On the asymptotic behaviour of Wightman functions in space-like directions

On the Asymptotic Behaviour of Wightman Functions in Space-Like Directions

by **H. Araki*)**, **K. Hepp** and **D. Ruelle**
Eidgenössische Technische Hochschule, Zürich

(1. II. 1962)

Abstract. The exponential decrease of the truncated vacuum expectation value of a product of field operators (each field being smeared out over a compact set) at large separation of their arguments along a fixed space-like hyperplane is proved under the assumptions of translation invariance, stability of the vacuum, existence of a lowest non-zero mass and local commutativity, but without assuming full Lorentz invariance or temperedness of the fields. A result is also obtained for the case of lowest mass zero.

1. Introduction and statement of the results

Since its introduction by Haag[1]) the asymptotic condition in space-like directions has been studied by several authors[2–5]). The problem is to determine the asymptotic behaviour of the Wightman functions of the basic fields $A(x)$

$$(\Omega, A(x_1) \ldots A(x_n)\,\Omega) \tag{1.1}$$

or of the vacuum expectation values (VEV) of smeared out fields $B(x)$

$$(\Omega, B(x_1) \ldots B(x_n)\,\Omega) \tag{1.2}$$

for large space-like separation of some of the arguments x_i.

The discussions by Araki[3]) and Jost and Hepp[4]) are mainly based on the assumptions of Lorentz invariance and spectral conditions. They show that, if the points x_i are separated into two clusters, (1.2) approaches a limit faster than any (negative) power of the separation distance between the clusters, when this distance is space-like and tends to infinity.

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*) On leave of absence from the Department of Nuclear Engineering, Kyoto University, Kyoto, Japan.

ARAKI further shows that, in the case of the dilatation of Jost points, (1.1) converges exponentially towards its limit*).

On the other hand the discussion of RUELLE[5]) is based on translation invariance, spectral conditions and local commutativity but without further use of Lorentz invariance or positiveness of the metric in Hilbert space. He proves the vanishing of the truncated VEV corresponding to (1.2) faster than any (negative) power of the diameter of the point set $\{x_i\}$, when the x_i are in a space-like plane and the testing functions are in $\mathcal{S}$.

We present here a proof of the exponential approach of (1.2) to its limit for testing functions in $\mathcal{D}$ under the following assumptions**), which are similar to those of RUELLE.

(T) There exists a unitary representation $T(a)$ of the translation group satisfying

$$T(a)\, A(x)\, T(-a) = A\,(x + a)\,. \tag{1.3}$$

(S) There exists a unique vacuum state Ω satisfying

$$T(a)\, \Omega = \Omega\,. \tag{1.4}$$

In $\Omega^{\perp}$ the spectral measure E of the unitary representation $T(a)$ defined by

$$T(a) = \int e^{i(p,a)}\, dE(p)\,, \tag{1.5}$$

$$(p, a) = p^0 a^0 - \sum_{i=1}^{3} p^i a^i \tag{1.6}$$

has its support in

$$\overline{V}_+^M = \{p\colon (p, p) \geqq M^2,\ p^0 > 0\}\,, \qquad M > 0\,, \tag{1.7}$$

(C) $$[A(x_1)\,, A(x_2)] = 0 \quad \text{if} \quad (x_1 - x_2, x_1 - x_2) < 0\,. \tag{1.8}$$

Our main idea is simple and may be sketched as follows for the case of the separation into two clusters (the proofs of the different steps will be

*) Theorem I of ref. [3]) is stated for points x_i with the same time component and assumes local commutativity, but the exponential vanishing of

$$(\Omega, A(x_1)\, E_0^{\perp}\, A(x_2) \ldots E_0^{\perp}\, A(x_n)\, \Omega)$$

can be proved in the same way without local commutativity, for $x_i - x_{i+1} = \xi_i + \lambda\, \xi_i'$, where $\xi_i + \lambda\, \xi_i'$ should be a Jost point for sufficiently large λ and $\lambda \to \infty$, ξ_i, ξ_i' fixed (with at least one $\xi_i' \neq 0$).

**) The operator valued distribution $A(x)$ is not assumed to be tempered unless explicitly stated. Apart from the Lorentz invariant support conditions implied by the spectral condition and by locality, no use of invariance with respect to Lorentz rotations is made.

given in later sections). We investigate the behaviour for large space-like $\xi = x_1 - x_2$ of the function

$$h_{12}(\xi) = \langle B_1(x_1)\, B_2(x_2)\rangle_0 - \langle B_1(x_1)\rangle_0\, \langle B_2(x_2)\rangle_0$$
$$= \langle B_1(0)\, T(-\xi)\, E_0^{\perp}\, B_2(0)\rangle_0 , \qquad (1.9)$$

where $\langle \ldots \rangle_0$ denotes the vacuum expectation value, $E_0^{\perp}$ is the orthogonal projection on $\Omega^{\perp}$ and $B_i(x_i)$ is defined by

$$B_i(x_i) = \int dx'_1 \ldots dx'_{r(i)}\, \varphi_i(x'_1, \ldots x'_{r(i)}) \prod_{j=1}^{r(i)} A\,(x_i + x'_j)\,, \qquad (1.10)$$
$$\varphi_i \in \mathcal{D}_{4\,r(i)}\,, \qquad i = 1, 2\,.$$

Now, due to (C), the function

$$h(\xi) = h_{12}(\xi) - h_{21}(-\xi) = \langle [B_1(x_1), B_2(x_2)]\rangle_0 \qquad (1.11)$$

vanishes for $\xi \in D(\varphi_1, \varphi_2)'$, where

$$D(\varphi_1, \varphi_2) = \{x'_2 - x'_1 \colon\ x'_i \in \operatorname{supp} \varphi_i\,,\ i = 1, 2\} \qquad (1.12)$$

and D' is the set of all points which are space-like to every point in D. Furthermore, due to (S), the Fourier transform $\tilde{h}(p)$ of $h(\xi)$ vanishes for $(p, p) < M^2$. Therefore, $h(\xi)$ has the JOST-LEHMANN-DYSON[6]) representation

$$h(\xi) = \int_{M^2}^{\infty} d(\varkappa^2) \int_{D_1} d\xi'\, [\Delta_\varkappa(\xi - \xi')\, \varrho_1(\xi', \varkappa) + \frac{\partial}{\partial \xi^0} \Delta_\varkappa\,(\xi - \xi')\, \varrho_2(\xi', \varkappa)] \qquad (1.13)$$

where the three-dimensional region D_1 is compact.

The function $h_{12}(\xi)$, being the positive frequency part of $h(\xi)$, is obtained if $\Delta_\varkappa$ is replaced by $\Delta_\varkappa^{(+)}$ in (1.13).

The exponential vanishing of $h_{12}(\xi)$ at large space-like distances is finally derived from the following asymptotic formulae valid for $(\xi, \xi) < 0$, $|\xi| = [-(\xi, \xi)]^{1/2} \to \infty$:

$$\Delta_\varkappa^{(+)}(\xi) = -\,i \left(\frac{\varkappa}{32\,\pi^3\,|\xi|^3}\right)^{1/2} e^{-\varkappa|\xi|} \left[1 + 0\left(\frac{1}{\varkappa|\xi|}\right)\right]. \qquad (1.14)$$

and

$$\frac{\partial}{\partial \xi^0}\, \Delta_\varkappa^{(+)}(\xi) = -\,i\, \xi^0 \left(\frac{\varkappa^3}{32\,\pi^3\,|\xi|^5}\right)^{1/2} e^{-\varkappa|\xi|} \left[1 + 0\left(\frac{1}{\varkappa|\xi|}\right)\right]. \qquad (1.15)$$

The result is the following

Theorem 1. (T), (S) *and* (C) *imply that*

$$|h_{12}(\xi)| \leqq C[\xi]^{-3/2} e^{-M[\xi]} \left(1 + \frac{|\xi^0|}{[\xi]}\right)$$

if $\xi \in D_1'$ *and* $[\xi] \geqq \delta > 0$, *where* $h_{12}(\xi)$ *is defined by* (1.9), C *is a constant independent of* ξ, $[\xi]$ *is the shortest space-like distance between* ξ *and a compact set* D_1 *and* D_1 *is the convex closure of the complement, in the plane* $\{\xi : \xi^0 = 0\}$, *of the intersection* $D' \cap \{\xi^0 = 0\}$, *where* D *is defined by* (1.11).

In short, if the fields in $B_1(x_1)$ and $B_2(x_2)$ are mutually separated by a large space-like distance R, $h_{12}(\xi)$ tends to zero at least as fast as $R^{-3/2} \times e^{-MR}$. The weight functions ϱ_1 and ϱ_2 are simply related to $\tilde{h}(p)$ and $p^0 \tilde{h}(p)$, which are bounded complex measures and, from this, C can be explicitly expressed by certain VEV's.

As a consequence of theorem 1 one has the following

Theorem 2. (T), (S) *and* (C) *imply that the truncated* VEV *of the product of the fields* $B_i(x_i)$ $(i = 1, \ldots n)$ *tends to zero at least as fast as* $R^{3/2} e^{-MR/(n-1)}$, *when the diameter* R *of the point set* $\{x_i\}$ *goes to infinity and* $x_1^0 = \ldots = x_n^0$.

Our method also yields a result for the case $M = 0$:

Theorem 3. (T), (S) *with* $M = 0$ *and* (C) *imply that*

$$|h_{12}(\xi)| \leqq C[\xi]^{-2} \left(1 + \frac{|\xi^0|}{[\xi]^2}\right)$$

if $\xi \in D_1'$ *where* $h_{12}(\xi)$ *is defined by* (1.9), C *is a constant independent of* ξ *and* $[\xi]$ *is again the shortest space-like distance between* ξ *and* D_1.

Remark: The formulation of the above theorems is not optimal because of the special rôle played by the space-like hyperplane $\{\xi : \xi^0 = 0\}$ in the definition of D_1 and in theorem 2. More adequate results can be easily obtained in particular cases.

2. The case of two clusters (proof of theorem 1)

First, we note that, due to (T), the Fourier transform $\tilde{h}_{12}(p)$ of $h_{12}(\xi)$ exists and is given by

$$\tilde{h}_{12}(p)\, dp = (2\pi)^{-2}\, d\, \langle B_1(0)\, E(p)\, B_2(0) \rangle_0 \,. \tag{2.1}$$

According to Jost and Hepp[4])

$$(p^0)^l \left(\prod_{i=1}^{3} (p^i)^{m_i}\right) \tilde{h}_{12}(p)$$

is a bounded complex measure for arbitrary positive integers l and m_i.

In particular, for any bounded continuous function $\chi(p)$ one has

$$\left| \int dp\ \chi(p)\ (p^0)^l\ \tilde{h}_{12}(p) \right| \leqq$$

$$\leqq \| (P^0)^l\ B_2(0)\ \Omega \| \cdot \| B_1(0)^*\ \Omega \|\ (2\pi)^{-2} \sup_{(p,p) \geqq M^2,\, p^0 > 0} |\chi(p)|\,, \qquad (2.2)$$

where P^0 is the energy operator and $(P^0)^l\ B_2(x)\ \Omega$ is obtained from $B_2(x)\ \Omega$ if $\varphi_2\ (x'_1, \ldots x'_{r(2)})$ is replaced in (1.10) by

$$\left(i \frac{\partial}{\partial x^0}\right)^l \varphi_2\ (x'_1 + x, \ldots x'_{r(2)} + x)\ |_{x=0}\,.$$

Correspondingly we obtain from (1,11):

$$\left| \int dp\ \chi(p)\ (p^0)^l\ \tilde{h}(p) \right| \leqq C_l \sup_{(p,p) \geqq M^2} |\chi(p)|\,, \qquad (2.3)$$

$$C_l = (2\pi)^{-2} \{ \| (P^0)^l\ B_2(0)\ \Omega \| \cdot \| B_1(0)^*\ \Omega \| +$$

$$+ \| (P^0)^l\ B_1(0)\ \Omega \| \cdot \| B_2(0)^*\ \Omega \| \}\,. \qquad (2.4)$$

Now, the boundedness and support properties of $\tilde{h}(p)$ imply that the function

$$H(\xi, s) = (2\pi)^{-2} \int dp\ e^{-i(p,\xi)} \cos\left(s \sqrt{(p,p)}\right) \tilde{h}(p) \qquad (2.5)$$

is infinitely continuously differentiable and satisfies

$$\left[\left(\frac{\partial}{\partial \xi^0}\right)^2 - \sum_{i=1}^{3} \left(\frac{\partial}{\partial \xi^i}\right)^2 - \left(\frac{\partial}{\partial s}\right)^2 \right] H(\xi, s) = 0\,, \qquad (2.6)$$

$$\frac{\partial^n}{\partial s^n} H(\xi, 0) = \begin{cases} 0 & \text{for odd } n\,, \\ \Box_\xi^{n/2}\ h(\xi) & \text{for even } n\,. \end{cases} \qquad (2.7)$$

Due to (C), $h(\xi)$ vanishes for ξ in $D(\varphi_1, \varphi_2)'$. $H(\xi, s)$ and all its derivatives therefore vanish along the time-like segment defined for fixed ξ by

$$s = 0\,, \qquad |\xi_0| < \min_{\xi' \in D_0} |\xi' - \xi| \qquad (2.8)$$

where D_0 is the complement in the plane $\{\xi : \xi^0 = 0\}$ of the intersection $D(\varphi_1, \varphi_2)' \cap \{\xi : \xi^0 = 0\}$ and is compact.

The uniqueness theorem for the solutions of (2.6) (see e.g. WIGHTMAN[7]) states that if $H(\xi, s)$ vanishes of infinite order along a time-like segment, it also vanishes in the double-cone spanned by this segment.

If we let ξ go to infinity in (2.8) and apply this theorem, we see that

$$\left(\frac{\partial}{\partial \xi^0}\right)^l H(\xi, s)\,|_{\xi^0=0} = 0 \quad \text{if} \quad \xi \notin D_1\,, \qquad l = 0, 1\,, \tag{2.9}$$

where D_1 is the convex closure of D_0 and is compact.

$H(\xi, s)$ may be expressed in terms of its Cauchy data in the plane $\{(\xi, s) : \xi^0 = 0\}$:

$$H(\xi, s) = -\int d\xi' \int ds' \left[\frac{\partial}{\partial \xi^0} \Delta^{(5)}(\xi - \xi', s - s')\, H(\xi', s')\,|_{\xi'^0=0} + \right.$$
$$\left. + \Delta^{(5)}(\xi - \xi', s - s') \frac{\partial}{\partial \xi'^0} H(\xi', s')\,|_{\xi'^0=0}\right] \tag{2.10}$$

where, due to (2,9), the ξ'-integration may be restricted to D_1 and

$$\Delta^{(5)}(\xi, s) = -i(2\pi)^{-4} \int dp\, d\varkappa\, e^{-i[(p,\xi) - \varkappa s]}\, \varepsilon(p^0)\, \delta(p^2 - \varkappa^2)\,. \tag{2.11}$$

Setting $s = 0$ in (2.10) and taking the $p^0 > 0$ part of (2.11), we obtain

$$h_{12}(\xi) = -(2\pi)^{-1} \int\limits_{D_1} d\xi' \int\limits_{-\infty}^{+\infty} ds' \int\limits_{-\infty}^{+\infty} d\varkappa\, e^{-i\varkappa s'} \left[\frac{\partial}{\partial \xi^0} \Delta_\varkappa^{(+)}(\xi - \xi') \times \right.$$
$$\left. \times H(\xi', s')\,|_{\xi'^0=0} + \Delta_\varkappa^{(+)}(\xi - \xi') \frac{\partial}{\partial \xi'^0} H(\xi', s')\,|_{\xi'^0=0}\right], \tag{2.12}$$

where we have

$$\Delta_\varkappa^{(+)}(\xi) = -i(2\pi)^{-3} \int dp\, e^{-i(p,\xi)}\, \theta(p^0)\, \delta(p^2 - \varkappa^2) \tag{2.13}$$

$$= -i\,(2\pi)^{-2}\, \varkappa\, K_1\left(\varkappa \sqrt{-(\xi,\xi)}\right) / \sqrt{-(\xi,\xi)} \quad \text{for} \quad (\xi,\xi) < 0\,. \tag{2.14}$$

We may thus write (see (2.5))

$$h_{12}(\xi) = -(2\pi)^{-2} \int\limits_{D_1} d\xi'\, f(\xi, \xi')\,, \tag{2.15}$$

$$f(\xi, \xi') = \int dp\, [\chi_1(p, \xi, \xi')\, \tilde{h}(p) - i\, \chi_2(p, \xi, \xi')\, p^0\, \tilde{h}(p)] \tag{2.16}$$

where*)

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*) Note that, in proceeding from (2.12) to (2.16), the exchange of the s'- and p-integration is allowed because

$$I = \int\limits_0^\infty \cos(\varkappa s')\, \Delta_\varkappa^{(+)}(\xi)\, d\varkappa = -i(8\pi)^{-1}\, (-(\xi, \xi) + s'^2)^{-3/2}$$

and $$I \sim |s'|^{-3}, \frac{\partial I}{\partial \xi^0} \sim |s'|^{-5} \quad \text{for} \quad |s'| \to \infty\,,$$

so that the convergence of the s'-integration is uniform in p in (2.17) and (2.18).

$$\chi_1(p, \xi, \xi') = \pi^{-1} e^{ip\xi'} \int_{-\infty}^{+\infty} ds' \cos\left(s' \sqrt{(p, p)}\right) \int_0^{\infty} d\varkappa \cos(\varkappa s') \frac{\partial}{\partial \xi^0} \times$$
$$\times \Delta_\varkappa^{(+)} (\xi - \xi') \big|_{\xi'^0 = 0} = e^{ip\xi'} \frac{\partial}{\partial \xi^0} \Delta^{(+)}_{\sqrt{(p, p)}} (\xi - \xi') \big|_{\xi'^0 = 0}, \qquad (2.17)$$

$$\chi_2(p, \xi, \xi') = \pi^{-1} e^{ip\xi'} \int_{-\infty}^{+\infty} ds' \cos\left(s' \sqrt{(p, p)}\right) \int_0^{\infty} d\varkappa \cos(\varkappa s') \times$$
$$\times \Delta_\varkappa^{(+)} (\xi - \xi') \big|_{\xi'^0 = 0} = e^{ip\xi'} \Delta^{(+)}_{\sqrt{(p, p)}} (\xi - \xi') \big|_{\xi'^0 = 0}. \qquad (2.18)$$

Applying (2.3) to (2.15), (2.16) we obtain the majorization

$$|h_{12}(\xi)| \leqq (2\pi)^{-2} V(D_1) \left[C_0 \sup |\chi_1(p, \xi, \xi')| + C_1 \sup |\chi_2(p, \xi, \xi')|\right], \qquad (2.19)$$

where $V(D_1)$ denotes the volume of D_1 and the supremum is taken over all p, ξ' such that $(p, p) \geqq M^2$, $\xi' \in D_1$.

On the other hand, from (1.14), (1.15) and (2.14) we get

$$|\xi|^{3/2} e^{M|\xi|} |\Delta_\varkappa^{(+)}(\xi)| < A_1,$$
$$|\xi|^{5/2} |\xi^0|^{-1} e^{M|\xi|} \left|\frac{\partial}{\partial \xi^0} \Delta_\varkappa^{(+)}(\xi)\right| < A_2$$

for $\varkappa \geqq M$ and space-like ξ such that $|\xi| = [-(\xi, \xi)]^{1/2} \geqq \delta > 0$.

These majorizations finally yield the formula

$$|h_{12}(\xi)| \leqq (2\pi)^{-2} V(D_1) [\xi]^{-3/2} e^{-M[\xi]} \left[C_0 A_1 + C_1 A_2 \frac{|\xi^0|}{[\xi]}\right], \qquad (2.20)$$

where $[\xi]$ is the shortest space-like distance between ξ and D_1.

3. The case of zero mass (proof of theorem 3)

We assume that Ω is the only eigenstate belonging to the eigenvalue 0 of P^μ. Thus its positive frequency part $\tilde{h}_{12}(p)$ may again be uniquely obtained from $\tilde{h}(p)$, since $\tilde{h}(p)$ has zero measure for the point $\{p = 0\}$, and every step in the previous section up to equation (2.8) is valid with M replaced by zero.

Furthermore, since

$$\Delta_0^{(+)}(\xi) = -i(2\pi)^{-2} (\xi, \xi)^{-1} \quad \text{for} \quad (\xi, \xi) < 0 \qquad (3.1)$$

and since $\varkappa K_1(\varkappa)$ and $\varkappa^3 \partial/\partial\varkappa [K_1(\varkappa)/\varkappa]$ are both bounded for $\varkappa \geqq 0$, we get

$$|\Delta_\varkappa^{(+)}(\xi)| < A_1' |\xi|^{-2}, \qquad (3.2)$$

$$\left| \frac{\partial}{\partial \xi^0} \Delta_\varkappa^{(+)}(\xi) \right| < A_2' \, |\xi^0| \cdot |\xi|^{-4} \tag{3.3}$$

for $\varkappa \geqq 0$ and space-like ξ. Therefore

$$|h_{12}(\xi)| \leqq V(D_1) \, [\xi]^{-2} \left[C_0 \, A_1' + C_1 \, A_2' \, \frac{|\xi^0|}{[\xi]^2} \right]. \tag{3.4}$$

4. The case of many clusters (proof of theorem 2)

We investigate now the behaviour of the truncated VEV

$$\langle B_1(x_1) \ldots B_n(x_n) \rangle_0^T \tag{4.1}$$

for large separation of the x_i, assuming $x_1^0 = \ldots = x_n^0$.

The main idea is the following. If $R(x) = \max_{i,j} |x_i - x_j|$, there exists a partition of the point set $\{x_i\}$ into two families such that the distance of their convex hulls is at least $R(x)/(n-1)$*).

Theorem 1 may then be applied, the constants C_l still depending on the configuration of points in each family. However, apart from the volume factor $V(D_1)$, C_l may be proved to be uniformly bounded, due to the following lemma, which is a direct consequence of the Schwarz inequality:

Lemma: For $x_1^0 = \ldots = x_n^0$ *the* VEV

$$\langle B_1(x_1) \ldots B_n(x_n) \rangle_0 \tag{4.2}$$

is a bounded function of its arguments.

Proof of the lemma.

a) Since (4.2) is a continuous function of the difference variables $x_i - x_{i+1}$, it is bounded in any region of the type $\max_{i,j} |x_i - x_j| \leqq (n-1) \, L$.

b) If $\max_{i,j} |x_i - x_j| > (n-1) \, L$, there exists a partition of the point set $\{x_i\}$ into two families F_1, F_2, such that $x_i \in F_1$ and $x_j \in F_2$ imply $|x_i - x_j| > L$. Furthermore, since $D(\varphi_i, \varphi_j)$ is compact for all i, j, it is possible to choose L such that $[B_i(x_i), B_j(x_j)] = 0$ whenever $|x_i - x_j| > L$. We may therefore rewrite (4.2) in the form

$$\langle B_1'(x_1') \ldots B_k'(x_k') \, B_1''(x_1'') \ldots B_{n-k}''(x_{n-k}'') \rangle_0 , \tag{4.3}$$

*) Let $\max_{i,j} |x_i - x_j|$ be obtained for $i = k, j = l$ and consider the plane orthogonal to $x_k - x_l$ through each x_i. Then there always exist two neighbouring planes with a distance not smaller than $R(x)/(n-1)$, and these planes divide the point set $\{x_i\}$ into two families F_1, F_2 with the required properties.

where

$$x_i' \in F_1, \qquad 1 \leqq i \leqq k; \qquad x_j'' \in F_2, \qquad 1 \leqq j \leqq n-k.$$

The absolute value of (4.3) is bounded by

$$\| B_k'(x_k')^* \dots B_1'(x_1')^* \Omega \| \cdot \| B_1''(x_1'') \dots B_{n-k}''(x_{n-k}'') \Omega \| . \tag{4.4}$$

c) Each factor of (4.4) is the square root of a VEV of type (4.2) which can be majorized again as above. This process has only to be repeated a finite number of times since (b) eventually leads to constant bounds with factors of the type $\langle (B_i(x_i) B_i(x_i)^*)^l \rangle_0$. The lemma is thus proved.

We present now the details of the proof of theorem 2.

If $\max_{i,j} |x_i - x_j| = R(x)$, there exists a partition of the x_i into two families F_1 and F_2 such that their convex hulls $C(F_1)$ and $C(F_2)$ have a distance not smaller than $R(x)/(n-1)$.

Let $R(x) \geqq (n-1) L$, then using the locality of the truncated VEV one may rearrange the $B_i(x_i)$ in (4.1) in the form

$$\langle B_1'(x_1') \dots B_k'(x_k') B_1''(x_1'') \dots B_{n-k}''(x_{n-k}'') \rangle_0^T \tag{4.5}$$

similar to (4.3). (4.5) may be written as a sum of products of ordinary VEV

$$\langle B_{i_1}'(x_{i_1}') \dots B_{i_r}'(x_{i_r}') B_{j_1}''(x_{j_1}'') \dots B_{j_s}''(x_{j_s}'') \rangle_0 . \tag{4.6}$$

This sum does not change if each factor (4.6) is replaced by

$$\langle B_{i_1}'(x_{i_1}') \dots B_{i_r}'(x_{i_r}') E_0^{\perp} B_{j_1}''(x_{j_1}'') \dots B_{j_s}''(x_{j_s}'') \rangle_0 , \tag{4.7}$$

when $r \cdot s \neq 0$, if it is left unchanged when $r \cdot s = 0$ and if the terms which contain only factors with $r \cdot s = 0$ are crossed out*).

We remain thus with terms containing at least one factor of the type (4.7). According to the above lemma the other factors are majorized by constants. The VEV (4.7) may be majorized according to (2.20), where $\xi^0 = 0$ and $[\xi] + L$ is not smaller than the distance of $C(F_1)$, $C(F_2)$, which is at least $R/(n-1)$. In the expression (2.4) for the constant C_l, $B_1(0)$ and $B_2(0)$ are to be replaced by $B_{i_1}'(x_{i_1}') \dots B_{i_r}'(x_{i_r}')$ and

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*) To see this, write the definition formula (see ref. [1]) for the truncated VEV):

$$\langle \dots \rangle_0^T = \langle \dots \rangle_0 - \Sigma \Pi \langle \dots \rangle_0^T , \tag{α}$$

where the summation extends over all partitions of $\{x_1, \dots x_n\}$ into several sets. From this follows immediately

$$\langle \dots \rangle_0^T = \langle \dots E_0^{\perp} \dots \rangle_0 - \Sigma' \Pi \langle \dots \rangle_0^T , \tag{β}$$

where the summation extends over the partitions which are not finer than (F_1, F_2). The property stated follows by iteration of (β).

$B''_{j_1}(x''_{j_1}) \ldots B''_{j_s}(x''_{j_s})$. Because of the lemma, C_l has again a bound independent of the x'_i, x''_j. On the other hand, the volume factor $V(D_1)$ is smaller than $(2R + L)^3$. This completes the proof of theorem 2.

5. The case of testing functions in $\mathcal{S}$

In this section we generalize theorem 1 and theorem 3 to the case where $\varphi_i \in \mathcal{S}$ in (1.10) using the additional assumption of the temperedness of the field $A(x)$ (yet still without using Lorentz invariance). For each $\varrho \geqq 1$ we introduce a partition of unity $\alpha^1_\varrho(x) + \alpha^2_\varrho(x) \equiv 1$ in $R^{4r(i)}$ such that $\alpha^1_\varrho(x)$ vanishes outside of the region $\max_j [|x^0_j| + |\boldsymbol{x}_j|] \leqq \varrho$ and $\alpha^2_\varrho(x)$ vanishes in the region $\max_j [|x^0_j| + |\boldsymbol{x}_j|] < \varrho - 1/2$. The derivatives of all order of α^1, α^2 are assumed to exist and to be bounded uniformly in ϱ. Introducing the functions

$$\varphi^1_{i\varrho} = \alpha^1_\varrho \varphi_i \in \mathcal{D}, \qquad \varphi^2_{i\varrho} = \alpha^2_\varrho \varphi_i \in \mathcal{S}$$

instead of φ_i in equation (1.10) we obtain fields $B^1_{i\varrho}$, $B^2_{i\varrho}$ such that

$$B_i = B^1_{i\varrho} + B^2_{i\varrho}.$$

Since $\varphi^1_{i\varrho} \in \mathcal{D}$, we may apply formula (2.20) to $B^1_{i\varrho}$ with the result

$$|\langle B^1_{1\varrho}(x_1)\, E^\perp_0\, B^1_{2\varrho}(x_2)\rangle_0| < E\, \varrho^3 [\xi]^{-3/2}\, e^{-M[\xi]} \left(1 + \frac{|\xi^0|}{[\xi]}\right) \tag{5.1}$$

for $0 < \delta \leqq [\xi]$.

The factor ϱ^3 comes from $V(D_1)$ and E is a constant which may be chosen independent of ϱ because of the boundeness*) of norms like $\| B^1_{1\varrho}(0)\, \Omega \|$ with respect to ϱ.

In the right-hand side of the relation

$$\begin{aligned} h_{12}(\xi) = {} & \langle B^1_{1\varrho}(x_1)\, E^\perp_0\, B^1_{2\varrho}(x_2)\rangle_0 + \langle B^1_{1\varrho}(x_1)\, E^\perp_0\, B^2_{2\varrho}(x_2)\rangle_0 + \\ & + \langle B^2_{1\varrho}(x_1)\, E^\perp_0\, B^1_{2\varrho}(x_2)\rangle_0 + \langle B^2_{1\varrho}(x_1)\, E^\perp_0\, B^2_{2\varrho}(x_2)\rangle_0 \end{aligned} \tag{5.2}$$

the first term is majorized by (5.1), the other terms tend to zero faster than any power of ϱ^{-1} when $\varrho \to \infty$ because of the temperedness of Wightman functions.

If we restrict ξ by $|\xi^0| \leqq \lambda |\boldsymbol{\xi}|$, $0 < \lambda < 1$, and take

$$2\varrho = |\boldsymbol{\xi}| \left(1 - \left(\frac{1+\lambda^2}{2}\right)^{1/2}\right) \tag{5.3}$$

*) $\varphi^1_{1\varrho}$ belongs to a bounded set of $\mathcal{S}$ when $1 \leqq \varrho < \infty$.

we find

$$[\xi] = [(|\xi| - 2\varrho)^2 - |\xi^0|^2]^{1/2} \geqq |\xi| \left(\frac{1+\lambda^2}{2} - \lambda^2\right)^{1/2} \geqq |\xi| \left(\frac{1-\lambda^2}{2}\right)^{1/2}. \quad (5.4)$$

Introducing (5.3) and (5.4) in (5.1), we see that all terms in (5.2) decrease faster than any power of $|\xi|$ when $|\xi| \to \infty$, $M > 0$.

Theorem 1'. (*T*), (*S*), (*C*) *and the temperedness of the field imply that the function* $h_{12}(\xi)$ *defined by* (1.9), *where* $\varphi_1, \varphi_2 \in \mathcal{S}$, *tends to zero faster than any power of* $|\xi|^{-1} = (-(\xi,\xi))^{1/2}$ *when* $|\xi| \to \infty$ *with the restriction* $|\xi^0| \leqq \lambda |\xi|$, $0 < \lambda < 1$.

In the case $M = 0$, (5.1) is replaced by

$$|\langle B^1_{1\varrho}(x_1)\, E^{\perp}_0\, B^1_{2\varrho}(x_2)\rangle_0| < E\, \varrho^3\, [\xi]^{-2} \left(1 + \frac{|\xi^0|}{[\xi]^2}\right) \quad (5.5)$$

(5.4) remains valid if 2ϱ is replaced by a quantity smaller than (5.3), for instance

$$\left[|\xi| \left\{1 - \left(\frac{1+\lambda^2}{2}\right)^{1/2}\right\}\right]^{\eta/3}, \qquad 0 < \eta < 3,$$

when

$$|\xi| > \left(1 - \left(\frac{1+\lambda^2}{2}\right)^{1/2}\right)^{-1},$$

and we have

Theorem 3'. (*T*), (*S*) *with* $M = 0$, (*C*) *and the temperedness of the field imply that the function* $h_{12}(\xi)$ *defined by* (1.9) *where* $\varphi_1, \varphi_2 \in \mathcal{S}$, *tends to zero as fast as* $|\xi|^{-2+\eta}$ *for any positive* η *when* $|\xi| \to \infty$ *with the restriction* $|\xi^0| \leqq \lambda |\xi|$, $0 < \lambda < 1$.

We are indebted to Professor R. Jost for helpful criticism and to the members of the Seminar für theoretische Physik, ETH, for creating a stimulating atmosphere. The first named author (H. A.) would like to thank Professors R. Jost and M. Fierz for their hospitality. Two of the authors (H. A. and K. H.) acknowledge the financial support received from the Schweizerischer Nationalfonds (K. A. W.).

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