Quantum theory in real Hilbert space. III, Fields of the first kind (linear field operators)

Quantum Theory in Real Hilbert Space III: Fields of the 1st kind (Linear Field Operators)

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Abstract: The method of RHS (real Hilbert space, see[1] [2])) is applied to the free scalar and spinor fields. We remark, that *two kinds of fields exist:*

Fields of the 1st kind commute with $\breve{J}$ ($\to i = \sqrt{-1}$ in CHS (complex Hilbert space)). In CHS, they are *linear operators.*

Fields of the 2nd kind anti-commute with $\breve{J}$. In CHS, they are *anti-linear operators.* The general formulas of this article are valid for both cases. In this publication, only the fields of the 1st kind are explicitly discussed. The relation between *statistics* and *strong time reflection* (CT) are clarified. Furthermore, a concise formulation of *contragredient four-spinors* is given. Some well known formulas are explicitly restated in RHS in order to show the difference between fields of the 1st kind and fields of the 2nd kind[3]).

§ 1. Field Operators

We consider the *scalar field* $w(x)$ ($\neq w^T(x)$) and the *N-component spinor field* $\psi^A(x)$ ($\neq \psi^{TA}(x)$, $AB\ldots = 12\ldots N$). Field operators satisfy the *wave* equation

$$(\Box - M^2)\, w(x) = (\Box - M^2)\, \psi^A(x) = 0\,, \tag{1.1}$$

$$\Box = -\operatorname{sig}(g^{nn})\, g^{\alpha\beta}\, \partial_\alpha\, \partial_\beta = \Delta - \partial_t^2\,,$$

$$\Delta = \sum_1^d (\partial_i)^2\,; \qquad d = n - 1\,; \qquad \partial_n = \partial_t\,. \tag{1.2}$$

Observables $F^{\alpha\cdots}(x)$ are *bi-linear forms* in $w(x)$ and $w^T(x)$ (*or* $\psi^A(x)$ and $\psi^{TA}(x)$) and their derivatives (involving numbers or $\breve{J}$-dependent operators):

$$w_\alpha(x) = \partial_\alpha w(x)\,; \qquad \psi_\alpha^A(x) \equiv \partial_\alpha\, \psi^A(x)\,. \tag{1.3}$$

There exist two kinds of observables, $F^{(1)\alpha\cdots}(x)$ and $F^{(2)\alpha\cdots}(x)$, depending on whether the transposed field operator operates *after* ($F^{(1)}$) or *before* ($F^{(2)}$)

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the untransposed operators. Observables are *symmetric operators* in RHS ($F^T = F$), which *commute* with $\breve{J}$. From

$$[A\,B, C] = A\,[B, C]_{\mp} \pm [A, C]_{\mp}\,B \qquad (1.4)^{*})$$

follows, that fields either *commute* (*fields of the 1st kind*)

$$[\breve{J}, w(x)] = 0\,; \qquad [\breve{J}, \psi^A(x)] = 0 \qquad (1.5, 1^{st}k.)$$

or *anti-commute* (*fields of the 2nd kind*) with $\breve{J}$.

$$(\breve{J}, \overline{w}(x)) = 0\,; \qquad (\breve{J}, \overline{\psi}^A(x)) = 0\,. \qquad (1.5, 2^{nd}k.)^{*})\ ^{**})$$

Using the operators

$$\breve{J} = \jmath \times 1\,, \qquad \overline{K} = k \times 1\,, \qquad \breve{\overline{L}} = l \times 1 \qquad (1.6)$$

where $\jmath$, k and l are the pseudoquaternions (see I A-4.9), one concludes from (we write w, for w and ψ^A)

$$w = 1 \times w_{(r)} + \jmath \times w_{(i)} + k \times w_{(k)} + l \times w_{(l)} \qquad (1.7)$$

(a form analogous to (I A-2.3)) and its transposed, that only

$$w = 1 \times w_{(r)} + \jmath \times w_{(i)} \qquad (1.8, 1^{st}k.)$$

or

$$\overline{w} = k \times w_{(k)} + l \times w_{(l)} = (1 \times w_{(k)} + \jmath \times w_{(l)})\,(k \times 1) = w\,\overline{K} \qquad (1.8, 2^{nd}k.)$$

can occur in the bilinear forms defining observables. *The present article is essentially restricted to fields of the 1st kind.* However some formulas, valid for either kind of fields are included. The discussions of fields of the 2nd kind is reserved to a later publication (see [3])).

For either kind of field, a *phase transformation*

$$'w(x) = e^{\lambda \breve{J}}\,w(x)\,; \qquad 'w^T(x) = w^T(x)\,e^{-\lambda \breve{J}} \qquad (1.9)$$

leaves the observables invariant.

§ 2. Quantization of the scalar field

We look for an observable $\theta^{\alpha\beta}(x)$, from which $\breve{\Pi}_\mu$ and $\breve{M}_{\mu\nu}$ may be constructed (see (I 0.25), (I 0.27)). In principle, bilinear observables, for example the scalar

$$F^{(1)}(x) = w^T(x)\,w(x) \qquad (2.1)$$

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*) $[A, B] = [A, B]_- = AB - BA\,;\ (A, B) = [A, B]_+ = AB + BA$.

**) Operators with a $^-$ ($\overline{K}, \overline{L}, \overline{w}, \ldots$) anti-commute with $\breve{J}$.

transform according to

$$'F^{(1)}('x) = (O_{(L)}^{-1}\, w^T('x)\, O_{(L)})\, (O_{(L)}^{-1}\, w('x)\, O_{(L)}) =$$

$$= F^{(1)}(L^{-1}\,'x) = F^{(1)}(x) \tag{2.2}$$

if

$$'w('x) = O_{(L)}^{-1}\, w('x)\, O_{(L)} = e^{\lambda \breve{J}}\, w\,(L^{-1}\,'x)\,. \tag{2.3}$$

For infinitesimal transformations, the phase λ must be zero. The proper Lorentz-group $\{L_{(\mathrm{cont})}\}$, is therefore generated in RHS by (I 5.6) with

$$[\breve{J}\,\breve{\Pi}_\mu,\, w(x)] = -\, w_\mu(x)\,, \tag{2.4}$$

$$[\breve{J}\,\breve{M}_{\mu\nu},\, w(x)] = -\,[x_\mu,\, \partial_\nu]\; w(x)\,. \tag{2.5}$$

Thus, fields of the 1st kind transform like scalar observables. (For fields of the 2nd kind, it is essential that $\breve{J}$ is *inside* of the commutators (2.4) and (2.5)!) For pseudochronous and pseudochorous transformations, a phase factor may occur.

We have, for the most general $\theta^{\alpha\beta}$, the form:

$$\theta^{\alpha\beta} = \alpha_1\, \theta^{(1)\,\alpha\beta} + \alpha_2\, \theta^{(2)\,\alpha\beta}\,, \tag{2.6}$$

$$\theta^{(1)\,\alpha\beta}(x) = \big(w^{T\alpha}\, w^\beta + w^{\beta T}\, w^\alpha - g^{\alpha\beta}(w_\varrho^T\, w^\varrho + M^2\, w^T\, w)\big)\,(x)\,, \tag{2.7(1)}$$

$$\theta^{(2)\,\alpha\beta}(x) = \big(w^\alpha\, w^{T\beta} + w^\beta\, w^{T\alpha} - g^{\alpha\beta}(w_\varrho\, w^{T\varrho} + M^2\, w\, w^T)\big)\,(x) \tag{2.7(2)}$$

and (2.4) takes the form

$$\begin{aligned}
&-[\breve{J}\,\breve{\Pi}_\mu,\, w(y')] = \\
&= -\int d\sigma_\alpha(y)\,\{\alpha_1\,[\breve{J}(w^{T\alpha}\, w_\mu)\,(y),\, w(y')] + \alpha_2\,[\breve{J}(w^\alpha\, w_\mu^T)\,(y),\, w(y')] \\
&\qquad + \alpha_1\,[\breve{J}(w_\mu^T\, w^\alpha)\,(y),\, w(y')] + \alpha_2\,[\breve{J}(w_\mu\, w^{T\alpha})\,(y),\, w(y')]\} \\
&\quad + \int d\breve{\sigma}_\mu(y)\,\{\alpha_1\,[\breve{J}(w_\alpha^T\, w^\alpha)\,(y),\, w(y')] + \alpha_2\,[\breve{J}(w^\alpha\, w_\alpha^T)\,(y),\, w(y')]\} \\
&\quad + M^2 \int d\breve{\sigma}_\mu(y)\,\{\alpha_1\,[\breve{J}(w^T\, w)\,(y),\, w(y')] + \alpha_2\,[\breve{J}(w\, w^T)\,(y),\, w(y')]\} \\
&\equiv w_\mu(y')\,.
\end{aligned} \tag{2.8}$$

We chose $\tau(y) = \tau(y') = 0$ i.e. y and y' are events on the same hypersurface with a time like normal $\breve{\nu}^\alpha(y)$ $(d\breve{\sigma}^\alpha(y) = \breve{\nu}^\alpha(y)\; d\sigma(y)$; $(\breve{\nu}^\alpha\,\breve{\nu}_\alpha)\,(y) = \mathrm{sig}\,(g^{nn}))$. w, w_μ, w_μ^T and w^T being linearly independent, all terms, except the first and fourth terms, have to cancel out. In particular, the last integral proportional to M^2 has to vanish. This leads to:

$$\alpha_1 [\breve{J}(w^T w)(y), w(y')] + \alpha_2 [\breve{J}(w w^T)(y), w(y')] = 0, \tag{2.9}$$

$$-\alpha_1 [\breve{J}(w^{T\alpha} w_\mu)(y), w(y')] - \alpha_2 [\breve{J}(w_\mu w^{T\alpha})(y), w(y')] = $$
$$= \breve{\delta}^\alpha(y\, y')\, w_\mu(y). \tag{2.10}$$

Where $\breve{\delta}^\alpha(y\,y')$ is the (pseudochronous) δ-function on the surface $\tau(y) = 0$:

$$\breve{\delta}^\alpha(y\, y') = \breve{\delta}^\alpha(y'\, y); \qquad \int d\breve{\sigma}_\alpha(y)\, \breve{\delta}^\alpha(y'\, y)\, f(y) = f(y'). \tag{2.11}$$

In order that the 2nd and 3rd term cancel against the second integral, the symmetry condition, compatible with (2.11)

$$\int d\breve{\sigma}_\mu(y)\, \breve{\delta}_\alpha(y\, y')\, f(y) = \text{sig}(g^{nn})\, (\breve{\nu}_\mu \breve{\nu}_\alpha)(y')\, f(y') \tag{2.12}$$

must be satisfied. (2.9) and (2.10) are necessary conditions for (2.4). We may integrate these two conditions, using the *pseudochronous invariant number*, defined by

$$\breve{D}^0(x\, y) = \breve{D}^0(x - y) = -\breve{D}^0(y\, x), \tag{2.13}$$

$$(\Box_x - M^2)\, \breve{D}^0(x\, y) = 0, \tag{2.14}$$

$$\breve{D}^0(y\, y') = 0; \qquad \partial^y_\alpha\, \breve{D}^0(y\, y') = -\breve{\delta}_\alpha(y\, y') \tag{2.15 *)}$$

obtaining

$$\alpha_1 [\breve{J}\, w^T(x)\, w(z), w(y')] + \alpha_2 [\breve{J}\, w(z)\, w^T(x), w(y')] = \breve{D}^0(x\, y')\, w(z). \tag{2.16}$$

A somewhat lengthy calculation shows, that (2.16) is a sufficient condition for (2.5). The CR (commutation relation) (2.16) (although more general) is analogous to the CR proposed by GREEN[4]) and VOLKOV[5]) for spinor fields (see (8.8)).
Let us consider fields of the 1st kind. Applying (1.4), we find:

$$\breve{J}\, \alpha_1 \{w^T(x)\, [w(z), w(y)]_\mp \pm [w^T(x), w(y)]_\mp\, w(z)\} +$$
$$+ \breve{J}\, \alpha_2 \{w(z)\, [w^T(x), w(y)]_\mp \pm [w(z), w(y)]_\mp\, w^T(x)\} =$$
$$= \breve{D}^0(xy)\, w(z). \tag{2.17, 1st k.}$$

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*) The particular choice $\tau(y) = y^n - y'^n = 0$, $d\breve{\sigma}_\alpha(y) = (00\ldots 0\, d^d y)$; $\breve{\delta}^\alpha(y\,y') = (00\ldots 0\, \delta(\vec{y} - \vec{y}'))$ leads to the usual definition of $\breve{D}^0(x\,y)$ (metric (6.14)).

The most simple solution is

$$\boxed{[w(x),\, w(y)]_{\mp} = 0} \qquad (2.18, 1^{st}\,k.)$$

$$\breve{J}\,[w^T(x),\, w(y)]_{\mp} = \breve{D}^0(x\,y) \qquad (2.19, 1^{s}\,k.)$$

$$\boxed{\pm\,\alpha_1 + \alpha_2 = 1} \qquad (2.20, 1^{st}\,k.\ \mp)$$

It will be more convenient to write (2.19) in the more usual form:

$$\boxed{\breve{J}\,[w(x),\, w^T(y)]_{\mp} = \pm\,\breve{D}^0(x\,y)} \qquad (2.19, 1^{st}\,k.)$$

§ 3. The Charge Operator for Scalar Fields

The observables

$$\breve{j}^{(1)\alpha}(x) = \big((\breve{J}\,w)^T\,w^\alpha + w^{T\alpha}(\breve{J}\,w)\big)\,(x)\,, \qquad (3.1^{(1)})\,*)$$

$$\breve{j}^{(2)\alpha}(x) = -\,\big(\breve{J}\,w\,w^{T\alpha} + w^\alpha\,(\breve{J}\,w)^T\big)\,(x) \qquad (3.1^{(2)})\,*)$$

satisfy the continuity equation. We form:

$$\breve{j}^\alpha(x) = \beta_1\,\breve{j}^{(1)\alpha}(x) + \beta_2\,\breve{j}^{(2)\alpha}(x) \qquad (3.2)$$

defining thus a $\tau(y)$-independent scalar

$$Q = \beta_1\,Q^{(1)} + \beta_2\,Q^{(2)} = \int d\breve{\sigma}_\alpha(y)\,\breve{j}^\alpha(y) \qquad (3.3)$$

called the *charge* Q. If we require, that the phase transformation (1.9) is an orthogonal transformation in RHS

$$'w(x) = O^{-1}(\lambda)\,w(x)\,O(\lambda) = e^{\lambda\breve{J}}\,w(x)\,, \qquad (3.4)$$

$$O(\lambda) = e^{\breve{J}\lambda Q}. \qquad (3.5)$$

We need the CR:

$$-\,[\breve{J}\,Q,\, w(y')] =$$

$$= -\int d\breve{\sigma}_\alpha(y)\,\{\beta_1\,[(\breve{J}(\breve{J}\,w)^T w^\alpha)\,(y),\, w(y')] + \beta_1\,[(\breve{J}\,w^{T\alpha}\,\breve{J}\,w)\,(y),\, w(y')] +$$

$$+\,\beta_2\,[(w\,w^{T\alpha})\,(y),\, w(y')] - \beta_2\,[(\breve{J}\,w^\alpha(\breve{J}\,w)^T)\,(y),\, w(y')]\} \equiv \breve{J}\,w(y')\,. \qquad (3.6)$$

*) The signes have been chosen so as to give, for fields of the 1st kind:

$$\breve{j}^{(1)\alpha}(x) = \breve{J}^{-1}\,(w^T\,w^\alpha - w^{T\alpha}\,w)\,(x)\,, \qquad (3.1^{(1)},\ 1^{st}\,k.)$$

$$\breve{j}^{(2)\alpha}(x) = \breve{J}^{-1}\,(w\,w^{\alpha T} - w^\alpha\,w^T)\,(x)\,. \qquad (3.1^{(2)},\ 1^{st}\,k.)$$

For fields of the 1st kind, this condition reduces to

$$- [Q, w(y')] =$$

$$= \int d\breve{\sigma}_\alpha(y) \{\beta_1 [\breve{J}(w^T w^\alpha)(y), w(y')] - \beta_2 [\breve{J}(w^\alpha w^T)(y), w(y')] -$$

$$- \beta_1 [\breve{J}(w^{T\alpha} w)(y), w(y')] + \beta_2 [\breve{J}(w\, w^{T\alpha})(y), w(y')] =$$

$$= \int d\breve{\sigma}_\alpha(y)\, \breve{\delta}^\alpha(y\, y')\, w(y) = w(y') . \qquad (3.6, 1^{st}\,k.)$$

Comparison with (2.16) shows, that the first two terms do not contribute if $\beta_1 = \lambda \alpha_1$ and $\beta_2 = -\lambda \alpha_2$, while the second two terms equal the integral in the third member, if

$$\beta_1 = \alpha_1 , \qquad \beta_2 = -\alpha_2 . \qquad (3.7, 1^{st}\,k.)$$

Thus, even for the general CR (2.16), Q defined by (3.7) is the generator of the infinitesimal phase transformation for fields of the 1st kind.

§ 4. Charge Conjugation for the Scalar Field of the 1st kind

For fields of the 1st kind and for the most simple CR's (2.18₋) and (2.19₋), follows that:

$$'w(x) = O_{(C)}^{-1}\, w(x)\, O_{(C)} = w^T(x) , \qquad (4.1)$$

$$'w^T(x) = O_{(C)}^{-1}\, w^T(x)\, O_{(C)} = w(x) \qquad (4.1)^T$$

is an orthogonal transformation in RHS. The CR's lead to BE-statistics. In other words: $O_{(C)}$*-covariance requires BE-statistics.* We have further:

$$O_{(C)}^{-1}\, \theta^{(1)\alpha\beta}(x)\, O_{(C)} = \theta^{(2)\alpha\beta}(x) , \qquad (4.2^{(1)})$$

$$O_{(C)}^{-1}\, \theta^{(2)\alpha\beta}(x)\, O_{(C)} = \theta^{(1)\alpha\beta}(x) . \qquad (4.2^{(2)})$$

Thus, chosing $\alpha_1 = \alpha_2 = 1/2$ in (2.20₋), we find, that

$$\theta^{\alpha\beta}(x) = \frac{1}{2} (\theta^{(1)\alpha\beta} + \theta^{(2)\alpha\beta})(x) \qquad (4.3)$$

is invariant with respect to $O_{(C)}$. Furthermore, from

$$O_{(C)}^{-1}\, \breve{j}^{(1)\alpha}(x)\, O_{(C)} = \breve{j}^{(2)\alpha}(x) , \qquad (4.4^{(1)})$$

$$O_{(C)}^{-1}\, \breve{j}^{(2)\alpha}(x)\, O_{(C)} = \breve{j}^{(1)\alpha}(x) \qquad (4.4^{(2)})$$

follows, that on account of (3.7) and (3.3)

$$O_{(C)}^{-1}\, Q\, O_{(C)} = -\, Q \qquad (4.5)\,^{*})$$

i.e. $\breve{j}^{\alpha}$ and Q change sign under $O_{(C)}$.
We may now define two kinds of time reversal

(1) $T \to O_{(T)}$, *weak time reversal* (T)

$$'w('x) = O_{(T)}^{-1}\, w('x)\, O_{(T)} = w(T^{-1}\, 'x) \qquad (4.7)\,^{**})$$

with respect to which $\breve{j}^{\alpha}(x)$ is a pseudochronous vector, and Q a scalar;

(2) $T \to O_{(C)}\, O_{(T)} \equiv O_{(C\,T)}$, *strong time reversal* (CT)

$$'w('x) = O_{(C\,T)}^{-1}\, w('x)\, O_{(C\,T)} = w^{T}(T^{-1}\, 'x) \qquad (4.8)$$

with respect to which $j^{\alpha}(x)$ is an (ortho)vector and $\breve{Q}$ a pseudochronous scalar.

The second definition seems more appropriate because *classical particle theory* (see STUECKELBERG[8]) defines

$$j^{\alpha}(x) = \int_{-\infty}^{+\infty} d\lambda\, \dot{z}^{\alpha}(\lambda)\, \delta\big(x - z(\lambda)\big)\,,$$

$$\dot{z}^{\alpha}(\lambda) = \frac{d}{d\lambda}\, z^{\alpha}(\lambda)\,; \qquad (\dot{z}_{\alpha}\, \dot{z}^{\alpha})\,(\lambda) = \mathrm{sig}\,(g^{n\,n})\,,$$

$$\breve{Q} = \int d\breve{\sigma}_{\alpha}(y)\, j^{\alpha}(y) = \mathrm{sig}\big(\dot{z}^{n}(\lambda)\big)\,.$$

Thus $O_{(C)}$-covariance or strong time reversal $(O_{(C\,T)})$-covariance decide for BE-statistics in the case of scalar fields (see SCHWINGER[6]) and PAULI[7])).

§ 5. The Development of the Scalar Field (1st kind) in Terms of Positive Frequency Wave Packet Operators

Let us define integral operators Ω and $\Omega^{1/2}$, operating on space functions $f(\vec{x})$ which vanish sufficiently strong for $|\vec{x}| \to \infty$:

$$\Omega\, f(\vec{x}) = (M^2 - \Delta)^{1/2}\, f(\vec{x}) = \int d^d y\, \Omega(|\vec{x} - \vec{y}|)\, f(\vec{y})\,, \qquad (5.1)$$

$$\Omega^{1/2}\, f(\vec{x}) = (M^2 - \Delta)^{1/4}\, f(\vec{x}) = \int d^d y\, \Omega^{1/2}(|\vec{x} - \vec{y}|)\, f(\vec{y})\,. \qquad (5.2)$$

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*) $$O_{(C)}^{-1}\, \breve{j}^{\alpha}(x)\, O_{(C)} = -\, \breve{j}^{\alpha}(x)\,. \qquad (4.6)$$

**) $'x = Tx \to \{'x^i = x^i;\ 'x^n = -x^n\}$. $T^{-1} = T$. The arbitrary phase factor may be left out.

The kernels $\Omega(|\vec{z}|)$ are essentially Hankel functions decreasing $\propto \exp(-|M||\vec{z}|)$ for $|\vec{z}| \gg |M|^{-1}$. We note especially

$$\int d^d x\, g(\vec{x}) \cdot \Omega f(\vec{x}) = \int d^d x\, g(\vec{x})\, \Omega \cdot f(\vec{x}) = \int d^d x\, (\Omega^{1/2} g)\, (\Omega^{1/2} f)\, (\vec{x}). \qquad (5.3)^*)$$

In terms of Ω, we define *two denumerable sets* $\{u', u'', \ldots, u^{(\varrho)}, \ldots\}$ and $\{v', v'', \ldots, v^{(\varrho)}, \ldots\}$ of *positive frequency wave packet operators* (PFWP's) depending but on the operator $\breve{J}$ and vanishing for $|\vec{x}| \to \infty$. They satisfy:

$$-\partial_t u'(\vec{x}, t) = \partial^n u'(x) = \breve{J}\, \Omega\, u'(\vec{x}, t). \qquad (5.4)$$

These sets are *normalised* in terms of 'matrix elements'

$$j^\alpha(u', v')\,(x) = -j^\alpha(v', u')\,(x) = \frac{1}{2} \breve{J}^{-1} \left(u' \,(.\partial^\alpha - \partial^\alpha.)\, v'\right)(x), \qquad (5.5)^*)$$

$$Q(u', v') = \int \left(d\breve{\sigma}_\alpha\, j^\alpha(u', v')\right)(y) = \frac{1}{2} \breve{J}^{-1} \int d^d y \left(u' \,(.\partial^n - \partial^n.)\, v'\right)(y) =$$

$$= \frac{1}{2} \int d^d y \left(u' \,(.\Omega - \Omega.)\, v'\right)(\vec{y}, t) = 0, \qquad (5.6)$$

$$Q(u''^T, u') = -Q(u', u''^T) = \frac{1}{2} \int d^d y \left(u''^T \,(.\Omega + \Omega.)\, u'\right)(\vec{y}, t) =$$

$$= \int d^d y \left((\Omega^{1/2} u''^T)\,(\Omega^{1/2} u')\right)(\vec{y}, t) \equiv \delta_{u'' u'} \geqq 0. \qquad (5.7)$$

Furthermore, each set is *complete* if the $\breve{J}$ dependent operators:

$$\mathbf{S}_{u'}\, u'(x)\, u'^T(y) = D^+(x\,y) = D^+(x - y), \qquad (5.8^+)$$

$$\mathbf{S}_{u'}\, u'^T(x)\, u'(y) = D^-(x\,y) = D^-(x - y) = D^+(x\,y) = D^{+\,T}(x\,y) \qquad (5.8^-)$$

depend but on $x - y$ and are invariant with respect to the subgroup $\{L_{(\text{ochr})}\}$. A PF-solution $f^+(x)$ of the wave equation may be expanded in terms of one of the sets:

$$f^+(x) = \mathbf{S}_{u'}\, u'(x)\, f^+_{u'} = \frac{1}{2} \breve{J}^{-1} \int d\breve{\sigma}_\alpha(y)\, D^+(x\,y)\, (.\partial^\alpha_y - \partial^\alpha_y.)\, f^+(y). \qquad (5.9^+)$$

For NF-solutions:

$$f^-(x) = \mathbf{S}_{v'}\, v'^T(x)\, f^-_{v'} = -\frac{1}{2} \breve{J}^{-1} \int d\breve{\sigma}_\alpha(y)\, D^-(x\,y)\, (.\partial^\alpha_y - \partial^\alpha_y.)\, f^-(y) \qquad (5.9^-)$$

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*) Operators $.\partial^\alpha$, $.\Omega$ (with point on the left) operate, in the usual way, *to the right*. Operators $\partial^\alpha.$, $\Omega.$ (with point on the right) operate *to the left*.

holds. The general solution of the wave equation may be written as

$$w(x) = 2^{-1/2} \left(\mathbf{S}_{u'} a_{u'} u'(x) + \mathbf{S}_{v'} b^T_{v'} v'^T(x) \right) =$$
$$= - \int d\breve{\sigma}_\alpha(y) \breve{D}^0(x\,y) \left(.\partial^\alpha_y - \partial^\alpha_y.\right) w(y) =$$
$$= \int d\breve{\sigma}_\alpha(y) \left(- \partial^\alpha_x \breve{D}^0(x\,y) \cdot w(y) - \breve{D}^0(x\,y)\, w^\alpha(y) \right) \qquad (5.10)$$

with

$$\breve{D}^0(x\,y) = \frac{1}{2} \breve{J} \left(D^+ - D^-\right)(x\,y) = - \breve{D}^0(y\,x) = \breve{D}^0(x\,y)\,. \qquad (5.11)$$

A comparison, for $x = y'$, shows, that the *pseudochronous number* (5.11) is identical with the number defined by (2.13), (2.14) and (2.15).

The CR's, resp. ACR's (2.18) and (2.19) imply

$$[a_{u'}, a_{u''}]_\mp = [b^T_{v'}, b^T_{v''}]_\mp = [a_{u'}, b^T_{v''}]_\mp = 0\,, \qquad (5.12)$$

$$[a_{u'}, a^T_{u''}]_\mp = \pm\, \delta_{u'\,u''}\,; \qquad [b^T_{v'}, b_{v''}]_\mp = \mp\, \delta_{v'\,v''}\,;$$
$$[a_{u'}, b_{v'}]_\mp = 0\,. \qquad (5.13)$$

As $a_{u'}\, a^T_{u'}$ and $a^T_{u'}\, a_{u'}$ are positive operators, (5.13_+) contains an algebraic contradiction. This is PAULI's[9]) argument for excluding ACR's and FD-statistics for scalar fields.

The CR's can be satisfied in terms of the creation- (a^T) and annihilation-(a)-operators:

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & . \\ 0 & 0 & \sqrt{2} & 0 & . \\ 0 & 0 & 0 & \sqrt{3} & . \\ . & . & . & . & . \end{pmatrix}; \quad a^T = \begin{pmatrix} 0 & 0 & 0 & . \\ 1 & 0 & 0 & . \\ 0 & \sqrt{2} & 0 & . \\ . & . & . & . \end{pmatrix}; \quad N = a^T a = \begin{pmatrix} 0 & 0 & 0 & . \\ 0 & 1 & 0 & . \\ 0 & 0 & 2 & . \\ . & . & . & . \end{pmatrix} \qquad (5.14)$$

writing

$$\left. \begin{aligned} a_{u(\varrho)} &= 1 \times (1 \times 1) \times (1 \times 1) \times \cdots \times (a \times 1) \times (1 \times 1) \times \cdots \\ b_{v(\varrho)} &= 1 \times (1 \times 1) \times (1 \times 1) \times \cdots \times (1 \times a) \times (1 \times 1) \times \cdots \end{aligned} \right\} \qquad (5.15)$$

The eigenvalues of $N_{u'} = a^T_{u'}\, a_{u'}$ are the non-negative integers and the charge has the form

$$Q = \mathbf{S}_{u'} \left(N_{u'} + \frac{1}{2}\right) - \mathbf{S}_{v'} \left(N_{v'} + \frac{1}{2}\right) = \mathbf{S}_{u'} N_{u'} - \mathbf{S}_{v'} N_{v'}\,. \qquad (5.16)$$

No 'zero-point charge' appears because a 1 to 1 correspondance between the sets $\{u'\}$ and $\{v'\}$ can be established.

Charge conjugation can be written explicitly, if the sets $\{u'\}$ and $\{v'\}$ are chosen *identical:*

$$'a_{u'} = O_{(C)}^{-1}\, a_{u'}\, O_{(C)} = b_{v'}\,,$$

$$'b_{v'} = O_{(C)}^{-1}\, b_{v'}\, O_{(C)} = a_{u'} \tag{5.17}$$

from which:

$$O_{(C)} = O_{(C)}^{-1} = O_{(C)}^{T} \tag{5.18}$$

follows. The operator is

$$O_{(C)} = \prod\nolimits_{u'} e^{-\pi/2\,(a_{u'}^T\, b_{v'} - b_{v'}^T\, a_{u'})}\, e^{\pm \breve{J} \pi N_{u'}}\,. \tag{5.19}$$

In order to quantize explicitly $\breve{\Pi}_\mu$, denumerable sets, satisfying

$$\partial^\alpha u'(x) \cong \breve{J}\, \breve{k}'^{\alpha}\, u'(x)\,,$$

$$\breve{k}'_\alpha\, \breve{k}'^{\alpha} + M^2 = 0\,; \qquad \breve{k}'^{n} \geqq |M| \tag{5.20}$$

have to be chosen. Taking in account some additional orthogonality relations, one finds

$$\breve{\Pi}^\mu \cong \mathbf{S}_{u'} \left(N_{u'} + \frac{1}{2}\right) \breve{k}'^{\mu} + \mathbf{S}_{v'} \left(N_{v'} + \frac{1}{2}\right) \breve{k}'^{\mu}\,. \tag{5.21}$$

(5.21) is symmetric for particle and antiparticle states. Aside from the infinite 'zero-point contribution', the energy spectrum has a lower limit: Thus, thermo-statistics with a positive absolute temperature can be applied[10]).

§ 6. Dual Spin-Spaces

In this section we deal exclusively with real and complex numbers: we distinguish between φ^A *contravariant* (ctr) and χ_A *covariant* (cov) *spinors.* φ^A and χ_A are two dual, N-dimensional *spin spaces* (SS). We shall distinguish between (real) RSS and (complex) CSS. The general case being CSS, all symbols φ^A, χ_A, ... should be written as $\hat{\varphi}^A$, $\hat{\chi}_A$, ...
For any complex number

$$\hat{\varrho} = \varrho_{(r)} + i\, \varrho_{(i)} \rightleftarrows \varrho = \varrho(\breve{J}) = \varrho_{(r)} + \breve{J}\, \varrho_{(i)} \tag{6.1*)}$$

gives the relation between the complex number $\hat{\varrho}$ and the $\breve{J}$-dependent operator $\varrho = \varrho\,(\breve{J})$ (see I, Annex 2).

*) We shall use $\hat{\varrho}$, $\hat{\sigma}$, ... for complex numbers (resp. for $\breve{J}$-dependent operators) and λ, μ, ... for real numbers.

SS or A-space is related to physical space-time or α-space by a *mixed A-space bi-spinor α-space vector* $\gamma^{\alpha A}{}_B$, satisfying

$$\gamma^{\alpha A}{}_C \gamma^{\beta C}{}_B + \gamma^{\beta A}{}_C \gamma^{\alpha C}{}_B = 2\, g^{\alpha\beta} \gamma^{0A}{}_B \tag{6.2}$$

or, in matrix notation

$$(\gamma^\alpha, \gamma^\beta) = 2\, g^{\alpha\beta} \gamma^0, \qquad (6.2\ \mathrm{M})\ ^{*)}$$

where $\gamma^{0A}{}_B = \delta^A_B$ is the identity bi-spinor. (6.2) with $\widehat{\gamma}^{\alpha A}_{\breve{B}}$ is an *algebraic equation* and, with $\gamma^{\alpha A}{}_B$, it is an *operator equation* depending but on J. Two contragredient spinors allow to define a scalar $\chi\varphi = \chi_A \varphi^A$ and a vector $\chi\gamma^\alpha\varphi = \chi_A \gamma^{\alpha A}{}_B \varphi^B$ in α-space.

We resume, without proof and references, a number of well known theorems:

(1) The ring $\Gamma = \{\gamma^r\}$, $r = 0, \alpha, [\alpha_1\alpha_2], \ldots, [\alpha_1\alpha_2\ldots\alpha_n] = 1, 2, \ldots, 2^n$ is formed from the totally antisymmetric α-space tensors

$$\gamma^{\alpha_1\ldots\alpha_\nu} = \gamma^{[\alpha_1\ldots\alpha_\nu]} = (\nu!)^{-1} \sum\nolimits_P (-1)^P\, P(\gamma^{\alpha_1}\ldots\gamma^{\alpha_\nu}) =$$
$$= \gamma^{\alpha_1}\ldots\gamma^{\alpha_\nu} \text{ if all } \alpha_1\ldots\alpha_\nu \text{ are different.} \qquad (6.3\ \mathrm{M})\ ^{**)}$$

The subsets $\Gamma^{(\nu)} = \{\gamma^{\alpha_1\ldots\alpha_\nu}\}$ contain $\binom{n}{\nu}$ independent elements.

(2) Two irreducible representations of (6.2) $'\gamma^\alpha$ and γ^α are related by

$$'\gamma^\alpha S = S\gamma^\alpha \qquad (6.4\ \mathrm{M})$$

where S is either the zero matrix or a non-singular square matrix. In particular, for $n = 2m$ (even), all irreducible representations are *equivalent* i.e. we have

$$'\gamma^\alpha = S\gamma^\alpha S^{-1}; \qquad S = \{S'^A{}_A\}; \qquad 'N = N. \qquad (6.5\ \mathrm{M})$$

For $n = 2m + 1$, two non equivalent representations exist. A matrix S commuting with the n γ^α's of an irreductible representation is always of the form:

$$[S, \gamma^\alpha] = 0; \qquad S = \varrho\,\gamma^0. \qquad (6.6\ \mathrm{M})$$

Furthermore if S and S' satisfy (6.5 *M*), we have

$$S' = \varrho\, S. \qquad (6.7\ \mathrm{M})$$

---

*) Matrix multiplication is always a *contraction over contragredient spinor indices*. Formulas with (... *M*) are matrix identities.

**) Σ_P implies summation over all $\nu!$ permutations P of $\alpha_1\ldots\alpha_\nu$ with $(-1)^P = +(-)1$ for even (odd) permutations.

(3) For $n = 2\,m$, the 2^n elements of the ring $\Gamma = \{\gamma^r\}$ are linearily independent.

(4) The relation

$$\gamma^r \gamma^s = \varepsilon^{rs} \gamma^t ; \qquad t = t(r\,s) ; \qquad \varepsilon^{rs} = \pm 1 \tag{6.8 M}$$

holds.

(5) If an irreducible representation of (6.2) for $n = 2\,m$ is found, we may form

$$\gamma_{n+1} = \varrho\, \gamma^{12\ldots n} , \qquad \varrho = 1 \text{ or } \breve{J} \tag{6.9 M}$$

and thus obtain an irreducible representation for $'n = n + 1 = 2\,m + 1$, because

$$(\gamma_{n+1}, \gamma^\alpha) = 0 ; \qquad \alpha = 12 \ldots 2\,m ; \qquad (\gamma_{n+1})^2 = \gamma^0 . \tag{6.10 M}$$

(6) The number ξ^r in

$$(w.\,s.)\ \gamma_r \gamma^r = \xi^r \gamma^0 = \xi^{(\nu)} \gamma^0 \tag{6.11 M) *}$$

depends but on the set $\Gamma^{(\nu)}$ and takes the values

$$\xi^{(\nu)} = (-1)^{\nu/2} \quad \text{for} \quad \nu = 2\,\mu , \tag{6.12}$$

$$\xi^{(\nu)} = (-1)^{(\nu-1)/2} \text{ for } \quad \nu = 2\,\mu + 1 \tag{6.13}$$

if

$$\text{signat}\,(g^{\alpha\beta}) = +\,(11 \ldots 1 - 1) . \tag{6.14}$$

(7) If the representation is given, to each L corresponds a transformation $S_{(L)}$

$$L \to \varrho\, S_{(L)} ; \qquad S \to L_{(L)} \tag{6.15}$$

(defined up to a factor ϱ) leaving $\gamma^{\alpha A}{}_B$ invariant:

$$\gamma^{'\alpha' A}{}_{'B} = L^{'\alpha}{}_\alpha\, S^{'A}_{(L)\,A}\, \gamma^{\alpha A}{}_B\, S^{-1\,B}_{(L)\ 'B} , \tag{6.16}$$

$$\gamma^{'\alpha} = L^{'\alpha}{}_\alpha\, S_{(L)}\, \gamma^\alpha\, S^{-1}_{(L)} . \tag{6.16 M}$$

This relation in SS is perfectly analogous to (I, 3.9) in RHS; instead of the arbitrary phase factor in (I, 3.12), the arbitrary $\hat{\varrho}$ appears in (6.15).

(8) To the infinitesimal transformation

$$L^{'\alpha}{}_\alpha = \delta^{'\alpha}{}_\alpha + \frac{1}{2}\, \delta\, \omega^{\mu\nu}\, \Sigma_{\mu\nu}{}^{'\alpha}{}_\alpha \tag{6.17}$$

*) ($w.\,s.$) means 'without summation over indices in contragredient positions'.

corresponds

$$S_{(L)} = \gamma^0 + \frac{1}{4}\,\delta\,\omega^{\mu\nu}\,\gamma_{\mu\nu}\,. \tag{6.18 M}$$

(9) To the transformations $L = (P, T, PT)$*) corresponds, for $n = 2m$ $(d = n - 1)$**)

$$S_P = \varrho\,\gamma^n; \qquad S_T = \varrho\,\gamma^{12\ldots d}; \qquad S_{PT} = \varrho\,\gamma_{n+1};$$

$$S_P^{-1} = \varrho^{-1}\,\gamma_n; \quad S_T^{-1} = \varrho^{-1}\,\gamma_{d\ldots 21}; \quad S_{PT}^{-1} = \varrho^{-1}\,\gamma^{n+1}. \tag{6.19 M}$$

(10) If the Dirac-equations are written as

$$(\gamma^{\alpha A}{}_B\,\partial_\alpha + M\,\gamma^{0A}{}_B)\,\varphi^B(x) = (\gamma^\alpha\,\partial_\alpha + M\,\gamma^0)\,\varphi(x) = 0\,, \tag{6.20 ctr}$$

$$\chi_A(x)\,(\gamma^{\alpha A}{}_B\,\partial_\alpha\cdot - M\,\gamma^{0A}{}_B) = \chi(x)\,(\gamma^\alpha\,\partial_\alpha - M\,\gamma^0)\,. = 0\,, \tag{6.20 cov}$$

the momentum-energy-density tensor is

$$\theta^{\alpha\beta}(x) = T^{\alpha\beta}(x) + \frac{1}{2}\,\partial_\varrho\,s^{\varrho\alpha\beta}(x)\,, \tag{6.21}$$

$$\partial_\varrho\,s^{\varrho\alpha\beta}(x) = (T^{\beta\alpha} - T^{\alpha\beta})\,(x)\,, \tag{6.22}$$

$$T^\alpha{}_\beta(x) = \frac{1}{2}\,(\chi\,\gamma^\alpha\,(.\partial_\beta - \partial_\beta.)\,\varphi)\,(x)\,, \tag{6.23}$$

$$\partial_\alpha\,T^\alpha{}_\beta(x) = 0\,, \tag{6.24}$$

$$s^{\alpha\beta\gamma}(x) = \frac{1}{2}\,(\chi\,\gamma^{\alpha\beta\gamma}\,\varphi)\,(x) \tag{6.25}$$

and the current-charge-density vector

$$j^\alpha(x) = (\chi\,\gamma^\alpha\,\varphi)\,(x)\,, \tag{6.26}$$

$$j^\alpha(x) = j^\alpha_{(\text{convection})}(x) - \partial_\varrho\,m^{\varrho\alpha}(x)\,, \tag{6.27}$$

$$j^\alpha_{(\text{convection})}(x) = -\,(2\,M)^{-1}\,(\chi\,(.\partial^\alpha - \partial^\alpha.)\,\varphi)\,(x)\,, \tag{6.28}$$

$$m^{\alpha\beta}(x) = -\,(2\,M)^{-1}\,(\chi\,\gamma^{\alpha\beta}\,\varphi)\,(x)\,. \tag{6.29}$$

In order that $\varphi^A(x)$ and $\chi_A(x)$ satisfy the wave equation (1.1), M must be a *real number* and the signature must have the form (6.14).

---

*) $P \to \{'x^i = -x^i,\ 'x^n = x^n\}$, $T \to \{'x^i = x^i,\ 'x^n = -x^n\}$.
$PT \to \{'x^\alpha = -x^\alpha\}$ for $n = 2\,m$.

**) For $n = 2\,m+1$, only $S_{(PT)}$ can be defined. As we require the full L-group, our discussion is limited to $n = 2m =$ even, $d = n - 1 =$ odd.

In addition to these well known theorems, we add the two following theorems which follow from a theorem of FROBENIUS and SCHUR[11][12] (given in Annex):

(11) For $n = 2\,m$, the number of dimensions of an irreducible representation is

$$N = 2^{n/2} = 2^m . \tag{6.30}$$

(12) For

$$n = 2{,}4 \pmod 8 = 2,\, 4;\, 10,\, 12;\, \ldots \tag{6.31}$$

all representations are equivalent to a *real representation*. For

$$n = 6{,}8 \pmod 8 = 6,\, 8;\, 14,\, 16;\, \ldots \tag{6.32}$$

all representations are *necessarily complex*. However a matrix C, the *Pauli-Matrix*[9]), exists, relating

$$\widehat{\gamma}^{*r} = \widehat{C}\, \widehat{\gamma}^r\, \widehat{C}^{-1} \quad \text{or} \quad \gamma^{Tr} = C\, \gamma^r\, C^{-1} . \tag{6.33}$$

(13) Every IMG (irreducible matrix group, see Annex) $\{\pm\, '\widehat{\gamma}^r\}$ is equivalent to a unitary representation $\{\pm\, \widehat{\gamma}^r\}$. In RHS, this implies for every $\{\pm\, '\widehat{\gamma}^r\}$, there exists an S

$$'\gamma^r = S\, \gamma^r\, S^{-1} \tag{6.34 M}$$

so as to have

$$\gamma^{\sim Tr} \equiv \gamma^{\times r} \overset{*}{=} (\gamma^r)^{-1} = (w.\, s.)\, \xi^r\, \gamma_r . \tag{6.35 M) *)$$

(14) For $n = 2{,}4 \pmod 8$, where $\{'\widehat{\gamma}^r\}$ may be chosen real, the unitary representation may be chosen *real and orthogonal:*

$$\gamma^{\sim r} \overset{*}{=} (\gamma^r)^{-1} = (w.\, s.)\, \xi^r\, \gamma_r . \tag{6.36 M) *)$$

---

*) $\overset{*}{=}$ signifies 'equal in a particular representation'.

$\sim$ is the *transposed matrix* in SS (^{T} being reserved for transposed operator in RHS, see I).

$$\gamma_A^{\sim r\, B} = \gamma^{r\, B}{}_A . \tag{6.37}$$

$\sim$ interchanges the spinor-indices 'left $\rightleftarrows$ right', leaving their covariance unchanged!

$\times$ is the *hermitian conjugate* matrix in CSS

$$\widehat{\gamma}_A^{\times r\, B} = \widehat{\gamma}^{*r B}{}_A . \tag{6.38}$$

To it corresponds, in a $\breve{J}$-dependent representation, the operator relation in RHS

$$\gamma_A^{\times r\, B} = \gamma^{Tr B}{}_A . \tag{6.39}$$

§ 7. The fundamental spinors $\overset{\smile}{\eta}_{AB}$ and $\overset{\frown}{\eta}_{AB}$

In order to form observables $\theta^{\alpha\beta}$, $T^{\alpha}{}_{\beta}$, $s^{\alpha\beta\gamma}$ and j^{α} ((6.21)–(6.25)) from a contravariant field operator $\psi^{A}(x)$, we need a *non-singular fundamental spinor* η_{AB} from which we may form a *covariant operator* $\psi^{T}_{A}(x) = \eta_{AB}\,\psi^{TB}(x)$

$$\eta^{-1\,AC}\,\eta_{CB} = \eta_{BC}\,\eta^{-1\,CA} = \gamma^{0\,A}{}_{B}\,, \tag{7.1}$$

$$\eta^{-1}\,\eta = \eta\,\eta^{-1} = \gamma^{0}\,. \tag{7.1 M}$$

We shall assume a *real representation* of $\gamma^{\alpha\,A}{}_{B}$ i.e. restrict our considerations to $n = 2{,}4$ (mod. 8). However the theory can also be written for $\overset{\smile}{J}$-dependent $\gamma^{\alpha\,A}{}_{B}$ (thus for $n = 6{,}8$ (mod. 8), but the calculations are much longer*).

Analogous to (6.16), we require invariance of η with respect to the group $S_{(L)}$ (i.e. $c(L) = 1$ or $=\operatorname{sig}(\det(L'^{i}{}_{i})$ or $=\operatorname{sig}(L'^{n}{}_{n})$)

$$\eta_{'A\,'B} = c(L)\;\eta_{AB}\;S^{-1\,A}_{(L)\;\;'A}\;S^{-1\,B}_{(L)\;\;'B}\,, \tag{7.2}$$

$$\eta = c(L)\;S^{-1\sim}_{(L)}\;\eta\;S^{-1}_{(L)}\,. \tag{7.2 M}$$

For $L_{(\mathrm{cont})}$ (cf. (6.18)) we have

$$\gamma^{\sim\mu\nu}\,\eta + \eta\,\gamma^{\mu\nu} = 0\,, \tag{7.3 M}$$

which allows the *two possibilities:*

$$\gamma^{\sim\alpha} = \mp\,\overset{\smile}{\overset{\frown}{\eta}}\,\gamma^{\alpha}\,\overset{\smile}{\overset{\frown}{\eta}}{}^{-1} \quad \text{i. e.} \quad \gamma^{\sim\alpha}\,\overset{\smile}{\overset{\frown}{\eta}} = \mp\,\overset{\smile}{\overset{\frown}{\eta}}\,\gamma^{\alpha}\,. \tag{7.4 M$\overset{\smile}{\frown}$}$$

We shall now show that $\overset{\smile}{\eta}$ is pseudochronous while $\overset{\frown}{\eta}$ is pseudochorous. From (7.2), with (6.19 *M*) ($\varrho = \lambda$), follows

$$\overset{\smile}{\overset{\frown}{\eta}} = c(P)\,\lambda^{-2}\,\gamma^{\sim}_{n}\;\overset{\smile}{\overset{\frown}{\eta}}_{n}\,\gamma_{n} = \mp\,c(P)\,\lambda^{-2}\,\overset{\smile}{\overset{\frown}{\eta}}(\gamma_{n})^{2} = \pm\,c(P)\,\lambda^{-2}\,\overset{\smile}{\overset{\frown}{\eta}}\,. \tag{7.5 M}$$

Thus we have $\lambda^{2} = 1$ and $c(P) = +\,(-)\,1$ for $\overset{\smile}{\eta}$ $(\overset{\frown}{\eta})$. $\overset{\smile}{\eta}$ is thus orthochorous and $\overset{\frown}{\eta}$ is pseudochorous. Further

$$\begin{aligned}\overset{\smile}{\overset{\frown}{\eta}} &= c(T)\,\lambda^{-2}\,\gamma^{\sim}_{d\ldots 1}\,\overset{\smile}{\overset{\frown}{\eta}}\,\gamma_{d\ldots 1} = \\ &= \mp\,c(T)\,\lambda^{-2}\,\overset{\smile}{\overset{\frown}{\eta}}(\gamma_{d})^{2}\cdots(\gamma_{1})^{2} = \mp\,c(T)\,\lambda^{-2}\,\overset{\smile}{\overset{\frown}{\eta}}\end{aligned} \tag{7.6 M}$$

leads again to $\lambda^{2} = 1$, and $c(T) = -\,(+)\,1$ for $\overset{\smile}{\eta}$ $(\overset{\frown}{\eta})$. $\overset{\smile}{\eta}$ is thus pseudochronous and $\overset{\frown}{\eta}$ orthochronous.

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*) The statement given in a previous communication[13]), that only real representations can be used, is therefore erroneous!

Transposing (7.4), we find

$$\gamma^{\alpha} = \mp \overset{\smile}{\overset{\frown}{\eta}}{}^{-1\sim} \gamma^{\sim\alpha} \overset{\frown}{\eta}{}^{\sim}; \qquad \gamma^{\sim\alpha} = \mp \overset{\smile}{\overset{\frown}{\eta}}{}^{\sim} \gamma^{\alpha} \overset{\smile}{\overset{\frown}{\eta}}{}^{\sim -1}. \qquad (7.7\ \mathrm{M})$$

(7.4) and (7.7 M) are both changes of representation of the type (6.5 M) (because $'\gamma^{\alpha} = -\gamma^{\alpha}$ is a change of representation). Thus, on account of (6.7 M), we have

$$\overset{\smile}{\overset{\frown}{\eta}} = \lambda \overset{\smile}{\overset{\frown}{\eta}}. \qquad (7.8\ \mathrm{M})$$

Transposing this equation, we find $\lambda^2 = 1$; $\overset{\smile}{\eta}$ *and* $\overset{\frown}{\eta}$ *are either symmetric or antisymmetric matrices.* Furthermore we have

$$\overset{\smile}{\eta}{}^{-1} \overset{\frown}{\eta} = S_{(PT)} = \pm \gamma^{12\ldots n} = \pm \gamma_{n+1}. \qquad (7.9\ \mathrm{M})$$

In order to find the symmetry of $\overset{\smile}{\eta}$, we choose an orthogonal representation (6.36):

$$\gamma^{\sim i} \overset{*}{=} (\gamma^{i})^{-1} = \gamma_{i} = \gamma^{i}, \qquad (7.10\ \mathrm{M})$$

$$\gamma^{\sim n} \overset{*}{=} (\gamma^{n})^{-1} = \gamma_{n} = -\gamma^{n}. \qquad (7.11\ \mathrm{M})$$

From the ACR's (6.2 M) follows that $\overset{\smile}{\eta} \overset{*}{=} \lambda \gamma^{n}$ satisfies (7.4 $\mathrm{M}^{\smile}$), $\overset{\smile}{\eta}$ is therefore an *anti-symmetric matrix* in this particular representation. Now, in analogy to (6.34 M), we have in a general representation

$$'\overset{\smile}{\eta}_{'A\,'B} = \overset{\smile}{\eta}_{AB}\, S^{-1A}{}_{'A}\, S^{-1B}{}_{'B} \qquad (7.12)$$

conservation of this antisymmetry: $\overset{\smile}{\eta}_{AB}$ is, *in all (real) representations, antisymmetric:*

$$\overset{\smile}{\eta}{}^{\sim}_{AB} = \overset{\smile}{\eta}_{BA} = -\overset{\smile}{\eta}_{AB}; \qquad \overset{\smile}{\eta}{}^{\sim} = -\overset{\smile}{\eta}. \qquad (7.13);\ (7.13\ \mathrm{M})$$

We choose, in particular

$$\overset{\smile}{\eta}_{AB} \overset{*}{=} \gamma^{nA}_{B} \qquad (7.14)$$

in order to have for the charge-density

$$\overset{\smile}{j}{}^{(1)n} = \overset{\smile}{\psi}{}^{T} \gamma^{n} \psi = \overset{\smile}{\psi}{}^{T}_{C} \gamma^{nC}{}_{B} \psi^{B} = \psi^{TA} (-\overset{\smile}{\eta}_{AC} \gamma^{nC}{}_{B}) \psi^{B} \equiv$$

$$\equiv \psi^{TA} \psi^{B} (-\overset{\smile}{\gamma}{}^{n}_{AB}) \overset{*}{=} \textstyle\sum_{A} \psi^{TA} \psi^{A} > 0; \qquad -\overset{\smile}{\gamma}{}^{n}_{AB} \overset{*}{=} \delta^{A}_{B} \qquad (7.15)$$

a *positive operator*. We may now determine the symmetries of the *pseudo-chronous covariant bi-spinors:*

$$\overset{\smile}{\gamma}{}^{r}_{AB} = \overset{\smile}{\eta}_{AC} \gamma^{rC}{}_{B}; \qquad \overset{\smile}{\gamma}{}^{r\,(\mathrm{cov})} = \overset{\smile}{\eta} \gamma^{r}. \qquad (7.16);\ (7{,}16\ \mathrm{M})$$

We have, on account of (7.4 M),

$$\breve{\gamma}^{\sim\,\alpha_1\dots\alpha_\nu} = \gamma^{\sim\,\alpha_\nu}\dots\gamma^{\sim\,\alpha_1}\,\breve{\eta}^{\sim} = -\,\gamma^{\sim\,\alpha_\nu}\dots\gamma^{\sim\,\alpha_1}\,\breve{\eta} = -\,(-1)^\nu\,\breve{\eta}\,\gamma^{\alpha_\nu}\dots\gamma^{\alpha_1}. \tag{7.17}$$

This gives the following table:

sym.:		$\breve{\gamma}^{\alpha}$	$\breve{\gamma}^{\alpha_1\alpha_2}$			$\breve{\gamma}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$	
antisym.:	$\breve{\gamma}^0 = \breve{\eta}$			$\breve{\gamma}^{\alpha_1\alpha_2\alpha_3}$	$\breve{\gamma}^{\alpha_1\alpha_2\alpha_3\alpha_4}$		...

(7.18)

From (7.9 M) and (7.17) follows:

$$\overset{\frown}{\eta}{}^{\sim} = \pm\,\overset{\frown}{\eta} \quad \text{for} \quad n = \begin{cases} 2 \pmod 8 \\ 4 \pmod 8 \end{cases}. \tag{7.19}$$

§ 8. Quantization of the Spinor Field

If $\psi^A(x)$ satisfies (6.20 ctr), it follows from the symmetries of $\breve{\gamma}^{\alpha}_{A\,B} = \breve{\gamma}^{\alpha}_{(A\,B)}$ and $\breve{\gamma}^{0}_{A\,B} = \breve{\gamma}^{0}_{[A\,B]}$, that

$$\breve{\psi}_A(x) = \breve{\eta}_{A\,B}\,\psi^B(x) = -\,\psi^B(x)\,\breve{\eta}_{B\,A} \tag{8.1}$$

satisfies (6.20 cov). The same is true for $\psi^{T\,A}(x)$ and $\psi^T_A(x)$, because we consider but real representations ($n = 2{,}4 \pmod 8$). We form the tensors $T^{\alpha}{}_{\beta}(x)$ and $s^{\alpha\beta\gamma}(x)$, either posing

$$\chi_A = (\breve{J}\,\breve{\psi}_A)^T\,; \qquad \varphi^A = \psi^A:$$

$$T^{(1)\alpha}{}_{\beta}(x) = \frac{1}{2}\,((\breve{J}\,\breve{\psi})^T\,\gamma^\alpha\,\psi_\beta - (\breve{J}\,\breve{\psi}_\beta)^T\,\gamma^\alpha\,\psi) =$$

$$= \frac{1}{2}\,(\psi^{T\,A}\,\breve{J}^T\,\psi^B_\beta + \psi^{T\,B}_\beta\,\breve{J}\,\psi^A)\,(x)\,(-\,\breve{\gamma}^{\alpha}_{A\,B}) \tag{8.2(1)}$$

or, posing

$$\chi_A = \breve{\psi}_A\,; \qquad \psi^A = (\breve{J}\,\varphi^A)^T:$$

$$T^{(2)\,\alpha}{}_{\beta}(x) = \frac{1}{2}\,(\breve{\psi}\,\gamma^\alpha(\breve{J}\,\psi_\beta)^T - \breve{\psi}_\beta\,\gamma^\alpha(\breve{J}\,\psi)^T) =$$

$$= \frac{1}{2}\,(\psi^A\,\psi^{T\,B}_\beta\,\breve{J}^T + \breve{J}\,\psi^B_\beta\,\psi^{A\,T})\,(x)\,(-\,\breve{\gamma}^{\alpha}_{(A\,B)})\,. \tag{8.2(2)}$$

The second form of both equations shows a) that $T^{(1)\,\alpha}{}_{\beta}$ and $T^{(2)\,\alpha}{}_{\beta}$ are *observables* (on account of the *symmetry* of $\breve{\gamma}^{\alpha}_{(A\,B)}$) and b) that they are *orthochronous*, if we *define:*

$$'\psi'^A('x) = O^{-1}_{(L)}\, \psi'^A('x)\, O_{(L)} = e^{\breve{J}\lambda}\, S'^A_{(L)A}\, \psi^A(L^{-1}\,'x)\,. \qquad (8.3)\,{*})$$

The same is true for $s^{(1)\alpha\beta\gamma}$ and $s^{(2)\alpha\beta\gamma}$, on account of the antisymmetry of $\breve{\gamma}^{\alpha\beta\gamma}_{AB} = \breve{\gamma}^{\alpha\beta\gamma}_{[AB]}$. Thus, $\theta^{(1)\alpha\beta}$ and $\theta^{(2)\alpha\beta}$ are *orthochronous observables*.

It can be shown, using the Dirac-equations and partial integrations, that one may write

$$\breve{\Pi}^{(1)}_\mu = \int d\breve{\sigma}_\alpha(y)\, ((\breve{J}\,\breve{\psi})^T\, \gamma^\alpha\, \psi_\mu)\,(y)\,, \qquad (8.4^{(1)})$$

$$\breve{\Pi}^{(2)}_\mu = -\int d\breve{\sigma}_\alpha(y)\, (\breve{\psi}_\mu\, \gamma^\alpha (\breve{J}\,\psi)^T)\,(y)\,. \qquad (8.4^{(2)})$$

With

$$\breve{\Pi}_\mu = \alpha_1\, \breve{\Pi}^{(1)}_\mu + \alpha_2\, \breve{\Pi}^{(2)}_\mu\,,$$

the relation

$$-[\breve{J}\,\breve{\Pi}_\mu, \psi^{B'}(y')] = \psi^{B'}_\mu(y') \qquad (8.5)$$

is satisfied, if

$$-\alpha_1\,[(\breve{J}\,(\breve{J}\,\breve{\psi})^T\,\gamma^\alpha\,\psi_\mu)\,(y), \psi^{B'}(y')] + \alpha_2\,[(\breve{J}\,\breve{\psi}_\mu\,\gamma^\alpha(\breve{J}\,\psi)^T)\,(y), \psi^{B'}(y')]$$
$$\equiv \breve{\delta}^\alpha(y'\,y)\,\psi^{B'}_\mu(y)\,. \qquad (8.6)$$

This relation can be integrated, using the *invariant pseudochronous (real) number* (function)

$$\breve{S}^{0A}{}_B(x\,y) = (-\gamma^{\alpha A}{}_B\,\partial^x_\alpha + \gamma^{0A}{}_B\,M)\,\breve{D}^0(x\,y) \qquad (8.7)\,{**})$$

in the form

$$-\alpha_1\,[\breve{J}\,(\breve{J}\,\breve{\psi}_A)^T\,(x)\,\gamma^{\alpha A}{}_B\,\psi^B(z), \psi^{B'}(y')] +$$
$$+\alpha_2\,[\breve{J}\,\breve{\psi}_A(z)\,\gamma^{\alpha A}{}_B(\breve{J}\,\psi^B)^T\,(x), \psi^{B'}(y') \equiv \breve{S}^{0B'}{}_A(y'\,x)\,\gamma^{\alpha A}{}_B\,\psi^B(z)\,. \qquad (8.8)$$

In order to verify (8.6), we operate on (8.8) with ∂^z_μ and pose $x = z = y$. Further we use (2.15)

$$\breve{S}^{0B'}{}_A(y'\,y) = \gamma^{\beta B'}{}_A\,\breve{\delta}_\beta(y'\,y) \qquad (8.9)$$

and (2.12)

$$\int d\breve{\sigma}_\alpha(y)\,\breve{\delta}_\beta(y'\,y)\,(\gamma^\beta\,\gamma^\alpha)^{B'}_B\,f^B(y) = -\operatorname{sig}(g_{nn})\,(\breve{\nu}_\alpha\,\breve{\nu}_\beta)\,(y')\,(\gamma^\beta\,\gamma^\alpha)^{B'}_B\,f^B(y') =$$
$$= f^{B'}(y') = \int d\breve{\sigma}_\alpha(y)\,\breve{\delta}^\alpha(y'\,y)\,f^{B'}(y)\,. \qquad (8.10)\,{***})$$

---

*) We may, as in (2.3), introduce the arbitrary phase factor for pseudochorous and for pseudochronous transformations.

**) The *function* $\breve{S}^A{}_B(xy)$ is not to be confounded with the transformation matrix $S'^A{}_A$ or $S'^A_{(L)A}$.

***)
$$(\gamma^\beta\,\gamma^\alpha)^{B'}{}_B\,\breve{\delta}_\beta(y'\,y) = \gamma^{0B'}{}_B\,\breve{\delta}^\alpha(y'\,y)\,. \qquad (8.10\,a)$$

So far, all relations are valid for fields of the 1st and of the 2nd kind. Leaving the discussion of fields of the 2nd kind for a later publication[3]), we rewrite (8.3) for fields of the 1st kind ($[J, \psi^A] = 0$) using (1.4):

$$- \alpha_1 \{\breve{\psi}_A^T(x)\, \gamma^{\alpha A}{}_B\, [\psi^B(z), \psi^{B'}(y')]_\mp \pm [\breve{\psi}_A^T(x), \psi^{B'}(y')]_\mp\, \gamma^{\alpha A}{}_B\, \psi^B(z)\} +$$
$$+ \alpha_2 \{\breve{\psi}_A(z)\, \gamma^{\alpha A}{}_B\, [\psi^{T B}(x), \psi^{B'}(y')]_\mp \pm [\breve{\psi}_A(z), \psi^{B'}(y')]_\mp\, \gamma^{\alpha A}{}_B\, \psi^{T B}(x)\} =$$
$$= \breve{S}^{0 B'}{}_A(y'\, x)\, \gamma^{\alpha A}{}_B\, \psi^B(z)\,. \qquad (8.11,\ 1^{st}\ k.)$$

This is a generalised form of the CR's discussed by GREEN[4]) and VOLKOV[5]). The most simple solution is

$$\boxed{[\psi^A(x), \psi^B(y)]_\mp = 0} \qquad (8.12,\ 1^{st}\ k.)$$

and

$$\mp [\breve{\psi}_A^T(x), \psi^{B'}(y')]_\mp = \breve{S}^{0 B'}{}_A(y'\, x) \qquad (8.13,\ 1^{st}\ k.)$$

which give to the 1st term the right kind of structure. In the 2nd term, we use the identity, following from (8.1) and (7.4 M)

$$\breve{\psi}_A(z)\, \gamma^{\alpha A}{}_B\, \psi^{T B}(x) = \psi^B(z)\, \gamma^{\alpha A}{}_B\, \breve{\psi}_A^T(x)\,.$$

Using again (8.13) and (8.12), we find that (8.11) holds, provided

$$\boxed{\alpha_1 \mp \alpha_2 = 1\,.} \qquad (8.14,\ 1^{st}\ k.\ \mp)$$

We may rewrite (8.13) in the more usual form

$$\boxed{[\psi^A(x), \breve{\psi}_B(y)]_\mp = \breve{S}^{0 A}{}_B(x\, y)\,.} \qquad (8.13,\ 1^{st}\ k.)$$

§ 9. The Charge Operator for Spinor Fields

The two expressions for j^α (6.26) are

$$\breve{j}^{(1)\alpha}(x) = (\breve{\psi}^T \gamma^\alpha \psi)\,(x) = (\psi^{T A}\, \psi^B)\,(x)\,(-\breve{\gamma}^\alpha_{A\,B})\,, \qquad (9.1^{(1)})$$

$$\breve{j}^{(2)\alpha}(x) = (\psi\, \gamma^\alpha\, \psi^T)\,(x) = (\psi^A\, \psi^{T B})\,(x)\,(-\breve{\gamma}^\alpha_{A\,B})\,. \qquad (9.1^{(2)})$$

Forming again $Q = \beta_1 Q^{(1)} + \beta_2 Q^{(2)}$ (see (3.2)), we have, for fields of the 1st kind, analogous to (3.6, 1st k.),

$$- [Q, \psi^{B'}(y')] =$$
$$= - \int d\breve{\sigma}_\alpha(y) \{\beta_1 [(\breve{\psi}^T \gamma^\alpha \psi)(y), \psi^{B'}(y')] + \beta_2 [(\breve{\psi} \gamma^\alpha \psi^T)(y), \psi^{B'}(y')]\} \equiv$$
$$\equiv \int d\breve{\sigma}_\alpha(y) \breve{\delta}^\alpha(y' y) \psi^{B'}(y) = \psi^{B'}(y') . \qquad (9.2, 1^{st} k.)$$

For fields of the 1st kind, we compare (9.2) with (8.6), (the $\breve{J}$'s cancel out and we omit the index $_\mu$ of differentiation*)). (9.2) results, if we pose again (3.7, 1st k.) i.e.

$$\beta_1 = \alpha_1 ; \qquad \beta_2 = - \alpha_2 . \qquad (9.3, 1^{st} k.)$$

§ 10. Charge Conjugation for the Spinor Field (1st kind)

We see at once, rising the index of (8.13)

$$[\psi^A(x), \psi^{T\,B}(y)]_{\mp} = S^{0\,A\,B}(x\,y) \qquad (10.1)$$

that the invariant number (function)

$$S^{0\,A\,B}(x\,y) = (-\breve{\gamma}^{\alpha\,A\,B}\,\partial^x_\alpha + \breve{\gamma}^{0\,A\,B}\,M)\,\breve{D}^0(x\,y) =$$
$$= S^{0\,B\,A}(y\,x) = \breve{\eta}^{-1\,B\,C}\,\breve{S}^{0\,A}{}_C(x\,y) \qquad (10.2)\,**)$$

is symmetric with respect to $A, x \rightleftarrows B, y$. Therefore the substitution

$$'\psi^A(x) = O^{-1}_{(C)}\,\psi^A(x)\,O_{(C)} = \psi^{T\,A}(x) ,$$
$$'\psi^{T\,A}(x) = O^{-1}_{(C)}\,\psi^{T\,A}(x)\,O_{(C)} = \psi^A(x) \qquad (10.3)$$

is again an orthogonal transformation, if we choose the ACR's (8.12_+), (8.13_+), (10.1_+). *Thus FD-statistics for spinor fields is a consequence of $O_{(C)}$-covariance or of $O_{(C\,T)}$-covariance* (in perfect analogy to § 4). Furthermore we have, on account of (8.2), again the relations (4.2). Using (8.14_+), we have again, with $\alpha_1 = \alpha_2 = 1/2$, the expression (4.3): *$\theta^{\alpha\beta}(x)$ is in-*

*) Omitting the index $_\mu$ means, that we use the integrated equation (8.8)!

**) $$\breve{\gamma}^{r\,A\,B} = \breve{\eta}^{-1\,B\,D}\,\gamma^{r\,A}{}_D = \breve{\eta}^{-1\,A\,C}\,\breve{\eta}^{-1\,B\,D}\,\breve{\gamma}^r_{C\,D} , \qquad (10.4)$$
$$\breve{\gamma}^{r\,(\mathrm{ctr})} = -\gamma^r\,\breve{\eta}^{-1} = -\breve{\eta}^{-1}\,\breve{\gamma}^{r\,(\mathrm{cov})}\,\breve{\eta}^{-1} . \qquad (10.4\ M)$$

variant with respect to $O_{(C)}$*). The equations (4.4) are also valid. Therefore, using again (3.7, 1st k.) (= (9.3, 1st k.)) we have again (4.5) and (4.6)*): *$\breve{j}^\alpha$ and Q change sign under $O_{(C)}$*. The rest of the discussion is analogous to § 4.

§ 11. The Development of the Spinor Field in Terms of Positive Frequency Wave Packet Operators

We introduce again *two denumerable sets* of PFWP's $\{\varphi'^A, \varphi''^A, \dots \varphi^{(\varrho)A} \dots\}$ and $\{\chi'^A, \chi''^A \dots \chi^{(\varrho)A} \dots\}$ satisfying the Dirac equation (6.20) and (5.4). The sets are normalised in terms of

$$\breve{j}^\alpha(\varphi', \chi')(x) = \breve{j}^\alpha(\chi', \varphi')(x) = (\varphi' \breve{\gamma}^\alpha \chi')(x) = (\varphi'^A \chi'^B)(x)(-\breve{\gamma}^\alpha_{A\,B}) . \quad (11.1)^{**})$$

On account of the decomposition (6.27), one verifies that the *normalisation*

$$Q(\varphi', \chi') = 0 ; \qquad Q(\varphi'^T, \varphi'') = Q(\varphi'', \varphi'^T) = \delta_{\varphi' \varphi''} > 0 \quad (11.2)$$

is possible. *Completness* is assured, if the $\breve{J}$-dependent operators

$$\breve{S}^{+A}{}_B(x\,y) \equiv \mathbf{S}_{\varphi'} \varphi'^A(x)\, \breve{\varphi}'^T_B(y) = \breve{S}^{+A}{}_B(x-y) , \quad (11.3^+)^{**})$$

$$\breve{S}^{-A}{}_B(x\,y) \equiv \mathbf{S}_{\varphi'} \varphi'^{TA}(x)\, \breve{\varphi}'_B(y) = \breve{S}^{-A}{}_B(x-y) , \quad (11.3^-)^{**})$$

$$S^{-AB}(x\,y) = S^{+TAB}(x\,y) = S^{+BA}(y\,x) \quad (11.4)$$

are invariant with respect to $\{L_{(\text{ochr})}\}$. Again a PF-solution $f^{+A}(x)$ of (6.20) may be written in the form

$$f^{+A}(x) = \mathbf{S}_{\varphi'} \varphi'^A(x)\, f^+_{\varphi'} = \int d\breve{\sigma}_\alpha(y)\, \breve{S}^{+A}{}_B(x\,y)\, \gamma^{\alpha B}{}_C\, f^{+C}(y) \quad (11.5^+)$$

and

$$f^{-A}(x) = \mathbf{S}_{\chi'} \chi'^{TA}(x)\, f^-_{\chi'} = \int d\breve{\sigma}_\alpha(y)\, \breve{S}^{-A}{}_B(x\,y)\, \gamma^{\alpha B}{}_C\, f^{-C}(y) . \quad (11.5^-)$$

---

*) $\theta^{\alpha\beta}$ and $\breve{J}^\alpha$ can be expressed in terms of *commutators*:

$$T^\alpha{}_\beta(x) = \frac{1}{4} \breve{J}^{-1} ([\psi^{TA}, \psi^B_\beta] + [\psi^A, \psi^{TB}_\beta])(x)(-\breve{\gamma}^\alpha_{(A\,B)}) , \quad (10.6)$$

$$s^{\alpha\beta\gamma}(x) = \frac{1}{4} \breve{J}^{-1} [\psi^{TA}, \psi^B](x)(-\breve{\gamma}^{\alpha\beta\gamma}_{[A\,B]}) . \quad (10.7)$$

$$\breve{j}^\alpha(x) = \frac{1}{2} [\psi^{TA}, \psi^B](x)(-\breve{\gamma}^\alpha_{(A\,B)}) . \quad (10.8)$$

**) The pseudochronous sign in $\breve{\varphi}'_A$ and $\breve{S}^{+A}{}_B$ does not imply any covariance property but only indicates lowering of the index.

The general solution of (6.20) may be written as

$$\psi^A(k) = \mathbf{S}_{\varphi'} a_{\varphi'} \varphi'^A(x) + \mathbf{S}_{\chi'} b^T_{\chi'} \chi'^{TA}(x) =$$

$$= \int d\breve{\sigma}_\alpha(y)\, \breve{S}^{0A}{}_B(x\,y)\, \gamma^{\alpha B}{}_C\, \psi^C(y) \tag{11.6}$$

with

$$\breve{S}^{0A}{}_B(x\,y) = (\breve{S}^{+A}{}_B + \breve{S}^{-A}{}_B)\,(x\,y)\,. \tag{11.8}$$

If $x = y'$ is chosen on $\tau(y') = \tau(y) = 0$, it follows that

$$\breve{S}^{0B'}{}_C(y'\,y)\, \gamma^{\alpha C}{}_B = \gamma^{0B'}{}_B\, \breve{\delta}^\alpha(y'\,y)\,. \tag{11.9}$$

Now, this is exactly the condition (8.9), (8.10). Thus $\breve{S}^{0A}{}_B(x\,y)$, defined by (11.8) is identical with (8.7).

From the development (11.6) and (8.12), (8.13), follow the CR's or ACR's:

$$[a_{\varphi'}, a_{\varphi''}]_\mp = [b^T_{\chi'}, b^T_{\chi''}]_\mp = [a_{\varphi'}, b^T_{\chi'}]_\mp = 0\,, \tag{11.10$_\mp$}$$

$$[a_{\varphi'}, a^T_{\varphi''}]_\mp = \delta_{\varphi'\varphi''}\,; \qquad [b^T_{\chi'}, b_{\chi''}]_\mp = \delta_{\chi'\chi''}\,; \qquad [a_{\varphi'}, b_{\chi'}]_\mp = 0\,. \tag{11.11$_\mp$}$$

Taking the ACR's (on account of $O_{(C)}$-covariance), (10.8) leads to

$$Q = \mathbf{S}_{\varphi'}\left(N_{\varphi'} - \frac{1}{2}\right) - \mathbf{S}_{\chi'}\left(N_{\chi'} - \frac{1}{2}\right) = \mathbf{S}_{\varphi'} N_{\varphi'} - \mathbf{S}_{\chi'} N_{\chi'} \tag{11.12}$$

with $N_{\varphi'} = a^T_{\varphi'}\, a_{\varphi'}$. The ACR's (11.10) and (11.11) are satisfied in terms of the pseudo-quaternions (I, A-4.8)

$$a^T = \frac{1}{2}\,(l + j) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad a = \frac{1}{2}\,(l - j) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$N = a^T\, a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\,(1 - k) \tag{11.13}$$

with

$$\begin{aligned} a_{\varphi(\varrho)} &= 1 \times (k \times k) \times (k \times k) \times \cdots \times (a \times 1) \times (1 \times 1) \times \cdots \\ b_{\chi(\varrho)} &= 1 \times (k \times k) \times (k \times k) \times \cdots \times (k \times a) \times (1 \times 1) \times \cdots \end{aligned} \tag{11.14}$$

The eigenvalues are $\{N'_{\varphi'}\} = \{0,1\}$ i.e. FD-statistics holds. The explicit form of $O_{(C)}$ is again given by (5.19), if the sets $\{\varphi'\}$ and $\{\chi'\}$ are chosen to be identical. Quantization of $\breve{\Pi}_\mu$ leads to

$$\breve{\Pi}^\mu = \mathbf{S}_{\varphi'}\left(N_{\varphi'} - \frac{1}{2}\right) \breve{k}'^\mu + \mathbf{S}_{\chi'}\left(N_{\chi'} - \frac{1}{2}\right) \breve{k}'^\mu\,, \tag{11.15}$$

if the PFWP's satisfy (5.20). Thus, the energy is, apart from the negative infinite 'zero point contribution' (which is again symmetric for particle and anti-particle states), a positive definit operator: the spectrum has a lower limit, and statistical thermodynamics with a positive absolute temperature may be applied[10]).

The CR's (11.10_) and (11.11_) lead, whatever α_1 and α_2 (satisfying (8.14, 1st k._)) we choose, to an energy spectrum without lower or upper limit: thus statistical thermodynamics cannot be applied. This is PAULI's[9]) reason for excluding BE-statistics for spinor fields.

Annex I: The Theorem of Frobenius and Schur[11])[12]),

states that an *irreducible matrix group* (IMG) $\{\widehat{D}_i\}$ ($ik. \ldots = 12 \ldots h$) of order h belongs to one of the three kinds:

An IMG is of the 1st, 2nd or 3rd kind, depending on wether

$$h^{-1} \sum_i \operatorname{tr}(\widehat{D}_i^2) = \begin{Bmatrix} -1 \\ +1 \\ 0 \end{Bmatrix} \text{ for an IMG of the } \begin{Bmatrix} 1^{st} \\ 2^{nd} \\ 3^{rd} \end{Bmatrix} \text{ kind.} \qquad \text{(A–1.1)}$$

(1) An IMG is of *the 1st kind,* if it is *equivalent to a group of real matrices,* i.e. if for all representations $\{\widehat{D}_i\}$ a matrix $\widehat{S}$ exists, satisfying

$$\{'\widehat{D}_i\} = \{\widehat{S}\,\widehat{D}_i\,\widehat{S}^{-1}\} \quad \text{with} \quad '\widehat{D}_i^* = {}'\widehat{D}_i\,. \qquad \text{(A-1.2)}$$

(2) An IMG is of *the 2nd kind,* if it is *equivalent to its conjugate complex group,* i.e. if for all representations a matrix $\widehat{C}$ exists, satisfying

$$\{'\widehat{D}_i\} = \{\widehat{C}\,\widehat{D}_i\,\widehat{C}^{-1}\} \quad \text{with} \quad '\widehat{D}_i = \widehat{D}_i^*\,. \qquad \text{(A-1.3)}$$

NB.: *For an IMG of the 1st kind, a matrix $\widehat{C}$ exists a fortiori* ($\widehat{C} = \widehat{S}^{*-1}\,\widehat{S}$).

(3) An IMG is of the *3rd kind,* if it is not equivalent to its complex conjugate group i.e. *if no matrix $\widehat{C}$ satisfying (A-1.3) exists.*

Now the set $\{\pm\,\Gamma\} = \{\pm\,\gamma^r\}$ forms, for $n = 2\,m$, on account of (6.8 M) an IMG of order 2×2^n. In order to find out to which kind it belongs, we have to evaluate

$$(2 \times 2^n)^{-1} \sum_r \sum_{(\pm)} \operatorname{tr}\left((\pm\,\gamma^r)^2\right) = 2^{-n} \sum_r \operatorname{tr}\left((\gamma^r)^2\right). \qquad \text{(A-1.4)}$$

To evaluate this sum, we decompose each subset $\Gamma^{(\nu)}$ into two parts

$$\{\gamma^{\alpha_1 \cdots \alpha_2}\} = \{\gamma^{i_1 \cdots i_2}\} + \{\gamma^{i_1 \cdots i_{\nu-1}\,n}\}\,, \qquad \text{(A-1.5)}$$

each having ($d = n - 1$)

$$\binom{n}{\nu} = \binom{d}{\nu} + \binom{d}{\nu - 1} \tag{A-1.6}$$

elements. Using (6.12) and (6.13), we have

$$\sum_r \operatorname{tr} [(\gamma^r)^2] = \sum_\nu \left[\binom{d}{\nu} - \binom{d}{\nu-1}\right] \left\{ \begin{matrix} (-1)^{\nu/2} \\ (-1)^{(\nu-1)/2} \end{matrix} \right\} \operatorname{tr}(\gamma^0) =$$

$$= N \operatorname{Re} \sum_\nu \left[\binom{d}{\nu} - \binom{d}{\nu-1}\right] (1 - i)\, i^\nu =$$

$$= 4\, N \operatorname{Re} [(1 + i)^{n-3}] = 2^{(n+1)/2}\, N \cos\left((n - 3)\, \frac{\pi}{4}\right). \tag{A-1.7}$$

Me remark first, that our condition (A-1.1), (A-1.4) has the periodicity $'n = n \pmod 8 = \ldots, n - 8, n, n + 8, \ldots$ In particular, we find, on account of $\cos(\pi/4) = 2^{-1/2}$, $N = 2^{n/2}$ i.e. (6.30) and furthermore that our IMG is of the 1st kind for $n = 2,4 \pmod 8$ (6.31) and of the 2nd kind for $n = 6,8$ (mod 8) (6.32).

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