Domain of holomorphy of the three-point function

Domain of Holomorphy of the Three-Point Function

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Abstract: A new derivation of the results of G. KÄLLÉN and A. WIGHTMAN on the domain of holomorphy of the three-point function is given, using essentially a system of coordinates due to R. JOST, and the tube theorem.

1. Preliminaries

As is well known, the domain of analyticity of the three-fold vacuum expectation value in x-space or in p-space which follows from the following axioms: existence and uniqueness of the vacuum, Lorentz invariance, positiveness of the energy, and local commutativity has been computed by G. KÄLLÉN and A. WIGHTMAN[4]). The purpose of the following note is to present a new derivation of the results of K.-W. using a system of coordinates introduced by R. JOST and a technique of analytic completion essentially based on the tube theorem. These tools allow for a much simplified calculation of the domain of holomorphy of the three-point function*).

Our starting point will be the conclusion from the 'DANAD' formalism of R. JOST and A. WIGHTMAN[3]) that when one expresses the three-fold vacuum expectation values as functions of the scalar products one obtains the same possible results whether one starts from a four-dimensional space-time or from a two-dimensional space-time.

We are thus led to the study of Wightman functions

$$W_{ABC}(\zeta_3, \zeta_1)$$

where ζ_3 and ζ_1 are two complex two-dimensional vectors.

For reasons of symmetry we put $\zeta_2 = -(\zeta_3 + \zeta_1)$.

Let us introduce the characteristic coordinates of $\zeta_i : \zeta_i = (z_i, z_i')$. It may be remarked that if one takes $z_1 = 1$, W is a function of the three

*) Another derivation, using different methods has also been given by F. J. DYSON (unpublished).

independent variables z_1', z_2, z_2' which are equivalent to the three scalar products ζ_1^2, ζ_2^2, ζ_3^2. It is this new system of variables which simplifies to a large extent the calculations. We will however find it practical to keep z_1 as an independent variable, paying due attention to Lorentz invariance. The vectors ζ_i transform according to

$$\Lambda \zeta_i = (\lambda z_i, \lambda^{-1} z_i')$$

under two-dimensional complex Lorentz transformations, λ being an arbitrary complex number.

We adopt the following representation of (ζ_i) in the complex plane: z_i is represented by its affix i and z_i' by the symmetric i' of its affix with respect to the imaginary axis.

We call Δ and Δ' respectively the triples (i) and (i'). In the following we will only be interested in the Lorentz transformations with $\lambda = e^{i\alpha}$, α real. These obviously have the simple effect of turning Δ and Δ' by an angle α in the same sense around the origin of the complex plane.

The domain in which we initially know that W is analytic as a consequence of the above-mentioned axioms (the union of the permuted extended tubes) is given by the set of all (Δ, Δ') such that one side of Δ and the corresponding side of Δ' are in a half-plane limited by a straight line through the origin. This domain is limited by pieces of analytic hypersurfaces which we call, following K.-W., cuts, $F_{ij'}$ and S, S'.

We will now define these and other pieces of analytic hypersurfaces which will appear in the computation of the holomorphy envelope.

1. The i-cut appears when i and i' are collinear with and on opposite sides of the origin. Its equation is $z_i z_i' = \varrho$.

2. $F_{ij'}$ $(F'_{ij'})$ appears when i and j' are collinear with and on opposite sides (on the same side) of the origin. Its equation is

$$z_i z_j' = \varrho \qquad (z_i z_j' = -\varrho)\,.$$

3. $S(S')$ appears when $\Delta(\Delta')$ shrinks to a line. Its equation is

$$z_1 = \pm \varrho z_2 \quad \text{or} \quad z_2 = \pm \varrho z_3 \quad \text{or} \quad z_3 = \pm \varrho z_1$$
$$(z_1' = \pm \varrho z_2' \quad \text{or} \quad z_2' = \pm \varrho z_3' \quad \text{or} \quad z_3' = \pm \varrho z_1')$$

4. $\mathfrak{F}(\mathfrak{F}')$ has the equation

$$z_1 z_1' - z_2 z_3' \equiv z_2 z_2' - z_3 z_1' \equiv z_3 z_3' - z_1 z_2' = \varrho$$
$$(z_1 z_1' - z_3 z_2' \equiv z_2 z_2' - z_1 z_3' \equiv z_3 z_3' - z_2 z_1' = \varrho)$$

ϱ represents in all these definitions a real non-negative parameter.

For the computation of the holomorphy envelope we will essentially use the following lemma: *If the funtion $f(\alpha, \beta, \gamma)$ of three complex variables is analytic in the product* P *of the three upper half-planes, except maybe for* $\arg \alpha \leqslant \arg \beta \leqslant \arg \gamma$, *then* f *is analytic in* P.

To prove this lemma let us write $\log \alpha = u_1$, $\log \beta = u_2$, $\log \gamma = u_3$. Expressed in the variables u_i, the hypothesis of the lemma is that f is analytic in a certain tube, and the thesis of the lemma is that f is analytic in the convex hull of this tube, a fact which results from the tube theorem ([2]), Th. 9, p. 92).

In applications of this lemma, we will not exactly have a tube to start with, because of some extra singularities (cuts). These extra singularities will not however intersect with the singularities to be eliminated. From this it follows that, since the tube theorem may be proved by means of the continuity theorem*), it may be applied to the latter singularities. This process of course does not eliminate the extra singularities.

2. Analytic completion

Following K.-W. we perform the analytic completion in three steps

1. Continuation through S and S′

Figure 1 represents the original domain of analyticity in the variable z_1. We are interested in the part D of the excluded domain.

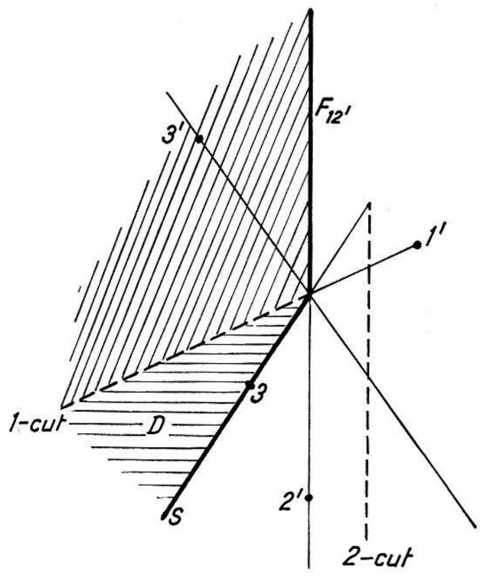

Fig. 1

*) Such a proof has been given by R. Jost and was reproduced in [5]). Similar proofs had been given earlier by K. Stein[6]) (for the case of two complex variables, see Hilfsatz 1, p. 557) and by G. Källén and A. Wightman (private communication of Prof. A. Wightman to Prof. R. Jost).

In order to show that D is contained in the holomorphy envelope we apply the lemma with the following variables $\alpha = z_1 z'_1$, $\beta = z_3 z'_1$, $\gamma = z'_2 z_3$. For fixed β, D is the product of the domains D_α and D_γ shown on Fig. 1′.

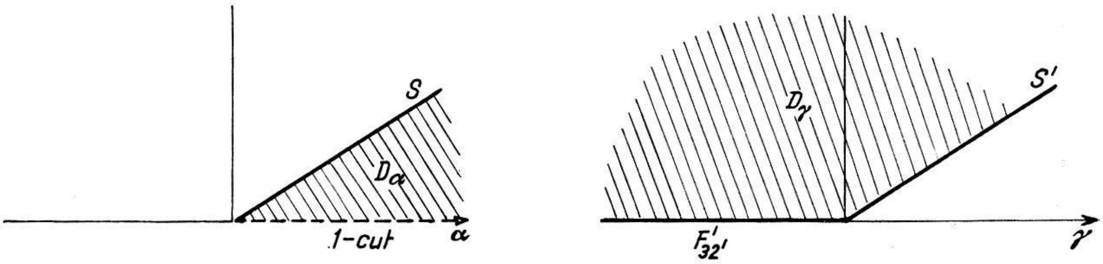

Fig. 1′

After this partial completion, the new boundary of the domain of analyticity is $F'_{32'}$ because if $\arg \gamma$ becomes bigger than π, the 2-cut intersects with D.

2. *Continuation through F and F′*

Figure 2 represents the new domain of analyticity in the variable z_1, and we are interested in the part D' of the excluded domain.

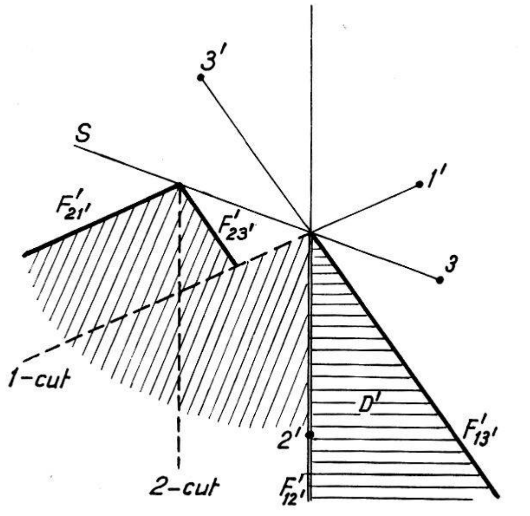

Fig. 2

In order to show that D' is contained in the holomorphy envelope, we again apply the lemma with the variables $\alpha = - z_1 z'_2$, $\beta = -z'_2/z'_3$, $\gamma = - 1/z_3 z'_3$. For fixed β, D' is the product of D'_α and D'_γ (cf. Fig. 2′).

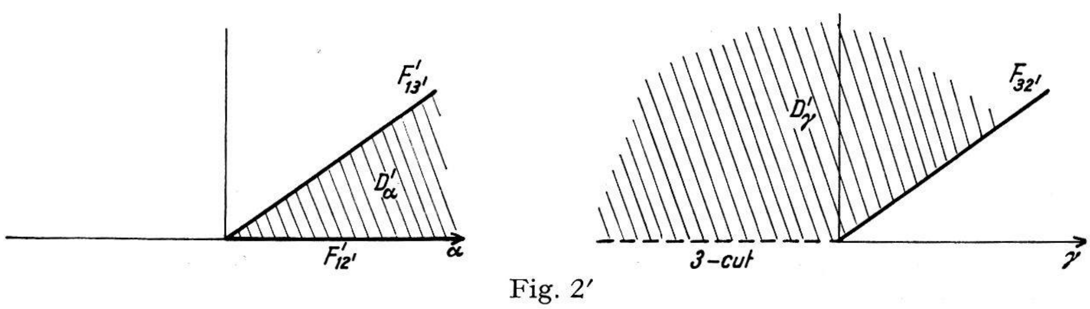

Fig. 2′

3. *Final completion*

After the second step, Figure 2 becomes Figure 3 and we are interested in the part D'' of the excluded domain.

Figure 3 is essentially identical to Figure 26 of K.-W. Following K.-W. we remark that the point $z = z_3 z_3'/z_2'$ is on the continuation of the boundary $\mathfrak{F}$ of D''. Adding the equations $z_1 z_1' = \varrho$ and $z_1 z_3' + z_3 z_3' = \varrho'$ of the 1-cut and $F'_{23'}$ respectively one finds indeed

$$z_1 = \frac{z_3 z_3'}{z_2'} - \frac{\varrho + \varrho'}{z_2'}$$

so that $\mathfrak{F}$ coincides with the hypersurface defined in the preliminaries.

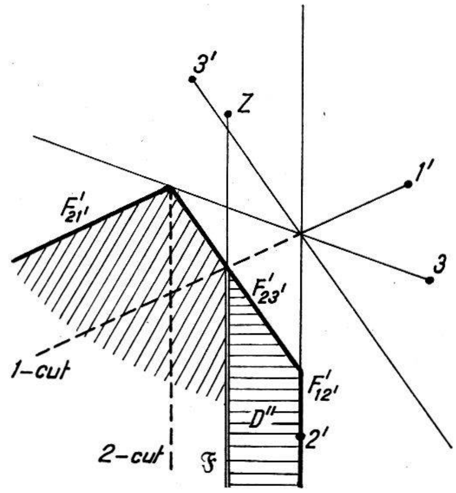

Fig. 3

In order to show that D'' is contained in the holomorphy envelope, we apply once more the lemma with the variables

$$\alpha = -(z_1 - z)\, z_2' \qquad \beta = -\frac{z_2'}{z_3'} \qquad \gamma = \frac{-1}{z_3 z_3'} .$$

For fixed β, D'' is contained in the product of D''_α and D''_γ as depicted in Figure 3′ and it is important to remark that the 1-cut and the 2-cut do not intersect with D''.

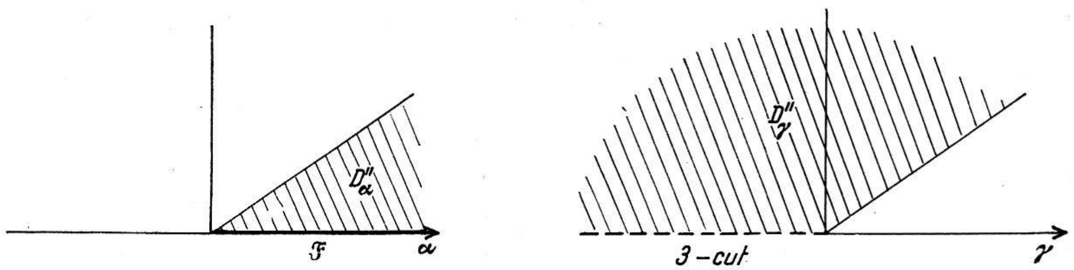

Fig. 3′

4. Conclusion

It is easy to show, using the Kantensatz (see [1]), Satz 20, p. 52) that the domain of analyticity finally obtained is a natural domain of holomorphy. It may be described as follows*) using our z_1-plots with fixed Δ' and z_3.

(a) If the 1-cut and the 2-cut do not intersect, they are the only present singularities.

(b) If the 1-cut and the 2-cut intersect, the situation is depicted either by Figure 3 or by Figure 4.

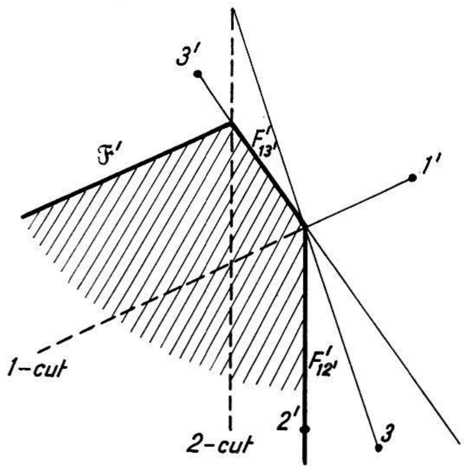

Fig. 4

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*) Private communication of Prof. R. JOST.