Zeitschrift: Helvetica Physica Acta

Band: 34 (1961)

Heft: [6]: Supplementum 6. Proceedings of the International Symposium on polarization phenomena of nucleons

Artikel: Stripping theory in operator form

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DOI: https://doi.org/10.5169/seals-541275

Stripping Theory in Operator Form[1])

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1. Introduction

When considering the possible stripping reaction experiments with beams of polarized deuterons or nucleons, and/or aligned nuclei, it is not necessary, or desirable, to consider a detailed model of the stripping process such as the Butler theory [1][2]). Rather, it is necessary to write the scattering matrix for stripping reactions in the most general form allowed by rotational invariance and parity conservation arguments; then it is possible to calculate the results of any conceivable experiment using the general methods of WOLFENSTEIN [2]. The most general form allowed within the framework of the shell model is worked out in Section 2 for the case in which the spin of the target nucleus is zero. The form given by the Butler theory is not the most general form.

In Section 3, the expressions for quantities of experimental interest are simplified by the use of operator techniques. A number of theorems, based on a general assumption underlying the Butler theory (but not on its other details), concerning, for example, the connection between (i) the left right asymmetry in the azimuthal distribution of protons produced by polarized deuterons and (ii) the polarization of protons produced by unpolarized deuterons, are proved. These theorems are then criticized on the basis of the more general form of the scattering matrix.

2. General Form of the Scattering Matrix

A. *Shell Model*

We assume that in a stripping reaction, a deuteron strikes a target nucleus (assumed to have spin zero in the following work), resulting in the production of a proton and a final nucleus in a state of definite total

[1]) This work performed under the auspices of the U. S. Atomic Energy Commission.

[2]) Numbers in brackets refer to References, page 371.

angular momentum J and definite parity p (the same or opposite) relative to the parity of the target nucleus. Thus we write the scattering matrix in the following form

$$M = (m_p \, J m_J \, p \mid m_d) \tag{1}$$

where m_p is the z component of the spin of the proton, m_J is the z component of J, and m_d is the z component of the spin of the deuteron. The operator which connects the initial and final states, is, of course, rotationally invariant and conserves parity, so that it is not explicitly shown in Eq. (1).

Within the framework of the shell model, M will have the form

$$M = \sum C\left(Jm_J;\, Lm_L;\, \tfrac{1}{2}\, m_n\right)(m_p m_n \mid O_L^{m_L}(p) \mid m_d) \tag{2}$$

where the $O_L^{m_L}(p)$ are operators which transform like the spherical harmonics $Y_L^{m_L}$ under spatial rotations. The C's are Clebsch-Gordan coefficients; the Σ runs over all m_L and m_n such that $m_L + m_n = m_J$. If the final nucleus has the same parity as the target nucleus, and L is even, $O_L^{m_L}(p)$ must have even parity; if L is odd, $O_L^{m_L}(p)$ must have odd parity. If the final nucleus has opposite parity from the initial nucleus, and L is even, $O_L^{m_L}(p)$ must have odd parity; if L is odd, $O_L^{m_L}(p)$ must have even parity.

That M will have this form follows from the fact that the right hand side of Eqs. (1) and (2) transform in the same way under spatial rotations or parity transformations, and the fact that within the framework of the shell model, J is formed from the orbital angular momentum and spin of the captured neutron. The quantity L which appears in Eq. (2) may be thought of as the orbital angular momentum of the captured neutron.

The operators $O_L^{m_L}(p)$ must be constructed from $\boldsymbol{\sigma}_p$, $\boldsymbol{\sigma}_n$, $\boldsymbol{k}_p$, and $\boldsymbol{k}_d$ (the spin operators for the proton and neutron, and the momenta of the proton in the final state and the deuteron in the initial state, respectively).

For $L = 0$, O_0^0 must be a scalar or pseudoscalar. The possible forms are

even parity (scalars)	odd parity (pseudoscalars)
$\boldsymbol{k}_p \cdot \boldsymbol{k}_d$[3]	$\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p$
$\boldsymbol{\sigma}_n \cdot (\boldsymbol{k}_p \times \boldsymbol{k}_d)$	$\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_d$
$\boldsymbol{\sigma}_p \cdot (\boldsymbol{k}_p \times \boldsymbol{k}_d)$	$\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p$
$(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p)$	$\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_d$,

[3]) It is understood that all scattering amplitudes may be functions of $\boldsymbol{k}_p \cdot \boldsymbol{k}_d$.

$$(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p) \qquad (\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d)$$

$$(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_d)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p) \qquad (\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_d)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d)$$

$$(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_d)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_d) \qquad (\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p)\,(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d)$$

$$(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d)\,(\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d) \quad (\boldsymbol{\sigma}_p \cdot \boldsymbol{k}_d)\,(\boldsymbol{\sigma}_n \cdot \boldsymbol{k}_p \times \boldsymbol{k}_d)\ .$$

For $L = 1$, $O_L{}^{m_L}(p)$ must be related to the components of a vector (or pseudovector) $\boldsymbol{V}$ as follows

$$O_1^1 = -\frac{V_x - iV_y}{\sqrt{2}}\,,\qquad O_1^0 = V_z\,,\qquad O_1^{-1} = \frac{V_x + iV_y}{\sqrt{2}}\,. \tag{4}$$

B. *Butler Theory*

We characterize the Butler theory by the assumption that the spin of the proton does not appear in the operators O. This single assumption drastically limits the possible form of the operators O.

For $L = 1$, for example, the wave function of the final nucleus may be constructed as follows

$$\sum C\Big(Jm_J\,;\, Lm_L\,;\, \frac{1}{2}\ m_n\Big)\ \boldsymbol{r}_n{}^{m_L}\ \chi_{1/2}{}^{m_n}\psi(r_n) \tag{5}$$

where $\boldsymbol{r}_n^1$, $\boldsymbol{r}_n^0$, and $\boldsymbol{r}_n^{-1}$ are given by Eq. (4), the χ's are spin functions for the neutron, and ψ does not depend on the direction of $\boldsymbol{r}_n$. In its simplest form, the Butler theory gives a matrix element

$$M \sim \sum C\Big(Jm_J;\ Lm_L\,;\, \frac{1}{2}\ m_n\Big) \times \tag{6}$$

$$\times \int d\boldsymbol{r}_n\, d\boldsymbol{r}_p \boldsymbol{r}_n{}^{m_L}\,\psi(r_n)\exp(-i\boldsymbol{k}_p\cdot\boldsymbol{r}_p)\,V(r_n)\,\varphi(|\,\boldsymbol{r}_p - \boldsymbol{r}_n\,|)\exp\Big(\frac{i\boldsymbol{k}_d}{2}\cdot(\boldsymbol{r}_p + \boldsymbol{r}_n)\Big)$$

where $V(r_n)$ is the potential the neutron experiences, and φ is the wave function for the ground state of the deuteron. The integral in Eq. (6) is easily evaluated by using $\boldsymbol{r}_n$ and $\boldsymbol{r}_p - \boldsymbol{r}_n$ as coordinates; the result is of the form of Eq. (2) with

$$\boldsymbol{O} = \boldsymbol{k}_p + \frac{\boldsymbol{k}_d}{2}\,. \tag{7}$$

By using distorted waves in place of

$$\exp(-\,i\boldsymbol{k}_p\cdot\boldsymbol{r}_p) \quad \text{or} \quad \exp\Big(i\,\frac{\boldsymbol{k}_d}{2}\cdot(\boldsymbol{r}_p + \boldsymbol{r}_n)\Big)\,,$$

it is possible to obtain a slightly different result; namely,

$$\boldsymbol{O} = \alpha \, \boldsymbol{k}_p + \beta \, \boldsymbol{k}_d \tag{8}$$

with α and/or β complex.

To obtain a form in which $\boldsymbol{\sigma}_p$ appears, it is necessary to allow the proton to experience a spin orbit potential or (what is the same thing) to allow for spin orbit effects in the distorted wave replacing $\exp(-\, i\boldsymbol{k}_p \cdot \boldsymbol{r}_p)$. In referring to the 'Butler theory', we mean that $O_L^{m_L}$ does not contain $\boldsymbol{\sigma}_p$ or, more explicitly for $L = 1$, that $\boldsymbol{O}$ has the form Eq. (8).

3. The Experiments

A. Theory of the Experiments

The polarization of the incident deuteron beam may be described in terms of the expectation values of ten operators

$$1$$

$$\boldsymbol{S} = \frac{1}{2} (\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n) \tag{9}$$

$$S_{ij} = -\frac{1}{6} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \, \delta_{ij} + \frac{1}{4} (\sigma_{ni} \, \sigma_{pj} + \sigma_{nj} \sigma_{pi}).$$

These are exactly the operators used by STAPP[4]) in his discussion of deuteron polarization.

The polarization of the outgoing proton beam is described by the expectation values of four operators

$$1 \text{ and } \boldsymbol{\sigma}_p. \tag{10}$$

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[4]) H. P. STAPP, thesis, UCRL 3098, unpublished. WOLFENSTEIN [2] normalizes these operators so that

$$\mathrm{Tr}\, S^\mu S^\nu = 3 \, \delta_{\mu\nu}$$

This is not a necessary assumption. WOLFENSTEIN assumes this normalization so that he can prove

$$\varrho = \sum_\nu \langle S^\nu \rangle \frac{S^\nu}{(2s + 1)\,(2s_t + 1)} \, .$$

STAPP shows that this equation remains true for the above definition of the S's even though the S_{ij}'s are not mutually orthogonal in the sense of the above equation; STAPP finds

$$\varrho = \frac{\langle 1 \rangle + 3/2 \, \langle \boldsymbol{S} \rangle \cdot \boldsymbol{S} + 3 \, \langle S_{ij} \rangle \, S_{ij}}{3} \, .$$

It is *as though* there were normalization factors $\sqrt{3/2}$ multiplying $\boldsymbol{S}$, and $\sqrt{3}$ multiplying S_{ij}.

The final nucleus may also be polarized; there exists a set of operators appropriate to the description of its polarization, and the complete set of operators for the state is the direct product of these operators with the proton spin operators; however, the relevant experiments are very difficult at the present time (while it is clear in principle how one might observe the spin of C^{13} by scattering off He^4 (or C^{12}), the high charge and mass of these particles make the experiments difficult; in an inverse experiment it might be more practical to align the spin of Be^9 or C^{13}), so that we do not have to exhibit these operators explicitly.

We need the analogue of Wolfenstein's Eq. (31), p. 62, of his article [2]; namely,

$$I \langle S_f^\mu \rangle = \sum_\nu \frac{\langle S_i^\nu \rangle \operatorname{Tr} M S_i^\nu M^+ S_f^\mu}{(2s_t + 1)(2s + 1)} \tag{11}$$

where the S_i^ν and S_f^μ are the spin operators for the initial and final state, respectively. We give the derivation of this analogue now, avoiding unnecessary detail, since the discussion follows Wolfenstein's article very closely.

The discussion beginning on page 60, section 4 (entitled General Formalism) of Wolfenstein's article certainly needs no modification before Eq. (31) (our Eq. (11) above) is obtained, provided the M's are kept in the form Eq. (1). In detail, Eq. (11) is (omitting the factor $(2s_t + 1)(2s + 1)$)

$$I \langle S_f^\mu \rangle = \sum_\nu \langle S_i^\nu \rangle \sum_{m_p} \sum_{m_J} \sum_{m_d} \sum_{m_d'} \sum_{m_p'} \sum_{m_J'} (m_p\, J\, m_J\, p \,|\, M \,|\, m_d) \times \tag{12}$$

$$\times (m_d \,|\, S_i^\nu \,|\, m_d') (m_d' \,|\, M^+ \,|\, m_p'\, J\, m_J'\, p) (m_p'\, J\, m_J'\, p \,|\, S_f^\mu \,|\, m_p\, J\, m_J\, p).$$

Eq. (2) may be substituted into this Eq. (12) to find $\langle 1_f \rangle$ or $\langle \boldsymbol{\sigma}_{pf} \rangle$ (the only quantities of interest if the spin of the final nucleus is not observed) with the result

$$I \langle 1_f \text{ or } \boldsymbol{\sigma}_{pf} \rangle = \sum_\nu \langle S_i^\nu \rangle \sum_{m_p} \sum_{m_J} \sum_{m_d} \sum_{m_d'} \sum_{m_p'} \sum_{m_J'} \times$$

$$\times \sum C\left(J m_J;\, L m_L;\, \tfrac{1}{2}\, m_n\right) (m_p m_n \,|\, O_L{}^{m_L}(p) \,|\, m_d) (m_d \,|\, S_i^\nu \,|\, m_d') \times \tag{13}$$

$$\times \sum C\left(J m_J;\, L m_L';\, \tfrac{1}{2}\, m_n'\right) (m_d' \,|\, O_L{}^{m_L'}(p) \,|\, m_p'\, m_n')^* (m_p' \,|\, 1_f \text{ or } \boldsymbol{\sigma}_{pf} \,|\, m_p).$$

For $L = 0$, the result is necessarily simple, since $m_L = m_L' = 0$, $m_n = m_n' = m_J$, and the Clebsch-Gordan coefficients are 1, so that

$$I \langle 1_f \text{ or } \boldsymbol{\sigma}_{pf} \rangle = \sum_{\nu} \langle S_i^\nu \rangle \sum_{m_p} \sum_{m_n} \sum_{m_p'} \sum_{m_n'} \sum_{m_d} \sum_{m_d'} \times$$

$$\times (m_p m_n \mid O \mid m_d)(m_d \mid S_i^\nu \mid m_d')(m_d' \mid O \mid m_p' m_n')^* \times \tag{14}$$

$$\times (m_p' \mid 1_f \text{ or } \boldsymbol{\sigma}_p \mid m_p)$$

which is formally identical with Eq. (11); that is, it may be written

$$I \langle 1_f \text{ or } \boldsymbol{\sigma}_{pf} \rangle = \sum_{\nu} \langle S_i^\nu \rangle \operatorname{Tr} O\, S_i^\nu\, O^+ (1 \text{ or } \boldsymbol{\sigma}_p) \tag{15}$$

For $L = 1$, it can be proved by use of the explicit values of the Clebsch-Gordan coefficients in Eq. (13), that

$$\text{for } J = \frac{1}{2},\ L = 1$$

$$I \langle 1_f \text{ or } \boldsymbol{\sigma}_f \rangle = \frac{1}{3} \sum \langle S_i \rangle \operatorname{Tr} \boldsymbol{\sigma}_n \cdot \boldsymbol{O}\, S_i^\nu\, \boldsymbol{O}^+ \cdot \boldsymbol{\sigma}_n (1 \text{ or } \boldsymbol{\sigma}_p) \tag{16}$$

$$\text{for } J = 3/2,\ L = 1$$

$$I \langle 1_f \text{ or } \boldsymbol{\sigma}_f \rangle = \sum \langle S_i \rangle \operatorname{Tr} [- \frac{1}{3} \boldsymbol{\sigma}_n \cdot \boldsymbol{O}\, S_i^\nu\, \boldsymbol{O}^+ \cdot \boldsymbol{\sigma}_n + \tag{17}$$

$$+ \boldsymbol{O} \cdot S_i^\nu\, \boldsymbol{O}^+] (1 \text{ or } \boldsymbol{\sigma}_p) .$$

The term $\boldsymbol{O} \cdot S_i^\nu \boldsymbol{O}^+$ is written as it is because the $\boldsymbol{O}$ may not commute with S_i^ν. Also, the order of the $\boldsymbol{O}$'s and $\boldsymbol{\sigma}_n$'s is correct in case they do not commute.

Since for the experiments of most interest, such as

$$d + C^{12} \rightarrow p + C^{13} \tag{18}$$

$$d + Be^8 \rightarrow p + Be^9 \text{ (actually, the inverse)}$$

the spin of the target nucleus is zero, and $J = 1/2$ or $J = 3/2$, it is not of great interest to attempt to generalize Eqs. (15)–(17).

For purposes of calculation, one further modification is convenient; namely, to alter Eq. (14) in such a way that $\sum_{m_d}$ and $\sum_{m'_d}$ are replaced by sums over $m_n\, m_p$ variables. This modification is easily accomplished by the introduction of the triplet projection operator

$$P_t = \frac{1}{4} (\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3) \tag{19}$$

All that is necessary is to replace O by OP_t; the matrix operations shown in Eqs. (15)–(17) are then the usual operations with 2×2 matrices.

B. The experiments in the case that the Butler theory is valid

(see last sentence in Section II B for a precise definition of the 'Butler Theory')

We can prove a series of theorems (some of which are well known).

Theorem 1. In the Butler theory, the proton polarization resulting from an unpolarized deuteron beam vanishes unless α or β is complex.

Proof (for $J = 1/2$, $L = 1$).

$$3I \langle \boldsymbol{\sigma}_p \rangle = \mathrm{Tr}\, \boldsymbol{\sigma}_n \cdot (\alpha\, \boldsymbol{k}_p + \beta\, \boldsymbol{k}_d) \frac{1}{4}(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3) \frac{1}{4}(\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3)\, \boldsymbol{\sigma}_n \cdot$$
$$\cdot (\alpha^* \boldsymbol{k}_p + \beta^* \boldsymbol{k}_d)\, \boldsymbol{\sigma}_p.$$

The factor 3 is $(2\, s_t + 1)\, (2\, s + 1)$. We take the traces in neutron or proton spin space using

$$\mathrm{Tr}\, (\boldsymbol{A} \cdot \boldsymbol{\sigma})\, \boldsymbol{\sigma} = 2\, \boldsymbol{A}$$
$$\mathrm{Tr}\, \boldsymbol{A} \cdot \boldsymbol{\sigma}\, \boldsymbol{B} \cdot \boldsymbol{\sigma}\, \boldsymbol{\sigma} = 2i\, \boldsymbol{A} \times \boldsymbol{B} \tag{20}$$
$$\mathrm{Tr}\, \boldsymbol{A} \cdot \boldsymbol{\sigma}\, \boldsymbol{\sigma}\, \boldsymbol{B} \cdot \boldsymbol{\sigma} = -\, 2i\, \boldsymbol{A} \times \boldsymbol{B}$$
$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = 2i\, \boldsymbol{\sigma}.$$

The result is

$$I \langle \boldsymbol{\sigma}_p \rangle = \frac{1}{3}\, \mathrm{i}\, (\alpha\, \boldsymbol{k}_p + \beta\, \boldsymbol{k}_d) \times (\alpha^* \boldsymbol{k}_p + \beta^* \boldsymbol{k}_d). \tag{21}$$

It is already clear from Eq. (21) that if α and β are both real, $\langle \boldsymbol{\sigma}_p \rangle$ vanishes.

Theorems about the maximum value of the polarization may also be obtained. The calculations are facilitated if we use orthogonal vectors instead of $\boldsymbol{k}_d$ and $\boldsymbol{k}_p$; namely, if we use

$$\boldsymbol{O} = \alpha'\, (\boldsymbol{k}_d \times \boldsymbol{k}_p) \times \boldsymbol{k}_d + \beta'\, \boldsymbol{k}_d \tag{22}$$

(we could just as well write $\boldsymbol{O} = \alpha'\, \boldsymbol{i} + \beta'\, \boldsymbol{k}$, where $\boldsymbol{i}$ and $\boldsymbol{k}$ are unit vectors in the x and z directions, respectively). We find

$$\text{for } J = \frac{1}{2} \quad \langle \boldsymbol{\sigma}_p \rangle = -\frac{1}{3}\, \mathrm{i}\, \frac{(\alpha' \beta'^* - \beta' \alpha'^*)}{\alpha' \alpha'^* + \beta' \beta'^*}\, \boldsymbol{j},$$
$$\text{for } J = \frac{3}{2} \quad \langle \boldsymbol{\sigma}_p \rangle = \frac{1}{6}\, \mathrm{i}\, \frac{(\alpha' \beta'^* - \beta' \alpha'^*)}{\alpha' \alpha'^* + \beta' \beta'^*}\, \boldsymbol{j}, \tag{23}$$

leading to the theorems.

Theorem 2. For $J = 1/2$, the maximum value of the proton polarization from an unpolarized deuteron beam is 1/3. For $J = 3/2$, the maximum value is 1/6. For the same α', β' (that is, if the $\psi(r_n)$ (see Eq. (5) are not too different for $J = 1/2$ and $J = 3/2$), the polarization from a $J = 1/2$ and $J = 3/2$ state are different in sign.

These results have been obtained by other methods (see reference 1, Chapter IX); the method is novel and the generality of the assumption under which they are obtained is somewhat remarkable.

Theorem 3. In the Butler theory, the tensor components (S_{ij}) of the deuteron spin polarization play no role in the angular distribution of the proton.

Proof: The details of the necessary calculation are given in Appendix A.

Theorem 4. In the Butler theory, the angular distribution of protons produced by polarized deuterons is (for $J = 1/2$)

$$I \langle 1 \rangle_f = I\,(1 + 3\,\boldsymbol{P}_d \cdot \boldsymbol{P}_p) \tag{24}$$

where $\boldsymbol{P}_d$ is the vector polarization of the deuterons (that is $\langle \boldsymbol{S} \rangle$; see Eq. (9) for the definition of $\boldsymbol{S}$ and footnote 4 for a warning about its normalization to make the factor 3 come out right), and $\boldsymbol{P}_p$ is the proton polarization produced by an unpolarized deuteron beam.

This theorem was first given by SATCHLER [3]. The proof follows by calculation from Eq. (16) and Eq. (20).

C. Criticism of the results of the preceding Section III B; the experiments when the Butler theory is not valid

It is known that the proton polarizations sometime exceed the maximum values quoted in Theorem 2 [4], so that the 'Butler theory' is, in fact, *not* valid. The question of most interest is: what happens to the Satchler result, Eq. (24) when the 'Butler theory' is not valid.

If we consider the case $J = 1/2$, $L = 1$ (certainly applicable to the $d + C^{12} \rightarrow p + C^{13}$ reaction), the most general pseudoscalar $(\boldsymbol{\sigma}_n \cdot \boldsymbol{O})$ that we can write is

$$\boldsymbol{\sigma}_n \cdot \boldsymbol{O}\, P_t = \boldsymbol{\sigma}_n \cdot \boldsymbol{V}_1 + \boldsymbol{\sigma}_p \cdot \boldsymbol{V}_2 + \boldsymbol{\sigma}_n \cdot \boldsymbol{V}_3\, \boldsymbol{\sigma}_p \cdot \boldsymbol{n} + \boldsymbol{\sigma}_p \cdot \boldsymbol{V}_4\, \boldsymbol{\sigma}_n \cdot \boldsymbol{n} \tag{25}$$

where $\boldsymbol{n}$ is a unit vector in the direction $\boldsymbol{k}_d \times \boldsymbol{k}_p$, and $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_4$ are arbitrary (possibly complex) vectors in the $\boldsymbol{k}_p$, $\boldsymbol{k}_d$ plane. We have included $\boldsymbol{P}_t$ in writing Eq. (25) for computational convenience; for the most general $\boldsymbol{\sigma}_n \cdot \boldsymbol{O}$, $\boldsymbol{\sigma}_n \cdot \boldsymbol{O}\, P_t$ is less general since it must vanish when applied to a singlet spin function; we find that this restriction implies that

$$-\,\boldsymbol{V}_1 + \boldsymbol{V}_2 + i\,(\boldsymbol{V}_3 \times \boldsymbol{n}) - i\,(\boldsymbol{V}_4 \times \boldsymbol{n}) = 0\,. \tag{26}$$

We find that the proton polarization (deuteron beam unpolarized) is

$$3I\,\langle \boldsymbol{\sigma}_p \rangle = 4(\boldsymbol{V}_1 \cdot \boldsymbol{V}_3{}^* + \boldsymbol{V}_3 \cdot \boldsymbol{V}_1{}^*)\,\boldsymbol{n} + 4i\,\boldsymbol{V}_2 \times \boldsymbol{V}_2{}^* + 4i\,\boldsymbol{V}_4 \times \boldsymbol{V}_4{}^*. \tag{27}$$

where

$$3I = 4(\boldsymbol{V}_1 \cdot \boldsymbol{V}_1{}^* + \boldsymbol{V}_2 \cdot \boldsymbol{V}_2{}^* + \boldsymbol{V}_3 \cdot \boldsymbol{V}_3{}^* + \boldsymbol{V}_4 \cdot \boldsymbol{V}_4{}^*). \tag{28}$$

We find that if the incident beam has a vector polarization $\boldsymbol{P}$, but no tensor polarization, the azimuthal distribution of the proton is

$$3I\,\langle 1 \rangle = 3I + \frac{3}{2}\,\frac{1}{2}\,\boldsymbol{P} \cdot \boldsymbol{P}' \tag{29}$$

where

$$\boldsymbol{P}' = 4(\boldsymbol{V}_1 \cdot \boldsymbol{V}_3{}^* + \boldsymbol{V}_3 \cdot \boldsymbol{V}_1{}^* + \boldsymbol{V}_2 \cdot \boldsymbol{V}_4{}^* + \boldsymbol{V}_4 \cdot \boldsymbol{V}_2{}^*)\,\boldsymbol{n} - \tag{30}$$

$$- 4i\,\boldsymbol{V}_1 \times \boldsymbol{V}_1{}^* - 4i\,\boldsymbol{V}_3 \times \boldsymbol{V}_3{}^* - 4i\,\boldsymbol{V}_2 \times \boldsymbol{V}_2{}^* - 4i\,\boldsymbol{V}_4 \times \boldsymbol{V}_4{}^*.$$

In Eq. (29), the factor 3/2 comes from the normalization referred to in footnote 4, and the factor 1/2 comes from the 1/2 in $\boldsymbol{S} = \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_p/2$. It is extremely tedious (but possible) to prove that

$$P' < \frac{4}{3}\,(3I)\,. \tag{31}$$

If Eq. (26) is not taken into account, one finds the obviously absurd possibility $P' = 2\,(3\,I)$.

There is *no connection* between Eq. (27) and Eq. (30) which might give an analogue of SATCHLER's theorem Eq. (24).

There is no point in attempting to prove a connection between $\boldsymbol{P}'$ and the vector polarization of deuterons produced by unpolarized protons in the inverse reaction, because the scattering amplitude for $d + C^{12} \to p + C^{13}$ is not the same as the scattering amplitude for $p + C^{13} \to d + C^{12}$; that is, these 'inverse' reactions are not 'inverse' in this sense. Such a theorem would not be of any use from the experimental point of view anyway.

Finally, we consider the effects of the tensor components of the deuteron polarization in the azimuthal distribution of the protons. If we define the tensor

$$t_{ij} \equiv \mathrm{Tr}\, M\, \sigma_{ni}\, \sigma_{nj}\, M^+ =$$

$$= 4\,[V_{1i}\, V_{2j}{}^* - i(V_1 \times V_3{}^*)_i n_j - i\,(V_1 \times n)_i\, V_{4j}{}^* - \tag{32}$$

$$- i\, V_{3i}{}^*\, (V_2 \times n)_j - i n_i\, (V_2 \times V_4{}^*)_j - (V_3 \times n)_i\, (n \times V_4)_j] +$$

$$+ \text{complex conjugate}\,.$$

Then

$$T_{ij} \equiv \mathrm{Tr}\, M\, S_{ij}\, M^+ = -\frac{1}{6}\,\mathrm{Tr}\, t\, \delta_{ij} + \frac{1}{4}\,(t_{ij} + t_{ji})\,. \tag{33}$$

By choosing coordinate axes $\boldsymbol{N}$, $\boldsymbol{P}$, $\boldsymbol{K}$ as STAPP has done

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{k}_d \times \boldsymbol{k}_p \\ \boldsymbol{P} &= \boldsymbol{k}_p + \boldsymbol{k}_d / |\boldsymbol{k}_p + \boldsymbol{k}_d| \\ \boldsymbol{K} &= \boldsymbol{k}_p - \boldsymbol{k}_d / |\boldsymbol{k}_p - \boldsymbol{k}_d| \end{aligned} \tag{34}$$

it is possible to write any of the vectors $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_4$ in the form

$$\boldsymbol{V} = V_P \boldsymbol{P} + V_K \boldsymbol{K} \tag{35}$$

and collect Eq. (33) in the form of STAPP'S Eq. (28) (p. 76, ref. 4). It is not necessary to repeat these results here.

One finds

$$\langle 1 \rangle = 1 + \frac{1}{2} tt' + \frac{2}{3} (uu' - vv') \cos \varphi + \frac{1}{6} ww' \cos 2\varphi. \tag{36}$$

Tables of t, u, v, w for $d + He^4$ scattering have been published elsewhere [5]. Because we do not refer to a specific model in this work, we cannot predict values of t', u', v', w' for the $d + C^{12}$ reaction (the Butler theory gives t', v', $w' = 0$). Experimentally, one looks for a difference between (left + right) and (up + down) scattering. Also, the total cross section (integral over the azimuth) may be different for polarized deuterons than for unpolarized deuterons.

D. Criticism of the use of C^{12} as an analyzer for the spin polarization of deuterons

The fact that there exists no rigorous analogue of the Satchler theorem means that the stripping reaction is unsuitable as a precision analyzer for the spin polarization of deuteron *before it has been 'calibrated'*. The quantity $\boldsymbol{P}'$ (Eq. (30)) cannot be measured without using deuterons of known vector polarization in an experiment designed to calibrate the reaction. One may, in a *preliminary* experiment, observe a left right asymmetry and use the Satchler theorem to interpret the result; there is no guarantee that the interpretation is correct.

4. Acknowledgement

I thank Professors LEONARD S. RODBERG, JOHN L. MCHALE, and R. M. THALER for stimulating discussions.

Appendix A

In this appendix we show that in the 'Butler theory' the T_{ij} defined in Eq. (33) vanish. We have, in the Butler theory, from Eq. (32)

$$t_{ij} = \text{tr}\, \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \frac{\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3}{4} \sigma_{n\,i} \sigma_{p\,j} \frac{\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3}{4} \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*. \tag{A.1}$$

It is best to break this into four terms

$$\begin{aligned}
&(\text{I}) &&= \frac{1}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \, \sigma_{n\,i} \sigma_{p\,j} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* \\
&(\text{II}) &&= \frac{3}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \, \sigma_{n\,i} \sigma_{p\,j} \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* \\
&(\text{III}) &&= \frac{3}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \sigma_{n\,i} \sigma_{p\,j} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* \\
&(\text{IV}) &&= \frac{9}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \sigma_{n\,i} \sigma_{p\,j} \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*
\end{aligned} \tag{A.2}$$

Term (IV) is obviously zero since Tr $\sigma_{pj} = 0$. Terms II and III are easily calculated by taking the proton trace first and using

$$\text{Tr (proton space) } \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \, \sigma_{p\,j} = 2\sigma_{n\,j} \,. \tag{A.3}$$

We find

$$(\text{II}) + (\text{III}) = \frac{6}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} (\sigma_{n\,i} \sigma_{n\,j} + \sigma_{n\,j} \sigma_{n\,i}) \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* \,. \tag{A.4}$$

Then using

$$\sigma_{n\,i} \sigma_{n\,j} + \sigma_{n\,j} \sigma_{n\,i} = 2\delta_{ij} \tag{A.5}$$

and

$$\text{Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* = 2\, \boldsymbol{V} \cdot \boldsymbol{V}^*, \tag{A.6}$$

it follows that

$$(\text{II}) + (\text{III}) = \frac{24}{16} \delta_{ij} \, \boldsymbol{V} \cdot \boldsymbol{V}^*. \tag{A.7}$$

(I) may be evaluated by using

$$\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \, \sigma_{n\,i} = (2\, \sigma_{p\,i} - \sigma_{n\,i} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p) \,. \tag{A.8}$$

Then

$$(\text{I}) = \frac{1}{16} \text{Tr}\, \boldsymbol{\sigma}_n \cdot \boldsymbol{V} (2\, \sigma_{p\,i} - \sigma_{n\,i} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p) \, \sigma_{p\,i} \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^* = \tag{A.9}$$

$$= (\text{I}_1) + (\text{I}_2) \text{ (the obvious separation into two terms).}$$

Term (I_1) may be evaluated using

$$\sigma_{p\,i} \sigma_{p\,j} = \delta_{ij} + i e^{ijk} \sigma_{p\,k} \tag{A.10}$$

($e^{ijk} = +1$ if ijk even permutation of 123, -1 if odd, 0 otherwise), so that

$$(\text{I}_1) = \frac{2}{16} \text{ Tr } \boldsymbol{\sigma}_n \cdot \boldsymbol{V} (\delta_{ij} + i e^{ijk} \sigma_{p\,k}) \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*. \tag{A.11}$$

Taking the trace in the proton space first, the term with δ_{ij} gives nothing because Tr $\boldsymbol{\sigma}_p = 0$. In the second term, σ_{pk} is replaced by $2\sigma_{nk}$, since

$$\mathrm{Tr}\ \boldsymbol{\sigma}_p \boldsymbol{\sigma}_p \cdot \boldsymbol{A} = 2\,\boldsymbol{A}. \tag{A.12}$$

Therefore

$$(I_1) = \frac{4}{16}\,\mathrm{Tr}\ \boldsymbol{\sigma}_n \cdot \boldsymbol{V}\, i e^{ijk}\, \sigma_{nk} \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*. \tag{A.13}$$

Term (I_2) is easily evaluated taking the proton space trace first, using

$$\mathrm{Tr}\ \boldsymbol{A} \cdot \boldsymbol{\sigma}_p\, \sigma_{pj}\, \boldsymbol{B} \cdot \boldsymbol{\sigma}_p = -\,2i(\boldsymbol{A} \times \boldsymbol{B})_j \tag{A.14}$$

$$(\boldsymbol{\sigma}_n \times \boldsymbol{\sigma}_n)_j = 2\,i\,\sigma_{nj}.$$

The result is

$$(\mathrm{I}_2) = -\,\frac{4}{16}\,\mathrm{Tr}\ \boldsymbol{\sigma}_n \cdot \boldsymbol{V}\, \sigma_{ni}\, \sigma_{nj}\, \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*. \tag{A.15}$$

Applying (A.10) gives

$$(\mathrm{I}_2) = -\,\frac{4}{16}\,\mathrm{Tr}\ \boldsymbol{\sigma}_n \cdot \boldsymbol{V}\,(\delta_{ij} + i e^{ijk}\, \sigma_{nk})\, \boldsymbol{\sigma}_n \cdot \boldsymbol{V}^*. \tag{A.16}$$

The second part of this cancels (I_1); the final result is

$$(\mathrm{I}_1) + (\mathrm{I}_2) = -\,\frac{8}{16}\,\delta_{ij}\, \boldsymbol{V} \cdot \boldsymbol{V}^* \tag{A.17}$$

which, when combined with (A.7) gives

$$t_{ij} = \boldsymbol{V} \cdot \boldsymbol{V}^*\, \delta_{ij}. \tag{A.18}$$

Eq. (33) leads immediately to

$$T_{ij} = 0. \tag{A.19}$$

It is also possible to obtain this result from Eq. (32) directly, using

$$\boldsymbol{\sigma}_n \cdot \boldsymbol{V}\, \frac{\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_p + 3}{4} = \frac{3}{4}\,\boldsymbol{\sigma}_n \cdot \boldsymbol{V} + \frac{1}{4}\,\boldsymbol{\sigma}_p \cdot \boldsymbol{V} +$$

$$+ \frac{i}{4}\,\boldsymbol{\sigma}_n \cdot (\boldsymbol{V} \times \boldsymbol{n})\ \boldsymbol{\sigma}_p \cdot \boldsymbol{n} - \frac{i}{4}\,\boldsymbol{p}_p \cdot (\boldsymbol{V} \times \boldsymbol{n})\ \boldsymbol{\sigma}_n \cdot \boldsymbol{n}\,. \tag{A.20}$$

That is, in the 'Butler theory',

$$\boldsymbol{V}_1 = \frac{3}{4}\,\boldsymbol{V},\ \ \boldsymbol{V}_2 = \frac{1}{4}\,\boldsymbol{V},\ \ \boldsymbol{V}_3 = \frac{i}{4}\,(\boldsymbol{V}, \times \boldsymbol{n})\ \ \boldsymbol{V}_4 = -\,\frac{i}{4}\,(\boldsymbol{V} \times \boldsymbol{n})\,. \tag{A.21}$$

The algebra is somewhat complicated.

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