|\,t\rangle$ is not necessarily zero (it is zero when $S(H_1)$ is gauge-invariant!). However, we could content ourselves in the non-local case with demanding that $\langle t\,|\,P_4^\theta\,|\,t\rangle$ be time independent, and zero in the local limit. The problem then is to find such a P_4^θ, and from (2.2) it can be seen that this is exactly the same problem as that of finding H_2 or $S_{e^2}(H_2)$, above. We now give the solution to this problem. It is

$$S_{e^2}(H_2) = -\int d^4(x\,x')\; \delta(x_0 - x_0')\; [Q_\mu(x), Q_0(x')] \left\{A_\mu(x) - \frac{1}{2} A(x),_\mu\right\} A(x') \tag{2.3}$$

where

$$A(x) = \sum_{\mu=0}^{\mu=3} \alpha_\mu\, \partial_\mu^{-1}\, A_\mu(x)\,,$$

where the α_μ are constants satisfying only $\sum_{\mu=0}^{\mu=3} \alpha_\mu = 1$. Thus $A(x)$ is defined so that under the gauge-transformation $[A_\mu(x) \to A_\mu(x) + \Lambda(x),_\mu]$, $A(x) \to A(x) + \Lambda(x)$. We have now to show (1) that $S_{e^2}(H_1 + H_2)$ is gauge-invariant and (2) that $S_{e^2}(H_2) \to 0$ in the local limit.

*) We assume that we already have gauge-invariance in first order i.e. that (1.6) is satisfied.

Let Δ denote the change in any quantity on making a gauge-transformation. Then

$$\Delta S_{e^2}(H_1 + H_2) = \Delta S_{e^2}(H_1) + \Delta S_{e^2}(H_2)\,,$$

$$= \Delta \int\limits_{x_0 > x_0'} d^4(x\,x')\; Q_\mu(x)\; Q_\nu(x')\; A_\mu(x)\; A_\nu(x') -$$

$$- \Delta \int\limits_{x_0 = x_0'} d^4(x,\,x')\; [Q_\mu(x),\, Q_0(x')] \left\{A_\mu(x) - \frac{1}{2}\, A(x),_\mu\right\} A(x')$$

$$= \int\limits_{x_0 > x_0'} d^4(x\,x')\; Q_\mu(x)\; Q_\nu(x')\; A_\mu(x)\; \Lambda(x'),_\nu$$

$+$ a similar term

$$- \int\limits_{x_0 = x_0'} d^4(x\,x')\; [Q_\mu(x),\, Q_0(x')] \left\{\frac{1}{2}\Lambda(x),_\mu\; A(x') + \left[A_\mu(x) - \frac{1}{2}\, A(x),_\mu\right] \Lambda(x')\right\}$$

$$= \int\limits_{x_0 = x_0'} d^4(x\,x')\; Q_\mu(x)\; Q_0(x')\; A_\mu(x)\; \Lambda(x')$$

$+$ a similar term

$$- \int\limits_{x_0 = x_0'} d^4(x\,x') \left\{\begin{array}{l} [Q_0(x),\, Q_\mu(x')] \left\{-\frac{1}{2}\, \Lambda(x)\;\; A(x'),_\mu\right\} \\ + \; [Q_\mu(x),\, Q_0(x')] \left\{A_\mu(x) - \frac{1}{2}\, A(x),_\mu\right\} \Lambda(x') \end{array}\right\}$$

using repeatedly, partial integration, and the condition $\partial_\mu\, Q_\mu(x) = 0$ for first order gauge-invariance [cf. (1.6)],

$$= \int\limits_{x_0 = x_0'} d^4(x\,x')\; [Q_\mu(x),\, Q_0(x')]\; A_\mu(x)\; \Lambda(x') -$$

$$- \int\limits_{x_0 = x_0'} d^4(x\,x')\; [Q_\mu(x)\; Q_0(x')]\; A_\mu(x)\; \Lambda(x') = 0\,, \tag{2.4}$$

as required. It remains to show that $S_{e^2}(H_2)$ vanishes in the local limit. It is tempting to assume this without any further ado, since in (2.3) a spacelike commutator appears and so, on putting $\Gamma_\mu(x - x',\, x - x'')$ equal to $\delta^4(x - x')\; \delta^4(x - x'')$ (in other words, letting the cut-off parameter λ tend to infinity), $S_{e^2}(H_2)$ vanishes. This is not a correct criterion, however, because in (2.3) there is sometimes an integration over virtual momenta to be carried out, and it turns out that, for certain processes, a different result is obtained according to whether the local limit is taken before or after this integration. (The ambiguity is due, of course, to the fact that in the local theory the corresponding integrals are divergent, or, at best, ambiguous). The only valid test of the behaviour of (2.3) therefore, is to take the local limit ($\lambda \to \infty$) *after* all the integrations over virtual momenta have been carried out. Only then can we really examine whether the departure from the local theory result is of order $1/\lambda$, or not.

Since (2.3) is of second order in e, let us discuss first of all the effects in this order. The two unambiguous second order effects of the local theory are the Klein-Nishina formula and Møller scattering. Now it is easily seen that (2.3) does not contribute at all to Møller scattering, apart from a vacuum contribution which should be neglected (it does not contain enough Fermion operators). It does contribute to the Klein-Nishina formula, but because the contribution *involves no integration over the momentum of virtual particles* the spacelike commutator in (2.3) does result in the contribution tending to zero with increasing cut-off parameter (the exact calculation is given in appendix A). The remaining second order effects are the self-energies of the electron and photon. It is to be noted that the local limit for these simply *does not exist*. What has to be checked therefore is that the contribution of $S_{e^2}(H_2)$ to these does not conflict with direct experimental evidence. That this is so, is shown below. The contribution of $S_{e^2}(H_2)$ for higher order effects (e.g. anomalous magnetic moment of the electron) will be discussed later and leads to no further difficulty.

We, therefore, conclude that at all events *gauge-invariance* can be obtained within the framework of the HA theory (in second order, but, therefore, probably in all orders). Before completing this section there are a number of remarks to be made concerning $S_{e^2}(H_2)$.

The first concerns the vacuum expectation values of $S_{e^2}(H_2)$. To obtain a zero vacuum expectation for it without interfering with its gauge-transformation properties, we must write it as a normal product (i.e. with destruction operators to the right) either with respect to the radiation field alone $[S_{e^2}(H_2)_n]$ or with respect to both the radiation and electron fields $[S_{e^2}(H_2)_N]$.

The second remark concerns the constants α_μ and the photon self-energy. Gauge-invariance alone demanded only $\sum_{\mu=0}^{\mu=3} \alpha_\mu = 1$. But if we calculate the contribution of $S_{e^2}(H_2)_n$ or $S_{e^2}(H_2)_N$ to the photon self-energy, the transverse character of the polarization of a free photon leads to an infinity unless $\alpha_i = 0$, $\alpha_0 = 1$ (appendix A). Hence, we must choose these values for the α_μ. But once we have chosen these values we find that $S_{e^2}(H_2)$ does not contribute *at all* to the photon self-energy. This result is of great importance and will be used later.

The last remark concerns the electron self-mass. As is known, this is generally velocity-dependant in the HA theory. The question is: does the contribution of $S_{e^2}(H_2)$ to the self-mass make the velocity-dependance worse (so as to conflict with experiment) or better (perhaps removing it altogether)? To investigate this question in the most simple manner possible we have carried out the calculations, with $\alpha_i = 0$, $\alpha_0 = 1$, using a special cut-off form factor [which is, of course, too primitive to be taken seriously as the correct type of cut-off, but nevertheless should give some indication of the rôle played by $S_{e^2}(H_2)$]. The results are (appendix A)

$$\delta m_{S_{e^2}(H_1)} = \delta m_0 \left[1 + \frac{1}{72} \frac{\vec{p}^2}{m^2} + \cdots\right] \text{*)} \tag{2.5}$$

where δm = self-mass, δm_0 = self-mass for $\vec{p} = 0$, m = mass, $\vec{p}$ = momentum of the electron. δm_0 is proportional to $\log \lambda$ where λ is the cut-off parameter.

$$\begin{aligned} \delta m_{S_{e^2}(H_2)_N} &= \delta m_0 \left[0 - \frac{1}{36} \frac{\vec{p}^2}{m^2} + \cdots\right], \\ \delta m_{S_{e^2}(H_2)_n} &\equiv 0 . \end{aligned} \tag{2.6}$$

Thus the contribution of $S_{e^2}(H_2)$ to δm is not large enough to conflict with experimental evidence at present, but at the same time it certainly does not compensate the deviation from strict relativistic invariance of δm. This shows that although $S_{e^2}(H_2)$ produces second order gauge-invariance in the HA theory it by no means produces second-order relativistic invariance.

3. Second-order relativistic invariance

In view of the fact that in § 2 a functional $S_{e^2}(H_2)$ could be constructed so that $S_{e^2}(H_1 + H_2)$ is gauge-invariant (though not relativistic invariant, as we have just seen) and such that $S_{e^2}(H_2) \to 0$ in the local limit, the question naturally arises as to whether an $S_{e^2}(H_2^L)$ could be found such that $S_{e^2}(H_1 + H_2^L)$ is relativistically invariant and such that $S_{e^2}(H_2^L) \to 0$ in the local limit (with or without $S_{e^2}(H_1 + H_2^L)$ being also gauge-invariant). *We have not succeeded in finding such an* $S_{e^2}(H_2^L)$ and it is extremely unlikely that such a functional exists, as we shall now attempt to show.

Writing $S_{e^2}(H_1)$ again as in (2.4), it has been shown in I, that $\delta_L S_{e^2}(H_1)$, the change in $S_{e^2}(H_1)$ on making a Lorentz transformation $x_\mu \to x_\mu + \delta\omega_{\mu\nu} x_\nu$, is

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*) A different result was obtained in another paper [3]) but in that case a different form-factor (not the simple spherical cut-off) was used.

$$\delta_L\, S_{e^2}(H_1) = \text{const.} \times \int\limits_{-\infty}^{\infty} dz\, d^4(x\, x')\, [H_1(x), H_1(x')] \times$$
$$\times\, \delta(\boldsymbol{n}\cdot\boldsymbol{x}+z)\, \delta(\boldsymbol{n}\cdot\boldsymbol{x}'+z)\, (n_\mu\, x'_\nu - n_\nu\, x'_\mu)\, \delta\omega_{\mu\nu}\,. \tag{3.1}$$

Let us try to find $S_{e^2}(H_2^L)$ such that $\delta_L S_{e^2}(H_2^L) = -\,\delta_L S_{e^2}(H_1)$ and $S_{e^2}(H_2^L) \to 0$ in the local limit. It is very difficult to see how one can avoid making $S_{e^2}(H_2^L)$ a functional of $[H_1(x), H_1(x')]$ since $\delta_L S_{e^2}(H_2^L)$ must contain this expression and $H(x)$ *is a scalar under a Lorentz transformation.* But then the most general Ansatz for $S_{e^2}(H_2^L)$ would be

$$S_{e^2}(H_2^L) = \int d^4(x\, x')\, dz\, [H_1(x), H_1(x')]\, f(n, x, x', z) \tag{3.2}$$

where f is, as yet, a completely undetermined *functional* of n, x, x' and z. Since, again, the *form* of (3.1) is relativistically invariant, it is difficult to see how $\delta_L S_{e^2}(H_2^L)$ can compensate (3.1) unless

$$f(n, x, x', z) = f\,(\boldsymbol{n}\cdot\boldsymbol{x}, \boldsymbol{n}\cdot\boldsymbol{x}', z)\,. \tag{3.3}$$

Assuming (3.2) and (3.3), however, we can show that $S_{e^2}(H_2^L)$ must be equal to $-\,S_{e^2}(H_1)$ if $\delta_L S_{e^2}(H_2^L)$ is equal to $-\,\delta_L S_{e^2}(H_1)$, and thus that $S_{e^2}(H_2^L)$ cannot vanish in the local limit. We have

$$\delta_L\, S_{e^2}(H_2^L) = \int d^4(x\, x')\, dz\, [H_1(x), H_1(x')] \left\{-\left(\frac{\partial f}{\partial(\boldsymbol{n}\cdot\boldsymbol{x})}\right)_{x\rightleftarrows x'} +\right.$$
$$\left. + \frac{\partial f}{\partial(\boldsymbol{n}\cdot\boldsymbol{x}')}\right\} (n_\mu\, x'_\nu - n_\nu\, x'_\mu)\, \delta\omega_{\mu\nu}\,. \tag{3.4}$$

For this to compensate (3.1) for *arbitrary* states the only possibility would seem to be

$$-\left(\frac{\partial f}{\partial(\boldsymbol{n}\cdot\boldsymbol{x})}\right)_{x\rightleftarrows x'} + \frac{\partial f}{\partial(\boldsymbol{n}\cdot\boldsymbol{x})} = -\,\delta(\boldsymbol{n}\cdot\boldsymbol{x}+z)\,\delta(\boldsymbol{n}\cdot\boldsymbol{x}'+z)\,. \tag{3.5}$$

With some manipulation the solution of this equation is seen to be

$$f(\boldsymbol{n}\cdot\boldsymbol{x}, \boldsymbol{n}\cdot\boldsymbol{x}', z) = -\frac{1}{4}\,\delta(\boldsymbol{n}\cdot\boldsymbol{x}+z)\,\varepsilon(\boldsymbol{n}\cdot\boldsymbol{x}'+z) + h(\boldsymbol{n}\cdot\boldsymbol{x}, \boldsymbol{n}\cdot\boldsymbol{x}', z) \tag{3.6}$$

where $\varepsilon = \pm 1$ according as its argument is $\gtrless 0$ and h is an arbitrary *symmetric* function of $\boldsymbol{n}\cdot\boldsymbol{x}$ and $\boldsymbol{n}\cdot\boldsymbol{x}'$. Inserting (3.6) in (3.2), however, one finds that h drops out, and the first part of (3.6) is just such that

$$S_{e^2}(H_2^L) = -\,S_{e^2}(H_1) \tag{3.7}$$

as mentioned.

It is, therefore, plausible that $S_{e^2}(H_2^L)$ satisfying both $\delta_L S_{e^2}(H_1+H_2^L)=0$ and $S_{e^2}(H_2^L) \rightarrow 0$ in the local limit, does not exist. But, of course, the above demonstration is no more than plausible and a more rigorous proof, or counter-proof, would be welcome.

It might be enquired if one can see at a glance why $S_{e^2}(H_2^L)$ does not exist while the corresponding $S_{e^2}(H_2)$ of the gauge-invariant case does. It seems to us that the reason for this state of affairs lies in the remark made above that under a Lorentz-transformation $H(x)$ is a *scalar*. This fact gives us very little scope in trying to construct $S_{e^2}(H_2^L)$ – as we saw above, only the functional form of $f(\boldsymbol{n} \cdot \boldsymbol{x}, \boldsymbol{n} \cdot \boldsymbol{x}', z)$ was at our disposal. In the gauge-invariance case, on the contrary, not only is $f(\boldsymbol{n} \cdot \boldsymbol{x}, \boldsymbol{n} \cdot \boldsymbol{x}', z)$ at our disposal, but also, since under a gauge transformation $H(x) = Q_\mu(x)\, A_\mu(x) \rightarrow Q_\mu(x)\, A_\mu(x) + Q_\mu(x)\, \Lambda(x),_\mu$ we have the variation of $A_\mu(x)$ to play around with. That this variation of $A_\mu(x)$ is, in fact, used in constructing $S_{e^2}(H_2)$ in § 2 can be seen from the fact that in $S_{e^2}(H_2)$ the rather peculiar function $A(x)$ [such that $A(x) \rightarrow A(x) + \Lambda(x)$] appears.

PART II

4. Comparison of HA theory with experiment in low order

We now go over to the general HA theory and we do not use in the following any of the results of Part I except where explicitly mentioned. In this section we wish to investigate how the HA theory compares with experiment for first and second orders of the perturbation expansion.

In first order (which includes effects such as the Dirac magnetic moment of the electron, Coulomb scattering without radiative corrections etc.) there is no difficulty in seeing that for energies less than the cut-off energy ($\approx$ mass of the nucleon), which are the only energies for which we have experimental evidence at present, the discrepancy between experiment and theory cannot be observed. Experiments using energies of the order 1 Bev, or higher (which may soon be possible, perhaps), should be capable of detecting the departure from the local theory cross-sections predicted by the HA theory. (The HA theory predicts a falling off of the cross-section for energies greater than 1 Bev. in the centre of mass system.) But to date there is no discrepancy in first order.

In second order of perturbation there are four possible processes – Møller scattering, Compton scattering, self-energy of the electron, self-energy of the photon. In the case of Møller and Compton scattering there is again no disagreement between the HA theory and experiment for energies less than 1 Bev in the centre of mass system. (For future experiments with energies greater than 1 Bev a difference between the HA and local theory should be observable, just as in first order effects, with the one difference that the falling off of the cross-section predicted by the HA theory would probably be *non*-relativistic in second order). The reason that for low energies there is no difference between the HA and local theories is that the theoretical calculation of the Møller and Compton scattering involves no integration over virtual momenta. A rigorous proof that no discrepancy can exist for low energies is easily constructed, and is analogous to that given by CHRISTENSEN and MØLLER[8]) in discussing the same problem in their theory. In the case of the self-energy of the electron, it has been shown in a previous paper[3]) that there is an appreciable dependence of the self-mass on velocity (which does *not* vanish as the cut-off tends to infinity), but that nevertheless the resultant deviation from relativistic invariance of the total mass is not big enough to be noticed in the direct experiments so far existing. Better experiments, especially at higher energies are most desirable. The last second order effect is the photon self-energy and here the HA theory meets with its first and most serious set-back, because it turns out that the photon self-energy, far from being zero, as required by experiment, is negative definite and does not tend to zero as the cut-off tends to infinity. Let us now discuss this question in more detail.

As is well-known, the photon self-energy is given essentially by the *ii*-component of the induction tensor $I_{\mu\nu}(k)$ with $k_0 = |\vec{k}|$, $k_i = 0$, where $\boldsymbol{k}$ is the momentum, and i is the direction of polarization, of the free photon ($i = 1, 2, 3$). What we find on calculation is that

$$I_{ii}(k)_{\substack{k_0 = |\vec{k}| \\ k_i = 0}} \neq 0 \ ^{*)} . \tag{4.1}$$

Furthermore,

$$\lim_{\vec{k}\to 0} I_{ii}(k)_{\substack{k_0 = |\vec{k}| \\ k_i = 0}} \neq 0 , \quad \text{although} \quad \lim_{\vec{k}\to 0} I_{\mu\nu}(k)_{\substack{k_0 = |\vec{k}| \\ k_i = 0}} = 0 \tag{4.2}$$

for the other components of $I_{\mu\nu}$.

Equation (4.2) shows clearly the lack of relativistic invariance of the theory. Note that the first equation of (4.2) already implies that a form factor involving the momenta $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{k}$ at the vertex shown in Fig. 1, has no advantage over a form-factor involving only $\boldsymbol{p}$ and $\boldsymbol{q}$, since in (4.2) $k = 0$. The important question now is: under what conditions (or for what form-factors) are (4.1) and (4.2) true? The answer is that (4.2) and (4.1) are true for *all* form-factors, even those which are q-numbers

*) It follows from the invariance of the HA theory under purely *spatial* rotations that (4.1) is a function of $\vec{k}^2$ only. This is most easily seen by expanding $I_{ij}(k)$ in powers of $\vec{k}$ and k_0.

in spinor space! The only assumptions that we need make to arrive at (4.2) and (4.1) are:

1) that the interaction Hamiltonian H, in the interaction picture, is linear in the radiation field,
2) that it is hermitian,
3) that it is translational invariant,
4) that the metric in Hilbert space is positive definite,
5) the usual assumptions for the free-fields, e.g. that the energy of a free-field is positive definite.

To see this, we need only recall that according to an elementary perturbation theory calculations (which we may well use since in a convergent theory all methods of calculation must yield the same result) the photon self-energy is given by

$$\sum_{i \neq 0} \frac{H_{0i} H_{i0}}{E_0 - E_i} \tag{4.3}$$

where 0 denotes the free photon state, i all possible intermediate states, and E_0 and E_i the respective energies. Note that since H is already assumed to be a normal product (destruction operators to the right) and since the photon is a Boson, (4.3) has not got to be modified by subtracting out vacuum expectation values. First of all, we can easily see that $E_i > E_0$. This is because, since the form factor involves no creation or destruction operators, the state i can only be a state with a positron and an electron just as in the local case. On the other hand the translational invariance of H guarantees the conservation of 3-momentum in the transition $0 \to i$. Hence

$$E_i - E_0 = \sqrt{\vec{p}^2 + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2} - |\vec{k}| > 0 \tag{4.4}$$

for all $\vec{p}$, where $\vec{p}$ is the 3-momentum of one of the particles in the state i.

If we now use the fact that H is Hermitian it becomes clear that *every term* in the sum of (4.3) is either negative or zero. Hence if (4.3) is to be zero, every term in the sum must be zero and so, for all $i \neq 0$,

$$H_{i0} = 0. \tag{4.5}$$

But this is obviously not possible since it would imply the impossibility of pair creation, which is observed directly in experiment. (Similarly if (4.3) were of the order $1/\lambda$ where λ is the cut-off parameter, the probability for pair creation would be much too small). Thus under the assumptions (1–5) listed the HA theory predicts a negative definite photon self-energy. The most restrictive assumption is, of course, (1).

An important consequence of the above discussion is that the local theory should also predict a negative definite photon self-energy, since that theory, also, satisfies assumptions (1–5). And so it does. But because (4.3) is *infinite* as well as negative definite in that case it is possible to make (4.3) zero by means of certain tricks[9]). It is interesting, perhaps, to describe one such trick in detail. In calculating $I_{\mu\nu}(k)$ in the local case, one arrives at an expression

$$I_{\mu\nu}(k) = \text{const.} \times \text{Tr} \int \gamma_\mu \frac{\boldsymbol{\gamma}\cdot\boldsymbol{p} - \boldsymbol{\gamma}\cdot\boldsymbol{k} + i\,m}{(\boldsymbol{p}-\boldsymbol{k})^2 + m^2} \gamma_\nu \frac{\boldsymbol{\gamma}\cdot\boldsymbol{p} + i\,m}{\boldsymbol{p}^2 + m^2}\, d^4p \tag{4.6}$$

(cf. JAUCH and RÖHRLICH (1955), p. 189). On taking the trace, integrating over p_0 and choosing the z-direction as that of $\vec{k}$, one finds that (4.6) leads to

$$I_{i\,i}(k)_{k_0=0} = \text{const.} \times \int \frac{d^3p}{|\vec{k}|} \left[\frac{p_z}{\vec{p}^2 + m^2} - \frac{p_z + |\vec{k}|}{(\vec{p}+\vec{k})^2 + m^2} \right] {}^{*}), \qquad k_z = |\vec{k}|\,. \tag{4.7}$$

One can then calculate (4.7) in two ways. The first is to assume $|\vec{k}| \ll m$ and to expand the integrand in powers of $|\vec{k}|$. In this case one sees immediately that the lowest term in the expansion – (the contribution for $\vec{k} = 0$) – is positive definite (and infinite, of course)! The other way is to make use of the fact that the limits of integration are infinite (ignoring the fact that the *integral* is then infinite) and simply making a change of variable $p_z + |\vec{k}| \to p'_z \to p_z$ in the second term of (4.7) so that it exactly cancels the first term and $I_{i\,i}(k)_{k_0=0}$ is zero for all values of $\vec{k}$, including $\vec{k} = 0$! The success of such tricks for removing unwanted terms depends, of course, on the very divergence of the theory.

Summing up the present section we may, therefore, say: In just one of the four possible second order processes the HA theory is, for the present at any rate, in disagreement with experiment. The same is true of the local theory. We next consider some higher order processes.

5. Comparison of HA theory with experiment in higher orders

We investigate in this section how the HA theory compares with experiment in the three higher order effects which are important experimentally – the anomalous magnetic moment of the electron, radiative corrections to scattering, and the Lamb shift.

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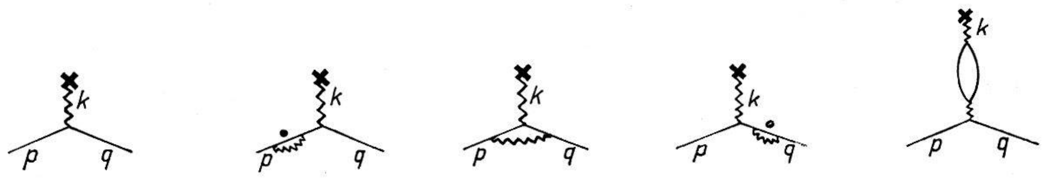
*) It is true that (4.7) is not exactly the photon self-energy since the latter is given by $I_{i\,i}(k)_{k_0=|\vec{k}|,\ k_i=0}$ and not $I_{i\,i}(k)_{k_0=0}$, but (4.7) is at any rate the contribution of the induction tensor to the magnetic moment of the electron, and as this should also be zero (and is not), it will suffice for our purpose of illustration.

Anomalous magnetic moment of the electron: In calculating this effect theoretically, some care must be taken. One cannot assume from the start that the magnetic moment is given by a higher order matrix element of the S-matrix which is proportional to a first order element that is known to correspond to the Dirac magnetic moment, because the S-matrix is primarily a scattering matrix, whereas the effect in question is an energy-shift. *However*, the energy shift will be given in the usual way by $\langle t\,|P_4|\,t\rangle - \langle -\infty\,|H_0|-\infty\rangle$ where $P_4 = H_0 + H$ and $t\rangle$ is a state with the external magnetic field fully switched on, and one can then prove (by a method analogous to that of UMEZAWA (1956) *pp.* 220–222) that this expression is, in fact, equal to the higher order matrix elements of S just mentioned, in the limit of k, the momentum of the external magnetic field, going to zero. These matrix elements (for second order radiative corrections) come from the following Feynmann graphs



1 2 3 4 5

Fig. 1

On calculating the matrix elements corresponding to these graphs (for one very simple form-factor at any rate) one finds in the local theory

$$1 + \frac{1}{2\pi}\left(\frac{e^2}{4\pi}\right) + 0\,. \tag{5.1}$$

In the non-local theory

$$1 + \frac{4}{3}\,\frac{1}{2\pi}\left(\frac{e^2}{4\pi}\right) + \lim_{\vec{k}\to 0}\left(\frac{I_{ii}(k)}{\boldsymbol{k}^2}\right)_{k_0 = k_i = 0} \tag{5.2}$$

where i is the direction of A, the potential vector of the external magnetic field. The Dirac magnetic moment has been normalized to unity, and $I_{\mu\nu}(k)$ is the induction tensor. The last term in (5.2) is deduced most easily by expanding $I_{ij}(k)$ in powers of k and using the invariance of the HA theory under *spacelike* rotations (so that $I_{ij}(k) = I^{(1)}(\vec{k}^2)\,g_{ij} + I^{(2)}(k^2, \vec{k}_0^2)\,k_i k_j$) as well as the Lorentz condition $\boldsymbol{k}\cdot\boldsymbol{A} = \vec{k}\cdot\vec{A} = 0$ on the external field. It can also be seen in this way that this term is a function of $\vec{k}^2$ only. Obviously (5.2) is in disagreement with experiment. The disagreement comes under two headings.

(1) In § 4 it was shown that $I_{ii}(k)_{k_0 = |\vec{k}|,\, k_i = 0} \neq 0$ even for $\vec{k} = 0$. The same obviously applies to $I_{ii}(k)_{k_0 = k_i = 0}$. But the magnetic moment is an effect

with $k_0 = 0$, $\vec{k} \to 0$. Hence the last term of (5.2) will give an *indefiniteley large* contribution to the magnetic moment. Further, although (5.2) has been calculated for only one particular form-factor, it was shown in § 4 that the result $I_{ii}(k) \neq 0$, $\boldsymbol{k} \to 0$ is true for *any* form-factor and so we see that the difficulty cannot be avoided simply by making a suitable choice of the form-factor.

(2) The second difficulty is that even if one manages to remove the $I_{ii}(k)_{k \to 0}$ difficulty just mentioned there still remain two *finite* discrepancies between the HA theory and experiment, namely the 4/3 term, and the remainder of the $I_{ii}(k)_{k_0 = k_i = 0}$ term, in (5.2). Now it *may* be possible that with a suitable form-factor these two discrepancies just cancel *but it is most unlikely* because with the form-factor chosen in (5.2) it turned out that the 4/3 term is (obviously) cut-off independent, whereas the remainder of the $I_{ii}(k)_{k_0 = k_i = 0}$ term is proportional to Log λ where λ is the cut-off parameter.

In sum, for the magnetic moment the HA theory encounters two difficulties, the most serious being (as in the case of the photon self-energy) that $I_{ii}(k) \neq 0$, $\boldsymbol{k} \to 0$. The fourth order radiative corrections to the magnetic moment will be discussed later and lead to no new difficulty.

Radiative corrections to scattering: In practice scattering is observed so inaccurately that radiative corrections have not been observable so far, and so, normally we could neglect them if we could be quite certain that they are really of order $e^2/4\pi$ compared with the uncorrected scattering and this will be true as long as the corrections do not become singular and so compensate the smallness of $e^2/4\pi$. In general, there is no reason why they should become singular but the result just found for the magnetic moment ($I_{ii}(k) / \boldsymbol{k}^2 \to \infty$ for $\boldsymbol{k} \to 0$) shows that we must investigate whether this singularity occurs also in scattering. Since we cannot consider every possible scattering process, we consider as examples the rôle played by the induction tensor in the 2nd order radiativ corrections to Møller scattering and to Coulomb scattering (Fig. 2).

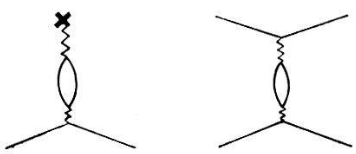


Fig. 2

We find (cf. appendix B):

(1) Coulomb scattering: since for the external field $\vec{A} = 0$, $A_0 \neq 0$ only $I_{0\mu}(k)$ plays a rôle and (mainly since $I_{0\mu}(k) \to 0$, $\boldsymbol{k} \to 0$ [cf. (4.2)]) there is no difficulty.

(2) Møller scattering: $I_{ii}(k)$ plays a rôle and there is a difficulty unless $I_{ii}(0) = 0$.

Thus with respect to scattering the HA theory runs into difficulties if $I_{ii}(0) \neq 0$ but is otherwise in agreement with experiment.

Lamb shift: This is again an energy shift, but like the magnetic moment, the calculation can be reduced to a calculation of the S-matrix. The calculation breaks up into two parts, a one-potential part and a many-potential part. The many-potential part (being convergent in the local case without renormalization etc.) is the same locally and non-locally within experimental error. The one-potential part must be treated more carefully. However, in the *observers* rest-system the relevant e.m. field is (as for the Coulomb effect) $\vec{A} = 0$, $A_0 \neq 0$, so that only $I_{0\mu}$ of the induction tensor plays a rôle. Consequently (just as for the radiative corrections to Coulomb scattering which is, in fact, essentially the same calculation) the singularity $I_{ii}(k)/\boldsymbol{k}^2$ for $\boldsymbol{k} \to 0$ does not occur here. At most, therefore, there will be a *finite* discrepancy between the HA theory and experiment for the Lamb shift. The removal of this finite discrepancy, in the event of its not being zero, will be discussed below.

6. Attempts to circumvent the difficulties of the HA theory

It has been seen in the previous two sections that the HA theory is in disagreement with experiment in the following way. In the case of the photon self-energy, the magnetic moment of the electron and the radiative corrections to Møller scattering the most serious disagreement occurs and in each case it is due to the fact that $I_{ii}(0) \neq 0$. Apart from this difficulty there are also (finite) discrepancies between the theory and experiment in the case of the photon self-energy and the magnetic moment, and possibly in the case of the Lamb shift. But the main difficulty is presented by $I_{ii}(0) \neq 0$. And this difficulty follows directly from the very simple assumptions of the theory listed in § 4.

At first sight, however, there would appear to be a number of ways of circumventing this difficulty (e.g. by renormalization). We wish now to examine this question in more detail. It turns out, in fact, that it is by no means easy to circumvent this difficulty. As an illustration of this we shall list in this section five fairly promising suggestions for removing it, all of which fail. In the following section we shall discuss a suggestion which *does* remove the difficulty and which removes all the other difficulties of the HA theory too, but which, on the other hand, is exceedingly ad hoc. Naturally the six suggestions just mentioned do not necessarily exhaust *all* the possibilities of removing the difficulty $I_{ii}(0) \neq 0$ and any further suggestion would be welcome. We list now the five suggestions which do *not* succeed.

(1) The first suggestion for removing the $I_{ii}(0) \neq 0$ difficulty is to introduce the operator $S_{e^2}(H_2)$ of § 2 which makes the S-matrix (and the energy-momentum vector) gauge-invariant in second order. This suggestion fails, because, as already remarked in § 2, $S_{e^2}(H_2)$ gives no contribution *at all* to the photon self-energy and therefore does not compensate the effect arising from H_1. Further, $S_{e^2}(H_2)$ cannot compensate the 4/3 etc. difficulties of the magnetic moment either (appendix E).

(2) The second suggestion is that the $I_{ii}(0)$ difficulty might be removed by a charge renormalization [similar to the charge renormalization of the local theory which removes the term proportional to $\boldsymbol{k}^2$ in $I_{\mu\nu}(k)$]. But it is easy to see that this is not possible, because for the magnetic moment an infinite renormalization factor $(1 + I_{ii}(k)/\boldsymbol{k}^2)_{\boldsymbol{k}\to 0}$ would be required and for Coulomb scattering or for the radiative corrections to the Millikan experiment, both of which effects involve only $I_{0\mu}(k)$ and not $I_{ii}(k)$, a factor of approximately unity would be necessary! The difference of the two factors needed, is, of course, a consequence of the non-relativistic nature of the theory.

(3) The third suggestion is to subtract out the induction tensor en bloc, by introducing the second order term in the Hamiltonian

$$-\int_{x_0 > x_0'} d^3(x\,x')\,dx_0'\,\langle 0 \mid Q_\mu(x)\,Q_\nu(x') \mid 0\rangle\,A_\mu(x)\,A_\nu(x')\,. \tag{6.1}$$

This would not only make $I_{ii}(0) = 0$ but also $I_{ii}(k)_{k_0 = |\vec{k}| \neq 0} = 0$ which, for the photon self-energy at any rate, would be most satisfactory. But such a term leads almost certainly to the wrong Lamb shift since in the local theory calculation of the latter the induction tensor (finite part) plays an important rôle. Furthermore it is easy to see from the form of (6.1) that it contributes to the magnetic moment *only* via the induction tensor. Thus it is incapable of removing the 4/3 difficulty from the theory.

(4) A fourth suggestion is to add to the first order Hamiltonian a second order term of the kind

$$-\frac{1}{2}\int_{x_0 > x_0'} d^3(x\,x')\,dx_0'\,[Q_\mu(x),\,Q_\nu(x')]\,A_\mu(x)\,A_\nu(x)\,. \tag{6.2}$$

Note that not $A_\nu(x')$ but $A_\nu(x)$ occurs in (6.2), a term which we can regard as being the first in an expansion

$$A_\nu(x') = A_\nu(x) + (x - x')\left[\frac{\partial}{\partial x'}\,A_\nu(x')\right]_{x'=x} + \cdots\,.$$

This would correspond roughly to a vertex renormalization. This suggestion fails because on calculating the contribution of (6.2) to the Klein-

Nishina formula it is found that for some choices of the gauge an *infinite* contribution is obtained (note that since $Q_\mu(x)$ is non-local, (6.2) is not gauge-invariant). The singularity is due to A_ν being a function of x, rather than x', since this leads to a term $\delta^3(p - q) / p_0 - q_0$ after space-time integration, instead of the usual $\delta^3(\vec{p} - \vec{q} - \vec{k}) / p_0 - q_0 - k_0$, and $\delta^3(\vec{p} - q) / p_0 - q_0$ is obviously singular. (The exact calculation is carried out in appendix D).

(5) The next suggestion is to modify (6.2) by taking the vacuum expectation value of it with respect to the electron field. Then (6.2) no longer contributes to the Klein-Nishina formula, or indeed to any other process except the induction tensor. It does just cancel $I_{ii}(0)$. Its dis advantage, however, is that it cancels *only* $I_{ii}(0)$ and no other part of $I_{ii}(k)$ since

$$\langle 1 \text{ photon}, \boldsymbol{k}, \boldsymbol{e} \,|\, A_\mu(x)\, A_\nu(x) \,|\, 1 \text{ photon}, \boldsymbol{k}, \boldsymbol{e} \rangle$$

is independent of $\boldsymbol{k}$ (apart from a normalizing factor $1/2\,\omega_k$). Similarly this term is incapable of removing the 4/3 difficulty of the magnetic moment.

Thus, five likely suggestions for improving the HA theory fail, which illustrates, as we have said, the non-trivial nature of the difficulties. Even a combination of some two or more of these suggestions would not improve matters very much. We now proceed to discuss a sixth suggestion which is more sucessful than the five suggestions of this section.

7. Modification of the HA theory

In this section we wish to propose a modification of the HA theory which will remove the $I_{ii}(0)$ difficulty from that theory and in addition will bring the theory into complete agreement with experiment. The suggestion consists of adding to the first order Hamiltonian a second order term of the from

$$H_{e^2}(x_0) = - e^2 \int_{-\infty}^{\infty} d^3(x\, x')\, dx_0'\, f_{\mu\nu}(x - x')\, A_\mu(x)\, A_\nu(x') \tag{7.1}$$

where the sixteen $f_{\mu\nu}(x - x')$ are just c-numbers, and do not necessarily form a tensor (so that the integrand of (7.1) is not necessarily a scalar). However, we restrict $f_{\mu\nu}(x - x')$ somewhat by stipulating that $f_{ij}(x - x')$ be a tensor in three-space, since we know already from § 5 that this is true of $I_{ij}(x - x')$. Let $f_{\mu\nu}(k)$ be the Fourier transform of $f_{\mu\nu}(x - x')$. We wish now to show that the functions $f_{\mu\nu}(k)$ can be so chosen (a posteriori) that the HA theory comes into complete agreement with experiment.

Let us consider first the photon self-energy. The contribution of the first order part of the Hamiltonian to this is, as was seen in § 4

$$I_{ii}(k)_{\substack{k_0=|\vec{k}|\\ k_i=0}} \tag{7.2}$$

and is not zero. It is easily seen that the corresponding contribution from (7.1) is

$$-f_{ii}(k)_{\substack{k_0=|\vec{k}|\\ k_i=0}} \,. \tag{7.3}$$

Hence to obtain a zero mass for the photon it is necessary and sufficient that $f_{\mu\nu}(k)$ be so chosen that

$$f_{ii}(k)_{\substack{k_0=|\vec{k}|\\ k_i=0}} = I_{ii}(k)_{\substack{k_0=|\vec{k}|\\ k_i=0}} \,. \tag{7.4}$$

Let us next consider the radiative corrections to scattering. We saw in § 5 that these would be in order if only

$$I_{ii}(0) \neq 0 \tag{7.5}$$

were compensated in some way. But with the addition of (7.1) to the theory (7.5) is obviously compensated (as a special case of (7.4) with $\vec{k}=0$). Thus with (7.4) the radiative corrections to scattering are automatically in order.

Next on the list comes the magnetic moment of the electron. The major difficulty in that case was again (7.5) and just as for the radiative corrections for scattering this difficulty is removed as a special case of (7.4) with $\vec{k}=0$. That leaves the contributions from $(4/3)\ (1/2\,\pi)\ (e^2/4\,\pi)$ and the remainder of the induction tensor still to be dealt with in calculating the magnetic moment. But clearly if we can choose $f_{\mu\nu}(k)$ further such that

$$\operatorname*{Lim}_{\vec{k}\to 0} \frac{I_{ii}(k)_{k_0=k_i=0} - f_{ii}(k)_{k_0=k_i=0}}{\vec{k}^2} + \frac{4}{3}\,\frac{1}{2\,\pi}\left(\frac{e^2}{4\,\pi}\right) = \frac{1}{2\,\pi}\left(\frac{e^2}{4\,\pi}\right), \tag{7.6}$$

where i is the direction of $\vec{A}$, the potential vector of the external field then this discrepancy vanishes also. That the first term of (7.6) represents the contribution of (7.1) to the magnetic moment of the electron, follows from the considerations given in § 5. It also follows from these considerations that $(I_{ii}-f_{ii})_{k_i=k_0=0}$ is a function of $\vec{k}^2$ only. The question is: are (7.4) and (7.6) compatible? And the answer is *yes*, because (7.4) is taken at the point $k_0=|\vec{k}|$ (all $\vec{k}$) and (7.6) at $k_0=0$ (all $\vec{k}$). This can best be seen by expanding $I_{ii}-f_{ii}$ in a power series according to $\vec{k}$ and k_0 separately*). In this way the magnetic moment (in order e^2) can be made to agree with experiment.

The last case which we must consider is the Lamb shift, and it might be thought that this case would provide a severe test of the validity of $f_{\mu\nu}(k)$ since this has been already determined to a very large extent by (7.4) and (7.6). This is not so. The reason is that (7.4) and (7.6) determine only $f_{ii}(k)$ whereas, as we shall see, the Lamb shift involves essentially only $f_{0\mu}(k)$. The Lamb shift consists of two calculations, a many-potential part which, because it is convergent in the local case, even without renormalization (and the main contribution is non-relativistic) yields practically the same contribution locally and non-locally, and a one-potential part, which must be handled much more carefully. The one-potential part consists essentially of taking the Coulomb scattering operator between two equal bound states. The point now is that in the rest-system of the observer (and in a non-relativistic theory this is the only system we are allowed to use!) the field of the nucleus is a pure Coulomb field $\vec{A} = 0$, $A_0 \neq 0$, so that whenever the induction tensor, and therefore $f_{\mu\nu}(k)$ occurs in the calculation, only $I_{0\mu}(k)$ and therefore $f_{0\mu}(k)$ will play a rôle. And $f_{0\mu}(k)$ is still left completely undetermined, by (7.4) and (7.6). The only question, therefore, is whether $f_{0\mu}(k)$ can now be determined so as to yield the correct Lamb shift for the HA theory? It might still be doubted whether this is possible because, in fact, $f_{0\mu}(k)$ depends only on $\vec{k}$, the momentum of the Coulomb field, and not on $\vec{p}$ the electron momentum (which has a certain given distribution in the bound state). However, it must be remembered that in the final analyses the Lamb shift is not a *function* of $\vec{k}$ and $\vec{p}$ but a *number* obtained after integrating (with given distribution functions) over $\vec{k}$ and $\vec{p}$. Hence, the non-dependence of $f_{0\mu}(k)$ on $\vec{p}$ is no handicap and there appears to be no reason, in principal, why one cannot chose the functions $f_{0\mu}(k)$ so as to obtain the correct Lamb shift. (Clearly if the HA theory without the term (7.1) already gives the correct Lamb shift, then we need only choose $f_{0\mu}(k) = 0$.)

We have, therefore, succeeded in showing that the term H_{e^2} of (7.1) is capable of removing all the difficulties encountered by the HA theory in comparing it with experiment (cf. § 4 and § 5). One serious objection can be levelled against the H_{e^2} term, of course, namely that it is *ad hoc* in the extreme. In the next section we shall discuss in more detail this and other matters connected with $H_{e^2}(x_0)$.

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*) Note that if it were not for the terms $(4/3)\,(1/2\,\pi)\,(e^2/4\,\pi)$, $(1/2\,\pi)\,(e^2/4\pi)$ in (7.6), $f_{ii}(k)$ would do nothing other than bridge the gap between the induction tensor obtained from the HA theory and the "rquired" induction tensor i.e. the finite induction tensor of the local theory obtained *after* the photon-mass and charge renormalization.

8. Discussion of second and higher order terms in the Hamiltonian

We have seen in the last section that a set of functions $f_{\mu\nu}(k)$ can be introduced in a second order part of the Hamiltonian so as to bring the HA theory into full agreement with experiment so far. As remarked in the introduction, $f_{\mu\nu}(k)$ is the non-local analogue of the photon mass 'renormalization' constant. In this section we wish to make a number of remarks in connection with $f_{\mu\nu}(k)$.

First it is clear from § 7 that $f_{\mu\nu}(k)$ is not entirely determined by the considerations of that section. But this obviously cannot be regarded as an objection to $f_{\mu\nu}(k)$. The same applies to the fact that even the parts of $f_{\mu\nu}(k)$ which *are* determined by § 7 are determined only up to the order m/λ (λ = cut-off, m = electron mass) or to within the experimental error.

Secondly it should be emphasized that the addition of the term H_{e^2} does not disturb the convergence of the HA theory in any way.

An interesting point is that H_{e^2} commutes with the first order part of the Hamiltonian if $f_{\mu\nu}(k)$ are *even* functions of k_0, a condition which is easily fulfilled without disturbing any of the results of § 7.

A question that might well be asked is whether the addition of terms to the Hamiltonian will stop at e^2, or whether there will be terms of order e^4, e^6 etc. We first discuss the e^4 terms. It turns out that terms of this order will in fact have to be added to the Hamiltonian. This is because in fourth order the contributions to any effect of the graphs of Fig. 3 play a rôle

Fig. 3

analogous to that of the induction tensor in second order. The only way to compensate certain unwanted parts of these contributions is by the addition of fourth order terms to the Hamiltonian. This can hardly be regarded as a handicap, however. On the contrary, it is probably an advantage, because the second graph of Fig. 3 contributes to fourth order radiative corrections to the magnetic moment of the electron and so the fourth order term can almost certainly be chosen, so that these radiative corrections too, can be brought into agreement with experiment (in a way completely analogues to that described in § 7 for the second order).

We next discuss the orders e^6, e^8, etc. At present, because of the lack of accurate enough experimental data, little can be said of these terms. But it seems probable that such additions to the Hamiltonian would be necessary if the experiments were fine enough. This, too, can hardly be considered as a valid objection to the HA theory, particularly in view of the fact that such fine experiments would very probably demand some such modification of the renormalized local theory also.

The one valid objection to these higher order terms in the Hamiltonian (and it is a serious objection) is their completely ad hoc nature. In fact, one introduces by means of these terms almost as many undetermined functions as there are experiments which involve radiative corrections! But at any rate the results of § 7 settle the question as to whether *in principle* the HA theory can be brought into agreement with experiments, in the affirmative, and we must not forget that this includes at least a qualitative account of the mass differences. We may, perhaps, interpret the higher order terms in the Hamiltonian as being an indication that the interaction picture, though an excellent starting point for obtaining convergence, is not the most natural starting point for the future correct theory. It could well be that with a different starting point, the higher order terms in the Hamiltonian would emerge as simple consequences of the correct theory. But, of course, this is only speculation.

In conclusion, it should perhaps be remarked that, to date, only attemps to build a new field theory *within* the normal framework of quantum field theory have been made. There still remains the possibility that by dropping one of the more fundamental postulates of field theory (e.g. hermiticity of the interaction Hamiltonian, or definiteness of the metric) more progress might be made. But these are questions which we cannot answer at present.

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Appendix A

Here we carry out explicitly the calculations in connection with the contribution of $S_{e^2}(H_2)$ to the following effects: Møller and Klein-Nishina (Compton) scattering, photon self-energy, electron self-energy. Since the commutator $[Q_\mu(x), Q_0(x')]$ plays a leading rôle in $S_{e^2}(H_2)$ we first calculate this explicitly. In all calculations the simplest type of form-factor is used, the c-number spherical cut-off given by a form-factor of the kind $f(\lambda, \vec{p}, \vec{q})$ where λ is the cut-off parameter and f decreases monotonically in $\vec{p}^2$ and $\vec{q}^2$. We use throughout these appendices the convention for metric etc. as given by Jauch and Röhrlich (1955). From the definition of $Q_\mu(x)$ in § 1 we have

$$[Q_\mu(x), Q_0(y)] = \int \bar{\psi}(x')\, \Gamma_\mu(x - x', x - x'')\, i\, S(x'' - y') \times$$
$$\times\, \Gamma_0(y - y', y - y'')\, \psi(y'')\, d^4(x'\, x''\, y'\, y'') - \text{V. V.} \qquad \text{(A.1)}$$

where V. V. (vice versa) denotes the same term with $x \rightleftarrows y$ and $\mu \rightleftarrows 0$. With the simple spherical cut-off, $x_0 = x_0' = x_0'' = y_0 = y_0' = y_0''$ and so

$$i\, S(x'' - y') = \left(\frac{i}{2\pi}\right)^3 \gamma_0\, \delta^3(x'' - y')\,. \qquad \text{(A.2)}$$

Inserting this in (A.1) and using $\Gamma_\mu(p, q) = \gamma_\mu\, f(p, q)$, we have, on transforming to momentum space,

$$[Q_\mu(x), Q_0(y)] =$$
$$\frac{i}{(2\pi)^6} \int d^4(p\, q)\, d^3 s\, \bar{\psi}(p)\, \gamma_\mu\, \psi(q)\, f(\lambda, \vec{p}, \vec{s})\, f(\lambda, \vec{s}, \vec{q})\, e^{-i\, \boldsymbol{p}\cdot\boldsymbol{x} + i\, \boldsymbol{q}\cdot\boldsymbol{y} + i\, \vec{s}\, (\vec{x} - \vec{y})} -$$
$$- \text{V. V.} \qquad \text{(A.3)}$$

We now have to consider the contribution of $S_{e^2}(H_2)$ to Møller and Klein-Nishina scattering. The contribution to Møller scattering is seen immediately to be identically zero because Møller scattering involves 4 electron operators altogether and in (A.3) there are only two. Hence, the only contribution of (A.3) could be as a vacuum effect which must be subtracted out. In the case of Klein-Nishina scattering the 1-electron states to the left and right of (A.3) change $\bar{\psi}(p)$ and $\psi(q)$ to $\bar{u}(p)$ and $u(q)$ respectively where $\bar{u}$ and u are the spinors for the free-particle of (fixed) momentum $\vec{p}$, $\vec{q}$. The radiation part of the operator in $S_{e^2}(H_2)$ contributes some constant (possibly zero) times a factor $e^{i\vec{k}(\vec{x}-\vec{y})}$ where $\vec{k}$ is the momentum transfer to or from the radiation field ($\vec{k}=\vec{p}-\vec{q}$). However in (A.3) the d^3x-integration has as consequence $\vec{s}=\vec{k}$. Thus all the momenta occuring in the form-factors are the finite external momenta (not integration variables). We therefore have $f=1$ up to order m/λ where m is the electron mass and so the two terms of (A.3) cancel up to order m/λ. Hence the contribution of $S_{e^2}(H_2)$ to the Klein-Nishina formula is of this order and vanishes in the local limit $\lambda\to\infty$.

The next question concerns the contribution of $S_{e^2}(H_2)$ to the photon self-energy or, more generally, to the induction tensor. We find first from (A.3)

$$\langle 0 \text{ electron} \mid [Q_0(x), Q_0(x')] \mid 0 \text{ electron}\rangle = 0\,. \tag{A.4}$$

$$\langle 0 \text{ electron} \mid [Q_i(x), Q_0(x')] \mid 0 \text{ electron}\rangle =$$

$$= 2\int d^3(p\,q)\,\frac{p_0\,q_i-p_i\,q_0}{\omega_p\,\omega_q}\,f(\lambda,\vec{p},\vec{q})\,f(\lambda,\vec{p},\vec{q})\,e^{i(\vec{p}-\vec{q})(\vec{x}-\vec{y})}\,,$$

$$\omega_p=\sqrt{\vec{p}^{\,2}+m^2}\,,\qquad \omega_q=\sqrt{\vec{q}^{\,2}+m^2}\,. \tag{A.5}$$

To calculate I_{00} we take the radiation part of $S_{e^2}(H_2)$ between two 1-photon states with 0-polarization. We find

$$\left\langle \begin{matrix}1 \text{ photon,}\\ \boldsymbol{k},\lambda=0\end{matrix} \middle| \left\{A_i(x)-\frac{1}{2}\,A(x)_{,i}\right\} A(x') \middle| \begin{matrix}1 \text{ photon,}\\ \boldsymbol{k},\lambda=0\end{matrix}\right\rangle =$$

$$=\frac{i\,\alpha_0^2}{4\,|\vec{k}|}\,\frac{k_i}{k_0^2}\,e^{-i\vec{k}(\vec{x}-\vec{x}')} \tag{A.6}$$

so that

$$I_{00}=2\,e^2\int d^3(x\,x')\,dx_0\int d^3(p,q)\,\frac{p_0\,q_i-p_i\,q_0}{\omega_p\,\omega_q}\,f(\lambda,\vec{p},\vec{q})\,f(\lambda,\vec{q},\vec{p})\,e^{i(\vec{p}-\vec{q})(\vec{x}-\vec{x}')}\times$$

$$\times\,\frac{i\,\alpha_0^2}{4\,|\vec{k}|}\,\frac{k_i}{|\vec{k}|^2}\,e^{-i\vec{k}(\vec{x}-\vec{x}')}$$

$$=\frac{i\,e^2\,\alpha_0\,k_i}{2\,|\vec{k}|^3}\,V\,T\int d^3p\,\frac{p_i\,\omega_{p+k}-(p+k)_i\,\omega_p}{\omega_p\,\omega_{p+k}}\,f(\lambda,\vec{p},\vec{p}+\vec{k})\,f(\lambda,\vec{p}+\vec{k},\vec{p}). \tag{A.7}$$

Similarly for I_{jj} we find

$$\left\langle \begin{matrix}1\ \boldsymbol{k}\\ \lambda=j\end{matrix} \middle| \left\{A_i(x)-\frac{1}{2}\,A(x)_{,i}\right\} A(x') \middle| \begin{matrix}1\ \boldsymbol{k}\\ \lambda=j\end{matrix}\right\rangle =$$

$$=\left(\frac{i\,\delta_{ij}\,\alpha_j}{|\vec{k}|\,k_j}-\frac{i\,\alpha_j^2\,k_i}{2\,|\vec{k}|\,k_j^2}\right)e^{-i\vec{k}(\vec{x}-\vec{x}')} \tag{A.8}$$

and so

$$I_{jj} = \left\{\frac{e^2\,\delta_{ij}\,\alpha_j}{|\vec{k}|\,i\,k_j} + \frac{i\,e^2\,\alpha_j^2}{2\,|\vec{k}|}\,\frac{k_i}{k_j^2}\right\} V\,T \int d^3p\,\frac{p_i\,\omega_{p+k} - (p+k)_i\,\omega_p}{\omega_p\,\omega_{p+k}} \times$$
$$\times\, f(\lambda, \vec{p}, \vec{p}+\vec{q})\, f(\lambda, \vec{p}+\vec{q}, \vec{p})\,. \qquad \text{(A.9)}$$

For a photon polarized in the direction j, $k_j = 0$. Hence I_{jj} is singular unless either $\alpha_j = 0$ or the integral in (A.9) is zero. Explicit calculation shows however that the integral is certainly not zero. Hence α_j must be zero. This applies equally well for $j = 1, 2, 3$. Hence $\vec{\alpha} = 0$, and only α_0 survives. Since $\sum_{\mu=0}^{\mu=3} \alpha_\mu = 1$ it follows that $\alpha_0 = 1$ and (A.7) implies no contradiction with this.

The last contribution of $S_{e^2}(H_2)$ we wish to discuss is its contribution to the electron self-energy. On account of what we have just found we have $A(x) = \partial_0^{-1} A_0(x)$. Then

$$\left\langle 0 \left| \left\{A_\mu(x) - \frac{1}{2}\,A(x),_\mu\right\} A(x') \right| 0 \right\rangle =$$

$$= g_0^\mu\,\langle 0\,|A_0(x)\,A(x')\,|\,0\rangle + g_i^\mu \left\langle 0 \left| -\frac{1}{2}\,A(x),_i\,A(x') \right| 0 \right\rangle$$

$$= \frac{g_0^\mu}{(2\,\pi)^3} \left\langle 0 \left| \int \frac{d^3k}{\sqrt{2\,|\vec{k}|}}\,a_0(k)\,\delta(k_0 - |\vec{k}|)\,e^{i\,k\cdot x} \int \frac{d^3k'}{\sqrt{2\,|\vec{k}'|}}\,\frac{1}{(i\,k_0')}\,a_0^*\,(-\,k') \times \right.\right.$$
$$\left.\left. \times\,\delta(k_0' + |\vec{k}'|)\,e^{i\,k'\cdot x'} \right| 0 \right\rangle + \text{a similiar term for } g_\mu^i$$

$$= \frac{i}{(2\,\pi)^3} \int \frac{d^3\,k}{2\,|\vec{k}|}\,e^{i\,\vec{k}\,(\vec{x}-\vec{x}')} \left[g_0^\mu\,\frac{1}{|\vec{k}|} + g_i^\mu\,\frac{k_i}{2\,|\vec{k}|^2}\right]. \qquad \text{(A.10)}$$

Next we multiply this with (A.3) taken between the two relevant 1-electron states and integrate over x and y. We find

$$\langle 1, \vec{p}\,|\,S_{e^2}(H_2)\,|\,1, \vec{p}\rangle =$$

$$= T \int d^3(x\,y) \left\{\frac{i}{(2\,\pi)^6}\,\bar{u}(p)\,\gamma_\mu\,u(p)\,f(\lambda, \vec{p}, \vec{s})\,f(\lambda, \vec{s}, \vec{p})\,e^{i\,\vec{p}\,(-\vec{x}+\vec{y}) + i\,\vec{s}\,(\vec{x}-\vec{y})}\,-\right.$$
$$\left. -\,\text{V.V.}\right\} d^3s\,\frac{i}{(2\,\pi)^3} \int \frac{d^3\,k}{2\,|\vec{k}|}\,e^{i\,\vec{k}\,(\vec{x}-\vec{x}')} \left\{\frac{g_0^\mu}{|\vec{k}|} + \frac{g_i^\mu\,k_i}{2\,|\vec{k}|^2}\right\}$$

$$= -\,\frac{V\,T}{(2\,\pi)^6}\,\bar{u}(p)\,\gamma_i\,u(p) \int d^3s\,f^2(\lambda, \vec{p}, \vec{s})\,\frac{p_i - s_i}{2\,\omega_{p-s}^2}\,. \qquad \text{(A.11)}$$

We now choose the direction of $\vec{p}$ as z-direction and calculate (A.11) up to order $\vec{p}^2/m^2$ in an expansion in $\vec{p}^2/m^2$. A straightforward calculation results in

$$\delta m = \frac{3\, m}{2\,\pi \cdot 137} \left[0 - \frac{2}{9}\, \frac{\vec{p}^2}{m^2} + O\, \frac{\vec{p}^4}{m^4} + O\, \frac{m^2}{\lambda^2}\right] \tag{A.12}$$

$$\left(\text{using} \quad \delta m = \frac{i}{2\,\pi}\, \langle S \rangle\right).$$

Appendix B

We wish to calculate the rôle played by the induction tensor in radiative corrections to scattering.

(1) Coulomb scattering: See Fig. 2. Since $\vec{A} = 0$ in this case only $I_{0\mu}(k)$ plays a rôle. Since, however, $I_{0\mu}(k) \to 0$ for $\boldsymbol{k} \to 0$ the main difficulty of the HA theory (infinite value of $I_{\mu\nu}(k)\,/\,\boldsymbol{k}^2$ for $\boldsymbol{k} \to 0$) does not arise here. The only possibility of a singularity would be if $\boldsymbol{k}^2 \to 0$ without $\boldsymbol{k} \to 0$ (i.e. $k_0 \to |\,\vec{k}\,|$). But this is not possible because we have the relations $\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{q}$, $\boldsymbol{p}^2 + m^2 = 0$, $\boldsymbol{q}^2 + m^2 = 0$, from which $\boldsymbol{k}^2 = 0$ implies $\boldsymbol{k} = 0$. Hence, for Coulomb scattering the radiative corrections contain no singularity and can be neglected.

(2) Møller scattering: In this case (Fig. 2) we have a contribution of the kind

$$J_\mu \int \frac{d^4 k}{\boldsymbol{k}^4}\, I_{\mu\nu}(k)\, J'_\nu$$

where J_μ, J'_ν are the respective electron currents. Since observers systems exist for which $\vec{J} \neq 0$, $\vec{J}' \neq 0$ (e.g. centre of mass system), $I_{ij}(k)$ plays an important rôle. Hence, unless $I_{ii}(k) = 0$ for $\boldsymbol{k} = 0$ we obtain a singularity. On the other hand, if $I_{ii}(0) = 0$, no further trouble is to be expected since just as for the Coulomb effect $\boldsymbol{k}^2 = 0$ implies $\boldsymbol{k} = 0$ on account of the electron-momenta being free particle momenta.

Appendix C

We wish to calculate the matrix elements of the graphs of figure 1, using the c-number spherical cut-off form-factor.

Graph (I): Using the rules for matrix elements given in JAUCH and RÖHRLICH (1955, p. 154) we find

$$M_{\mathrm{I}} = \frac{e}{\sqrt{2\,\pi}\,\sqrt{2\,\omega_k}}\, \bar{u}(p)\, \boldsymbol{A} \cdot \boldsymbol{\gamma}\, u(p + k)\,, \qquad \omega_k = |\,\vec{k}\,|\,. \tag{C.1}$$

Graph (III):

$$M_{\mathrm{III}} = \int d^4q \, \frac{1}{\sqrt{2\,\omega_k}} \, \frac{i\,e^3}{(2\,\pi)^{9/2}} \, \overline{u}(p)\, \gamma^\mu \, \frac{\boldsymbol{\gamma}\cdot\boldsymbol{p} - \boldsymbol{\gamma}\cdot\boldsymbol{q} + i\,m}{(p_0-q_0)^2 - \Omega^2_{p-q}} \, \boldsymbol{\gamma}\cdot\boldsymbol{A} \times$$

$$\times \frac{\boldsymbol{\gamma}\cdot\boldsymbol{p} + \boldsymbol{\gamma}\cdot\boldsymbol{k} - \boldsymbol{\gamma}\cdot\boldsymbol{q}}{(p_0-q_0+k_0)^2 - \Omega^2_{p+k-q}} \, \gamma_\mu \, u(p+k) \, \frac{1}{q_0^2 - q^2} \times$$

$$\times f(\lambda, \vec{p}, \vec{p} - \vec{q}) \, f(\lambda, \vec{p} - \vec{q}, \vec{p} - \vec{q} + \vec{k}) \, f(\lambda, \vec{p} - \vec{q} + \vec{k}, \vec{p} + \vec{k}) \,,$$

$$\Omega^2_{p-q} = (\vec{p} - \vec{q})^2 + m^2 \,, \qquad \Omega^2_{p-q+k} = (\vec{p} - \vec{q} + \vec{k})^2 + m^2 \,. \qquad \text{(C.2)}$$

Letting $\vec{p} = 0$, $a_0 = k_0 = 0$ the numerator can be reduced so that in the limit $\vec{k} \to 0$,

$$M_{\mathrm{III}} = \frac{1}{\sqrt{2\,\omega_k}} \, \frac{e^3}{(2\,\pi)^{9/2}} \, \frac{\overline{u}(p)\, \vec{\gamma}\cdot\vec{a}\; \vec{\gamma}\cdot\vec{k}\, u(p+k)}{m} \int d^4q \, \frac{(-q_0^2 - 2\,m\,q_0 + 2\,m^2 - \vec{q}^2)}{[(q_0-m)^2 - \Omega^2]^2 \, (q_0^2 - \vec{q}^2)} \times$$

$$\times f(\lambda, 0, \vec{q}) \, f(\lambda, \vec{q}, \vec{q}) \, f(\lambda, 0, \vec{q}) \qquad \text{(C.3)}$$

+ terms not involving the magnetic moment.

$$\Omega^2 = \vec{q}^2 + m^2 \,.$$

Graphs (II) and (IV): These are equal and are given by

$$M_{\mathrm{II}} + M_{\mathrm{IV}} =$$

$$= \frac{i\,e^3}{(2\,\pi)^{9/2}} \, \frac{1}{\sqrt{2\,\omega_k}} \int d^4q \, \overline{u}(p) \, \frac{\gamma^\mu \, (\boldsymbol{\gamma}\cdot\boldsymbol{p} - \boldsymbol{\gamma}\cdot\boldsymbol{q} + i\,m) \; \boldsymbol{\gamma}\cdot\boldsymbol{p} \; (\boldsymbol{\gamma}\cdot\boldsymbol{p} - \boldsymbol{\gamma}\cdot\boldsymbol{q} + i\,m) \, \gamma_\mu}{(p_0-q_0)^2 - \Omega^2_{p-q}} \times$$

$$\times \frac{1}{q_0^2 - \vec{q}^2} \, \frac{\boldsymbol{\gamma}\cdot\boldsymbol{p} + i\,m}{2\,m^2} \, \boldsymbol{\gamma}\cdot\boldsymbol{a} \, u(p+k) \, f(\lambda, \vec{p} - \vec{q}, \vec{p}) \, f(\lambda, \vec{p}, \vec{p} - \vec{q}) \, f(\lambda, \vec{p}, \vec{p} + \vec{k})$$

(cf. HEITLER 1954, p. 308), where the mass renormalization has been included. On reducing the numerator this becomes

$$M_{\mathrm{II}} + M_{\mathrm{IV}} = \frac{e^3}{(2\,\pi)^{9/2}} \, \frac{1}{\sqrt{2\,\omega_k}} \, \frac{\overline{u}(p)\, \vec{\gamma}\cdot\vec{a}\; \vec{\gamma}\cdot\vec{k}\, u(p+k)}{m} \times$$

$$\times \int d^4q \, \frac{(-2\,m\,q_0 - q_0^2 - \vec{q}^2 + 2\,m^2)}{[(q_0-m)^2 - \Omega^2]^2 \, (q_0^2 - \vec{q}^2)} \, f(\lambda, 0, \vec{q}) \, f(\lambda, \vec{q}, 0) \, f(\lambda, 0, 0) \qquad \text{(C.5)}$$

+ terms not involving the magnetic moment.

Adding (C.5) to (C.3) we obtain an integral which converges even without the form-factor. Hence, we can drop the form-factor thereby making an error of at most $\sim m/\lambda \approx 1/2000$. The integration of the convergent integral is perfectly straightforward and leads to

$$M_{\text{II}} + M_{\text{III}} + M_{\text{IV}} = \frac{1}{\sqrt{2\,\omega_q}}\,\frac{e^3}{(2\,\pi)^{9/2}}\,\frac{\overline{u}(p)\,\vec{\gamma}\cdot\vec{a}\,\vec{\gamma}\cdot\vec{k}\,u(p+k)}{m}\left(-\frac{4\,\pi^2}{3}\,i\right). \tag{C.6}$$

Using now the well-known relation

$$\overline{u}(p)\,\boldsymbol{\gamma}\cdot\boldsymbol{a}\;u(p+k) = \frac{1}{2\,i\,m}\,\overline{u}(p)\,\vec{\gamma}\cdot\vec{a}\;\vec{\gamma}\cdot\vec{k}\;u(p+k) \tag{C.7}$$

+ other terms not involving the magnetic moment, it is easily seen that the ratio of (C.6) to (C.1) is

$$\frac{4}{3}\left(\frac{\alpha}{2\,\pi}\right) \tag{C.8}$$

as required.

It remains only to calculate M_{V}. We have

$$M_{\text{V}} = \frac{1}{\sqrt{2\,\omega_k}}\,\frac{\overline{u}(p)\,\gamma_\mu\,A_\nu\,u(p+k)}{(2\,\pi)^{9/2}}\,\frac{i\,e^3}{\boldsymbol{k}^2}\times$$

$$\times\,\text{Tr}\int d^4q\,\gamma_\mu\,\frac{\boldsymbol{\gamma}\cdot\boldsymbol{q}+i\,m}{q_0^2-\vec{q}^{\,2}-m^2}\,\gamma_\nu\,\frac{\boldsymbol{\gamma}\cdot\boldsymbol{q}-\boldsymbol{\gamma}\cdot\boldsymbol{k}+i\,m}{(q_0-k_0)^2-(\vec{q}-\vec{k})^2}\,f^2(\lambda,\vec{q},\vec{q}-\vec{k})\,f(\lambda,\vec{p},\vec{p}+\vec{k}) \tag{C.9}$$

(cf. JAUCH and RÖHRLICH (1955, p. 189). On reduction we find

$$M_{\text{V}} = \frac{4\,i\,e^3}{(2\,\pi)^4}\,\frac{\overline{u}(p)\,\gamma_\mu\,A_\nu\,u(p+k)}{\boldsymbol{k}^2}\times$$

$$\times\int d^4q\,\frac{2\,q_\mu\,q_\nu-q_\mu\,k_\nu-q_\nu\,k_\mu+(-m^2-\boldsymbol{q}^2+\boldsymbol{q}\cdot\boldsymbol{k})g_{\mu\nu}}{(q_0^2-\Omega^2)\,[(q_0-k_0)^2-\Omega_{q-k}^2]}\,f^2(\lambda,\vec{q},\vec{q}-\vec{k})\,f(\lambda,\vec{p},\vec{p}+\vec{k})$$

$$= \frac{e\,\overline{u}\,\gamma_\mu\,A_\nu\,u}{\sqrt{2\,\omega_k}\,\sqrt{2\,\pi}}\,\frac{I_{\mu\nu}(k)}{\boldsymbol{k}^2}\,. \tag{C.10}$$

For the external magnetic field $k_0 = 0$ we can choose our axes such that $\vec{k} = (0, 0, |\vec{k}|)$. Integrating (C.10) for this case we find

$$I_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \quad \text{in all orders of } |\vec{k}|\,, \tag{C.11}$$

and

$$\frac{I_{11}}{\boldsymbol{k}^2} = \frac{I_{22}}{\boldsymbol{k}^2} = \frac{4\,\pi}{|\vec{k}|^2}\,\frac{8\,\pi\,e^2}{(2\,\pi)^4}\int\limits_0^\lambda \frac{q^2\,dq}{2\,\Omega^3}\left[\frac{q^2}{3}-\Omega^2\right]+$$

$$+\,\frac{e^2}{8\,\pi^2}\left\{\frac{4}{3}\,\text{Log}\left(\frac{\lambda+\sqrt{\lambda^2+m^2}}{m}\right)-\frac{32}{45}\right\}+O(|\vec{k}|^2)\,. \tag{C.12}$$

The first term of (C.12) is the diverging (for $|\vec{k}| \to 0$) negative definite term mentioned in the text. It is proportional to λ^2 where λ is the cut-off. The second term is the term leading to the 'finite' discrepancy in the text. It is proportional to Log λ. I_{33} (on account of the Lorentz condition $\vec{k} \cdot \vec{A}$ for the external field) and I_{ij} $(i \neq j)$ play no rôle in this calculation.

Appendix D

We wish to calculate the contribution of (6.2) to the Klein-Nishina formula. In other words the operator (6.2) is to be taken between the initial and final states for the Compton effect. Changing to momentum space in (6.2) and carrying out the x-integration we find

$$\left\langle \begin{matrix} 1 \text{ photon}, \boldsymbol{k}', \alpha \\ 1 \text{ electon}, p \end{matrix} \middle| (6.2) \middle| \begin{matrix} 1 \text{ photon}, \boldsymbol{k}'', \beta \\ 1 \text{ electron}, q \end{matrix} \right\rangle =$$

$$= \int d^4k \left\{ \begin{matrix} \bar{u}(p)\, \Gamma_\mu(-\boldsymbol{p}/\boldsymbol{q})\, S(\boldsymbol{k})\, \Gamma_\nu(-\boldsymbol{k}/\boldsymbol{q})\, \psi(q)\, \dfrac{\delta^3(\vec{k}-\vec{q})}{-k_0+q_0} \\ -\, \bar{u}(p)\, \Gamma_\nu(-\boldsymbol{p}/\boldsymbol{q})\, S(\boldsymbol{k})\, \Gamma_\mu(-\boldsymbol{k}/\boldsymbol{q})\, \psi(q)\, \dfrac{\delta^3(\vec{k}-\vec{p})}{-p_0+k_0} \end{matrix} \right\} \delta_{\mu\alpha}\, \delta_{\nu\beta} \tag{D.1}$$

where α and β are the polarization directions of the incoming and outgoing photon. Hence, for a c-number form-factor
$\langle$final state | (6.2) | initial state$\rangle$ is proportional to

$$\begin{aligned} &\bar{u}(p)\, \gamma \cdot \boldsymbol{e}_1\, (\gamma_0\, \Omega_q - \vec{\gamma} \cdot \vec{q} - i\, m)\, \gamma \cdot \boldsymbol{e}_2\, u(q) / \omega_q\, (\omega_q - \Omega_q) \\ &+ \bar{u}(p)\, \gamma \cdot \boldsymbol{e}_1\, (\gamma_0\, \Omega_q + \vec{\gamma} \cdot \vec{q} + i\, m)\, \gamma \cdot \boldsymbol{e}_2\, u(q) / \omega_q\, (\omega_q + \Omega_q) \\ &+ \bar{u}(p)\, \gamma \cdot \boldsymbol{e}_2\, (\gamma_0\, \Omega_p - \vec{\gamma} \cdot \vec{p} - i\, m)\, \gamma \cdot \boldsymbol{e}_1\, u(q) / \omega_p\, (\omega_p - \Omega_p) \\ &+ \bar{u}(p)\, \gamma \cdot \boldsymbol{e}_2\, (\gamma_0\, \Omega_p + \vec{\gamma} \cdot \vec{p} + i\, m)\, \gamma \cdot \boldsymbol{e}_1\, u(q) / \omega_p\, (\omega_p + \Omega_p) \end{aligned} \tag{D.2}$$

where $\boldsymbol{e}_1$, and $\boldsymbol{e}_2$ are the polarization vectors, and in order to handle the singularity for $k_0 = q_0$ we have taken

$$S(k) = (-\gamma_0\, k_0 + \vec{\gamma} \cdot \vec{k} + i\, m)\, \frac{1}{\omega_k}\, \{\delta(k_0 - \Omega_k) - \delta(k_0 + \Omega_k)\},$$

$$\Omega_k^2 = \vec{k}^2 + M^2, \qquad M = m + \varepsilon$$

where ε is small and is let go to zero at the end of the calculation. We take now the simplest case $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \boldsymbol{e}$ (i.e. no change of polarization on scattering) and calculate (D.2). We find

$\langle$final state | (6.2) | initial state$\rangle$ is proportional to

$$\bar{u}(p)\, \boldsymbol{\gamma} \cdot \boldsymbol{e}\, u(q) \left\{ \boldsymbol{e}' \cdot \boldsymbol{q} \left(\frac{1}{\omega_q^2} + \frac{1}{\omega_p^2} \right) - 2\, (\boldsymbol{e} \cdot \boldsymbol{q}) \left[\frac{1}{\omega_q(\omega_q - \Omega_q)} + \right.\right.$$
$$\left.\left. + \frac{1}{\omega_p} \frac{1}{(\omega_p - \Omega_p)} \right] \right\} \tag{D.3}$$

where $\boldsymbol{e}'_\mu = \boldsymbol{e}_\mu\, (-1)^{\delta_{\mu 0}}$. In the rest system of the incident electron (D.3) reduces to

$$\bar{u}(p)\, \boldsymbol{\gamma} \cdot \boldsymbol{e}\, u(0) \left\{ \boldsymbol{e}' \cdot \boldsymbol{q} \left(\frac{1}{\omega_q^2} + \frac{1}{\omega_p^2} \right) - \frac{2\, \boldsymbol{e} \cdot \boldsymbol{q}}{m\, \varepsilon} \right\}. \tag{D.4}$$

If the gauge is such that the incident photon is purely transverse in the rest system of the incident electron then (D.4) is zero since $\boldsymbol{e} \cdot \boldsymbol{q}\ (= \boldsymbol{e}' \cdot \boldsymbol{q}) = \vec{e} \cdot \vec{q} = 0$. However for any other gauge $e_0 \neq 0$ and

$$\boldsymbol{e} \cdot \boldsymbol{q} = -\, e_0\, q_0 + \vec{e} \cdot \vec{q} = -\, e_0\, q_0 \neq 0$$

and so (D.4) is infinite for $\varepsilon \to 0$.

Appendix E

We wish to calculate the contribution to the magnetic moment of

$$H_{e^2} = i\, e^2 \int d^3(x\, y)\, [Q_\mu(x),\, Q_0(y)] \left\{ A_\mu(x) - \frac{1}{2}\, A(x),_\mu \right\} A(y)\, . \tag{E.1}$$

The contribution comes from the S-matrix built with this term and the first order Hamiltonian involving the external field, i.e.

$$H_e^{\text{ext}} = \int d^3x\, Q_\mu(x)\, A_\mu^{\text{ext}}(x)\, . \tag{E.2}$$

Since the external field may be treated classically, and we are working only to the order e^3 we may take the photon vacuum value of (E.1) immediately. We further simplify (E.1) by using the usual simple form-factor. Carrying out the x-integration we obtain (cf. app. A)

$$\langle 0 \text{ photon} \,|\, H_{e^2} \,|\, 0 \text{ photon} \rangle =$$
$$= \frac{-\, i\, e^2}{(2\, \pi)^3} \int d^4(p\, q)\, \bar{\psi}(q)\, \gamma_i\, \psi(q)\, f(\lambda, \vec{p}, \vec{s})\, f(\lambda, \vec{s}, \vec{p})\, \frac{k_i}{2\, |\vec{k}|^3}\, e^{i\, (p_0 - q_0)\, x_0} \times$$
$$\times\, \delta^3(-\, p + s + k)\, \delta^3(q - s - k)\, . \tag{E.3}$$

It is not difficult to see that the only contribution of this term to the S-matrix in third order is

$$\left\langle \begin{matrix} 0 \text{ photon} \\ 1 \text{ electron} \end{matrix} \right| \frac{(-i)^2}{2\,!} \int_{\substack{-\infty \\ x_0 > x_0'}}^{\infty} d(x_0\, x_0')\, H_{e^2}(x_0)\, H_e^{\text{ext}}(x_0') +$$

$$+ H_e^{\text{ext}}(x_0)\, H_{e^2}(x_0') \left| \begin{matrix} 0 \text{ photon} \\ 1 \text{ electron} \end{matrix} \right\rangle . \tag{E.4}$$

Inserting the Fourier transforms of (E.2) and (E.3) into (E.4) we obtain after a long but straightforward calculation

$$\frac{1}{(2\,\pi)^2}\, \bar{u}(p)\, \gamma_\mu\, A_\mu^{\text{ext}}\, \frac{\boldsymbol{\gamma} \cdot \boldsymbol{p}' + i\, m}{\boldsymbol{p}'^2 + m^2}\, \frac{\vec{\gamma} \cdot \vec{k}}{2\,|\vec{k}|^3}\, u(q)\, f(\lambda, \vec{p}, \vec{p}')\, f(\lambda, \vec{p}', \vec{s})\, f(\lambda, \vec{s}, \vec{q}) \times$$

$$\times \left[\delta^4(p - p' - k)\, \delta(p_0' - q_0)\, \delta^3(p' - s - k)\, \delta^3(q - s - k)\right] . \tag{E.5}$$

Since, however, H_{e^2} contributes also to the mass of the electron another term corresponding to the mass renormalization must be substracted from (E.5). Using an argument similar to that of HEITLER (1954, p. 308) we find that this term is the same as (E.5) except that

$\delta^3(-p' + s + k)\, \delta^3(q - s - k)$ must be replaced by $\delta^3(p' - q)\, \delta^3(q - s - k)$.

Putting $p' = q\,(1 + \varepsilon)$ as in HEITLER (loc. cit.) one finds that the difference of the two sets of δ-functions leads to a term

$$\left\{ \frac{\partial}{\partial \varepsilon}\, \frac{1}{(2\,\pi)^2}\, \bar{u}(p)\, \gamma_\mu\, A_\mu\, \frac{\boldsymbol{\gamma} \cdot \boldsymbol{q} + \varepsilon\, \boldsymbol{\gamma} \cdot \boldsymbol{q} + i\, m}{2\, m^2}\, \frac{\vec{\gamma} \cdot \vec{k}}{2\,|\vec{k}|^3} \times \right.$$

$$\left. \times\, u(q)\, f(\lambda, \vec{p}, \vec{q} + \vec{q}\,\varepsilon)\, f(\lambda, \vec{q} + \vec{q}\,\varepsilon, -\vec{k} + \vec{q} + \vec{q}\,\varepsilon)\, f(\lambda, -\vec{k} + \vec{q} + \vec{q}\,\varepsilon, \vec{q}) \right\}_{\varepsilon = 0} \tag{E.6}$$

with $\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{k}$.

It is not difficult to see that this term $\to 0$ with $\vec{q} \to 0$ (and indeed is proportional to $\vec{q}^{\,2}$ for small $\vec{q}$). This result is only to be expected, since in Appendix A we showed that (E.1) contributed a term to the self-energy of the electron which was of order $\vec{q}^{\,2}/m^2$ for small $\vec{q}$. And the contribution of (E.1) to the magnetic moment is just the difference between the renormalized and unrenormalized self-energy, times another factor.