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The Relation between the Optical Model and the Direct Interaction Theory of Inelastic Scattering

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Abstract: Using Feshbach's *Unified Theory of Nuclear Reactions* the contribution to the optical potential due to the loss of particles through direct interaction inelasting scattering is calculated. The imaginary part is found to be consistent with the magnitude required to fit elastic scattering data. A relation connecting V, R, and a is derived and checked against some existing optical model results. This relation allows a shape parameter to be determined for the nucleon-nucleon potential and may be used to reduce the number of free parameters in optical model calculations.

I. Introduction

In recent years there have been several attempts to justify the optical model and to derive an expression for the imaginary part of the optical potential[1–4]). One of the most recent, and perhaps the most successfull is due to FESHBACH[5]). He is able to obtain some of the more common models of nuclear reactions from a single formalism which he calls the "Unified Theory of Nuclear Reactions". In this treatment resonances, and their effect upon the imaginary part of the optical potential, are considered in detail.

The present work was undertaken to obtain a numerical result for the optical potential and, in particular, to determine if a characteristic shape exists. Measurements of (d, p) and (p, p') angular distributions indicate that the direct interaction mechanism is the dominant one in most energy regions (see especially Ref. 6). Thus it may be thought that a consideration of the general properties of the optical potential need not concern itself with rapid energy variations in the cross section. For the present calculation the Feshbach formalism was used but without the resonance

*) This work was begun when the author was at The Rice University, Houston, Texas.

contributions. This then is essentially the strong coupling approximation as used by CHASE, WILETS, and EDMONDS[7]).

In section II the Unified Theory is reviewed briefly for the case of no resonances and a local form of the optical potential is obtained. This last is a necessary feature since only local forms are used in actual calculations. However, in order to obtain this potential the solution to the entire system of coupled equations must be known.

Section III gives the derivation of a relation between the depth of the real part of the optical potential and the radius and diffuseness of the density of nucleons in the nucleus. Comparison of this relation with existing optical model results shows a high degree of consistency and provides a method for determining the only parameter available aside from the nuclear wave functions. Section IV compares this parameter with the value predicted from low energy nucleon-nucleon scattering.

The general method of solution of the coupled equations is given in section V and section VI gives the results of the numerical calculations.

II. General Formalism

Let us consider a single simple particle (proton or neutron) incident upon a target nucleus consisting of A nucleons. Let Ψ represent the solution to the time independent Schrödinger equation for $A + 1$ particles with the proper boundry conditions. Thus Ψ is a function of $A + 1$ positions coordinates $(\bar{r}_0, \bar{r}_1, \bar{r}_2, \ldots \bar{r}_A)$ and $A + 1$ spin coordinates $(\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_A)$. We shall adopt the convention that the zero subscript refers to the incident particle and subscripts 1 through A refer to the particles making up the target nucleus. This formalism is general but will be used here only for spinless particles so we will drop the spin coordinates now for simplification of the notation. The origin of coordinates will be taken to be at the center of mass of the target nucleus.

If H_0 is the Hamiltonian of the target nucleus then the wave functions (χ_n) and the energies (ε_n) of the target nucleus are given by the solutions to

$$H_0 \chi_n = \varepsilon_n \chi_n . \tag{1}$$

Each χ_n is a function of A position coordinates and the collection of all χ_n forms a complete set of functions in these variables, we shall use this fact to expand Ψ as

$$\Psi = \sum_{n=0}^{\infty} \chi_n(\bar{r}_1, \bar{r}_2, \ldots r_A)\, U_n(\bar{r}_0) \tag{2}$$

where this relation defines U_n. Of course n becomes a continuous variable over part of its range.

Now let H represent the Hamiltonian of the total system. Then $H = H_0 + T + \mathrm{V}$ where T is the kinetic energy of the incident particle with respect to the center of mass of the target nucleus and V is the contribution to the potential energy due to the interaction of the incident particle with all of the target nucleons.

In general $V = V(\bar{r}_0, \bar{r}_1, \bar{r}_2, \ldots \bar{r}_A)$ while $T = T(\bar{r}_0) = (\hbar^2/2m)\nabla^2_{\bar{r}_0}$.

If E is the total energy of the system, Ψ must satisfy

$$H\Psi = E\Psi . \tag{3}$$

Substituting (2) into (3), multiplying by χ^*_m and integrating over $\bar{r}_1$ through $\bar{r}_A$ (assuming the χ_n are orthonormal) we have the set of coupled equations

$$T U_m + \sum_n V_{mn} U_n = E_m U_m , \qquad m = 0, 1, 2, \ldots \tag{4}$$

where

$$V_{mn}(\bar{r}_0) = \iint^{3A} \cdots \int \chi^*_m V \chi_n \, d\bar{r}_1 \, d\bar{r}_2 \ldots d\bar{r}_A \tag{5}$$

and $E_m = E - \varepsilon_m$.

In order to solve the set of coupled differential equations (4) let us make the following approximation.

a) Neglect all reaction channels except the inelastic ones. This means we are also neglecting exchange with the target. Thus there are no nuclear particles at infinity so that there are no continuum terms in the sum. Hence the set of equations (4) reduces to a finite number. It will now be possible to solve (4) if V_{mn} is known. In order to simplify (5) we will make assumption b.

b) Assume V may be represented as a sum of 2-body delta-function potentials i.e.

$$V = V_0 \sum_{i=1}^{A} \delta(\bar{r}_0 - \bar{r}_i) . \tag{6}$$

If we define

$$\varrho_i^{mn}(\bar{r}_i) = \iint^{3A-3} \cdots \int \chi^*_m(\bar{r}_1 \ldots \bar{r}_A) \, \chi_n(\bar{r}_1 \ldots \bar{r}_A) \, d\bar{r}_1 \, d\bar{r}_2 \ldots d\bar{r}_{i-1} \, d\bar{r}_{i+1} \ldots d\bar{r}_A$$

then

$$V_{mn}(\bar{r}_0) = V_0 \sum_{i=1}^{A} \varrho_i^{mn}(\bar{r}_0) . \tag{7}$$

$\varrho_i^{nn}(\bar{r}_0)$ is the probability density for finding particle i at $\bar{r}_0$ so that $\varrho^n = \sum_{i=1}^{A} \varrho_i^{nn}$ is the density of nucleons in the nucleus. The shape of this function is well known[8]) and is commonly taken to be that of the Woods-Saxon potential.

We may get an approximation to V_{mn} by assuming a model for the nucleus. If the nucleus is considered as a single nucleon bound to a core then we may write

$$\chi_n = \phi(\bar{r}_2, \bar{r}_3, \bar{r}_4, \ldots \bar{r}_A)\, \psi_n(\bar{r}_1)\,.$$

Here ψ_n is the wave function of one particle moving in the average potential of the core. Using this approximation in equation (7) we may see that

$$V_{mn}(\bar{r}_0) = V_0\, \psi_m(\bar{r}_0)\, \psi_n(\bar{r}_0)\,. \tag{8}$$

Returning to the solution of (4) it may be noted that there will be resonances corresponding to the energies at which some of the U_n are bound states while the dominant reaction mechanism will be that of direct interaction. The optical model may be obtained as follows.

Let the zero subscript denote the elastic channel. For the optical model we are interested in solving only for U_0. The equation for U_0 is

$$T\, U_0 + V_{00}\, U_0 = E_0\, U_0 - \sum_{n \neq 0} V_{0n}\, U_n\,.$$

This may also be written as

$$T\, U_0 + V_{00}\, U_0 = E_0\, U_0 - U_0 \left(\frac{\sum_{n \neq 0} V_{0n}\, U_n}{U_0} \right)$$

from which it may be seen that

$$V + i\, W = V_{00} + \frac{\sum_{n \neq 0} V_{0n}\, U_n}{U_0} \tag{9}$$

is a local form of the optical potential. Of course this may be calculated only after the entire system of equations has been solved.

The usual procedure is to try various guesses at this function until the elastic scattering data is fit. It may be possible to learn something about the general shape of this function by a study of the system of coupled equations. This is done in the later section of this paper.

III. Choice of V_0 and a Relation Between Optical Model Parameters

The strength of the zero range potential was chosen by comparison with optical model results. In expression (9) V_{00} should dominate the real part of the optical potential hence we may write

$$V_{00} = V_0 \sum_{i=1}^{A} \varrho_i^{00} = \frac{V_s}{e^{(r-R)/a} + 1}\,.$$

Integrating both sides over all space gives

$$V_0\, A = 4\pi\, V_s\, a^3 \left[\frac{1}{3} \left(\frac{R}{a} \right)^3 + \frac{\pi^2}{3} \frac{R}{a} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\, e^{-n R/a} \right].$$

If $R/a \lesssim 3$ the last sum amounts to less than 1% of the value of this expression and it will be neglected here.

Putting $R = r_0 A^{1/3}$ we obtain

$$V_0 = \frac{4\pi V_s}{3}\left[r_0^3 + \frac{a^2\pi^2 r_0}{A^{2/3}}\right] \equiv V_s f \tag{10a}$$

or

$$V_s = \frac{3 V_0}{4\pi\left(r_0^3 + \frac{a^2\pi^2 r_0}{A^{2/3}}\right)}, \tag{10b}$$

where the identity in (10a) defines $f(r_0, a, A)$ used in Figure 1.

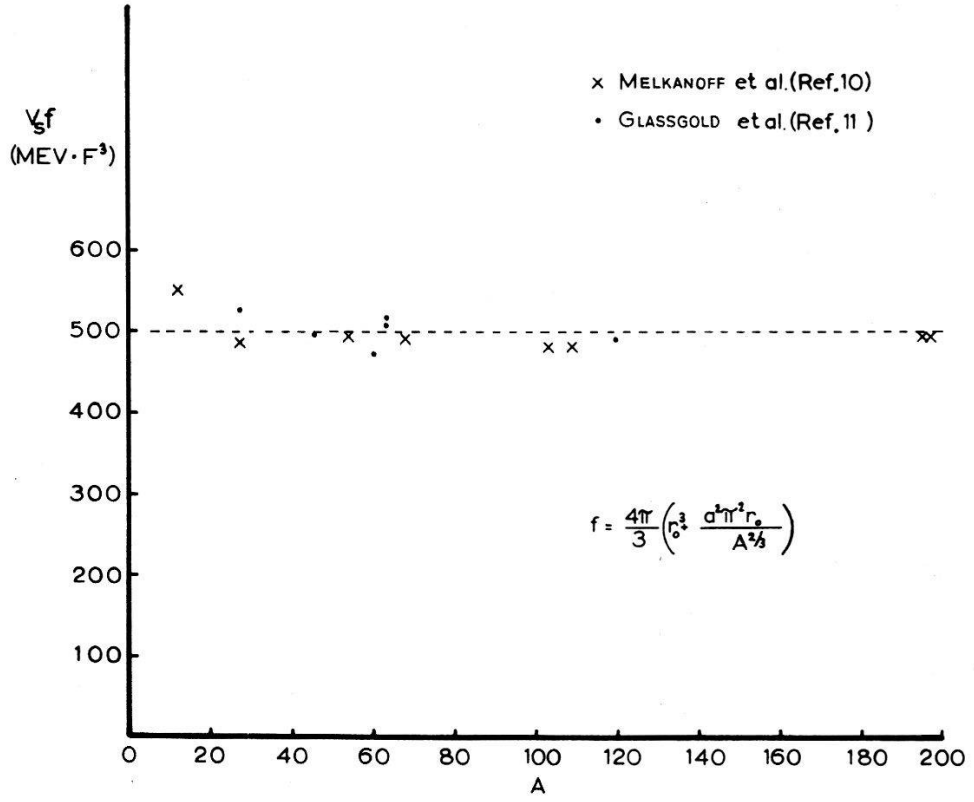

Fig. 1

A plot of $V_s f$ as a function of A. The dotted line shows the value of V_0 chosen

Figure 1 shows V_0 calculated from several optical model fits to data. The dotted line shows the value taken for V_0. Using this, one now has a relation between V_s, r_0, and a. Equation (10b) allows a prediction of V_s essentially the same as that actually used in 7 out of 9 cases and within 10% in the other 2.

IV. A Determination of a Shape Parameter from the Optical Model

If the nucleon-nucleon potential is taken to be of the form

$$V(r) = V'(r) + V^s(r)\,\bar{\sigma}_1 \cdot \bar{\sigma}_2$$

we may still carry through the same procedure to determine the depth of the optical potential. Since the nucleons in the nucleus have on an average as many spin up particles as spin down the second term will tend to cancel itself and for this reason will be neglected in the integral. The same equation (eq. (10a) is obtained but with V_0 defined by

$$V_0 = 4\pi \int_0^\infty V'(r)\, r^2\, dr\,.$$

This may serve to give another parameter to determine the shape of the nucleon-nucleon potential. BLATT and JACKSON[9]) have determined the parameters for three types of wells which will fit low energy scattering data. These are given in the form of singlet and triplet scattering potentials. The deduced value of V_0 is given in the Table for the n-p spin

Table

Integrated Potentials

Well	V_0 (MeV·Fermi³) (from n-p scattering)	V_0 (MeV·Fermi³) (from p-p scattering)
Exponential	− 228	− 217
Yukawa	− 517	− 470
Square Well	−1020	− 977

independent part as well as the integrated value of the proton singlet potential. We may see that this method will clearly distinguish between these three potentials and, at least as regards the wells considered, the Yukawa is clearly indicated.

These values are to be compared with the Optical Model prediction of $V_0 = -500$.

V. Method of Solution

The equations have been solved up to $n = 3$ by numerical methods for inelastic neutron scattering. The exact procedure used is given below.

The U_n were expanded in spherical harmonics.

$$U_n(\bar{r}) = \sum_{l=0}^{\infty} \sqrt{4\pi\,(2\,l+1)}\; i^l\; Y_l^0(\Omega_{\bar{r}})\; U_n^l(r)/r \tag{11}$$

With this form the boundry conditions for $r \to \infty$ may be written

$$U_0^l(r) \to r\, j_l(K_0\, r) + r\, \beta_l^0\, h_l^+(k_0\, r)\,, \qquad n = 0 \tag{12}$$

and

$$U_n^l(r) \to r\, \beta_l^n\, h_l^+(k_n\, r), \qquad n \neq 0\,.$$

The amplitude for scattering from channel n is given by

$$f_n(\theta) = \frac{1}{k_n\, i} \sum_{l=0}^{\infty} (2\,l + 1)\, \beta_l^n\, P_l(\cos\theta)\,. \tag{13}$$

The coupled equations may now be written as

$$\frac{d^2 U_n^l}{dr^2} + v_{nn}\, U_n^l = \phi_n^l \tag{14}$$

where

$$\phi_n^l = -\frac{2\,m}{\hbar^2}\, V_0 \sum_{l',\, j \neq n} U_j^{l'} \int Y_l^0(\Omega)\, \psi_n\, \psi_j\, Y_{l'}^0(\Omega)\, d\Omega \tag{15}$$

and

$$v_{nn} = k_n^2 - \frac{2\,m}{\hbar^2}\, V_{nn}\,.$$

For the case actually calculated it was assumed that ψ_n had no angular dependence so that

$$\phi_n^l = -\frac{2\,m\,V_0\,\psi_n}{\hbar^2} \sum_{j \neq n} U_j^l\, \psi_j\,.$$

It was also assumed that there was no coupling between the excited states. With this assumption we may write

$$\phi_0^l = -\frac{2\,m\,V_0}{\hbar^2}\, \psi_0 \sum_{j \neq 0} \psi_j\, U_j^l\,,$$

$$\phi_n^l = -\frac{2\,m\,V_0}{\hbar^2}\, \psi_0\, U_0^l\, \psi_n\,, \qquad n \neq 0\,.$$

In order to solve (14) it was first assumed that $\phi_0^l = 0$ and the elastic equation was solved with the appropriate boundry conditions (12). The the ϕ_n^l were computed for $n \neq 0$ and the corresponding equations solved by means of a Green's function discussed below. With these solutions ϕ_0^l was computed and the elastic equation solved again. This procedure was continued until convergence was reached. Convergence can easily be judged by the β_l^n which must be computed each time.

Each equation to be solved is of the form

$$\frac{d^2 U}{dr^2} + v\, U = \phi \tag{17}$$

with boundry conditions (12) at infinity and the condition that $U \to 0$ as $r \to 0$. Let $g_1(r)$ and $g_2(r)$ be two linearly independent solutions of

$$\frac{d^2 U}{dr^2} + v\, U = 0\,, \tag{18}$$

such that $g_1(r) \to 0$ as $r \to 0$. Further let W ($W = \text{const.} \neq 0$) be the Wronskian of the two solutions. It may easily be shown by direct substitution that the general solution to (17) which goes to zero at the origin is given by

$$U = g_1 \left(C + \frac{1}{W} \int_0^r g_2 \, \phi \, dr' \right) - \frac{g_2}{W} \int_0^r g_1 \, \phi \, dr' , \tag{19}$$

where C is a constant to be adjusted to fit the boundry conditions at infinity. Here $g_2(r)$ may be chosen to have some convenient asymptotic form.

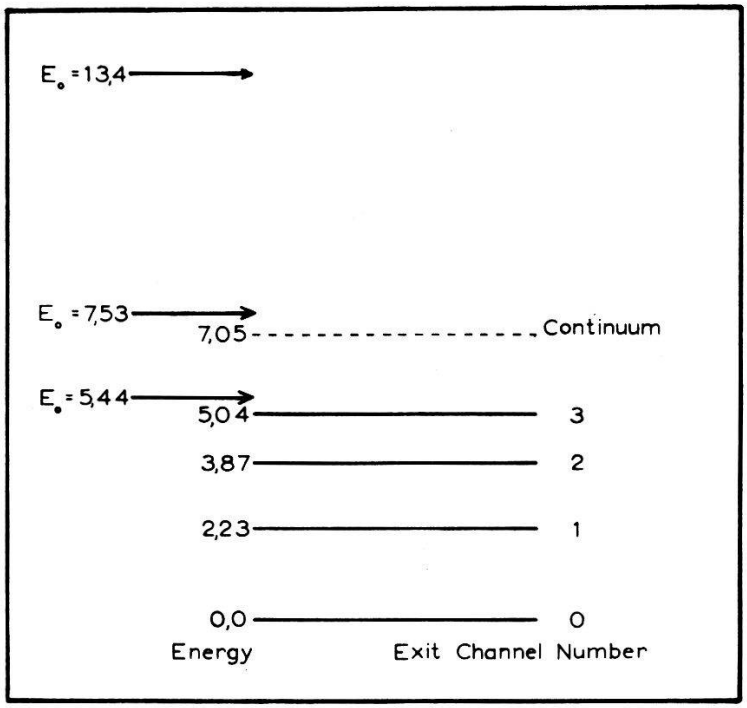

Fig. 2
Energy level diagram for the calculation that was done

Let us define

$$f_j(r) = \frac{1}{W} \int_0^r g_j \, \phi \, dr'$$

and

$$a_j = f_j(\infty) .$$

The procedure is to solve (18) for g_1 and g_2 and compute $f_j(r)$, then from

$$U \underset{r \to \infty}{\to} g_1(r) \, (C + a_2) - g_2(r) \, a_1 = r \, j_l(k\,r) + r \, \beta_l \, h_l^+(k\,r) , \qquad n = 0$$

or

$$U \underset{r \to \infty}{\to} g_1(r) \, (C + a_2) - g_2(r) \, a_1 = r \, \beta_l \, h_l^+(k\,r) , \qquad n \neq 0 \tag{20}$$

the values of C and β_l may be calculated. With C known U may be calculated directly from (19).

For convenience ψ_n was approximated by a function which resembles an $l = 0$ wave function. It was taken to be

$$\psi_n = \frac{N_n}{r} \sin\{(n+1)\,\pi\, e^{-k_n r}\}$$

where N_n was chosen to make

$$\int_0^\infty \psi_n^2\, dv = 1\,.$$

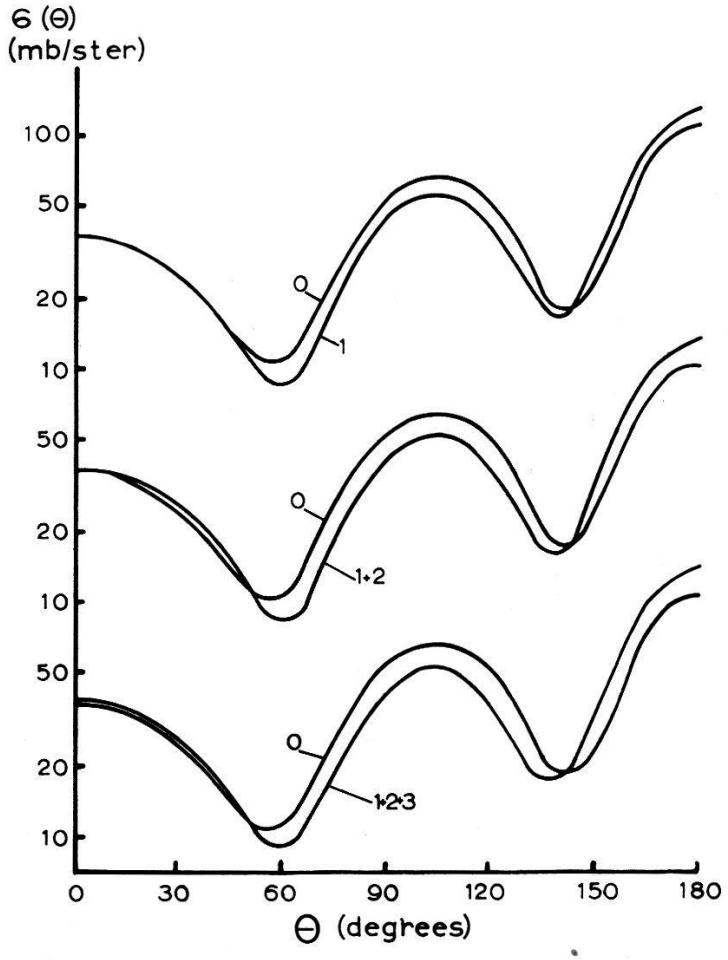

Fig. 3

Elastic cross sections. The curves labeled «0» denote scattering from a real Saxon potential with no coupling to other channels. «1» denotes coupling to one inelastic channel, «1 + 2» denotes coupling to two inelastic channels, and «1 + 2 + 3» denotes coupling to three inelastic channels

VI. Results

Figure 2 shows the level scheme of the target nucleus that was used in the calculation. The arrows at the left indicate the energies for which calculations were done.

The V_{nn} were taken to be the same for all levels. This should be very nearly true. The parameters used in the Saxon Well were $V = 50$ MeV, $R = 3{,}17$, and $a = 0{,}5$ f. V and a are average values and R was chosen to be compatible with the bound state wave functions (20). This last was necessary since these wave functions have no radius adjusting parameter.

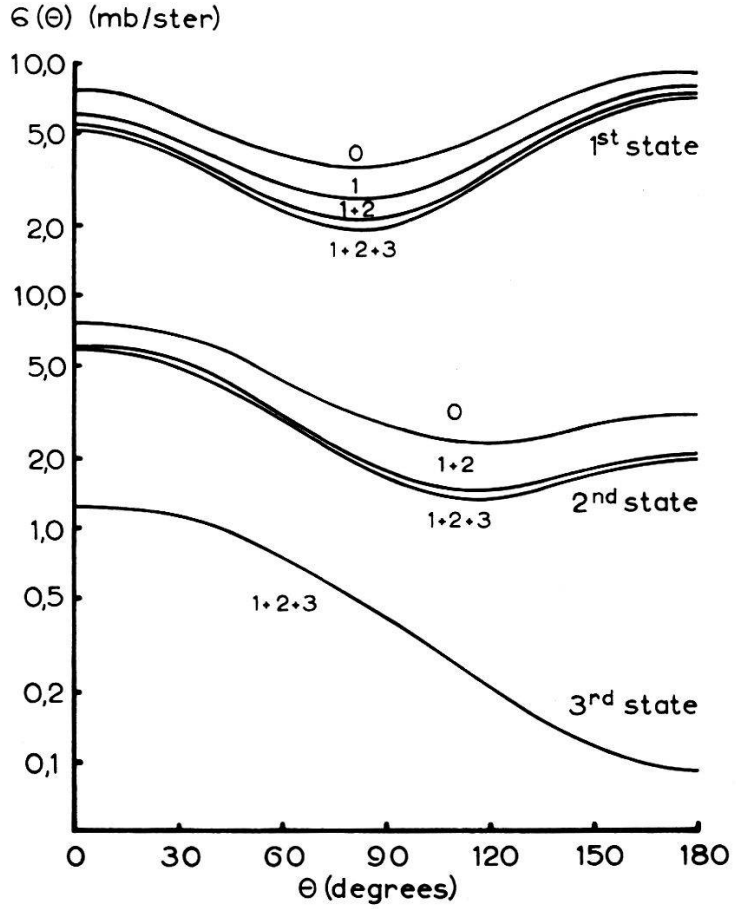

Fig. 4

Inelastic cross sections. The significance of the number «0» is explained in the text. The other numbers have the same significance as in fig. 3

Figure 3 shows the effect of the inelastic channels on the elastic cross section. Except at extreme forward and backward angles they have the same general effect as introducing an imaginary part to the potential. The effect at forward and backward angles is probably due to small alterations in the real part of the potential. There is also a shift in the positions of the minima.

The calculation was arranged so that the cross section could be calculated after the first iteration. This corresponds to the distorted wave Born approximation using a real potential only to distort the waves. This is shown in two cases as the zero curve in figure 4. The rest of the curves

have various numbers of wells coupled. It may be seen that the effect of additional channels is to multiply the cross section by a constant less than 1.

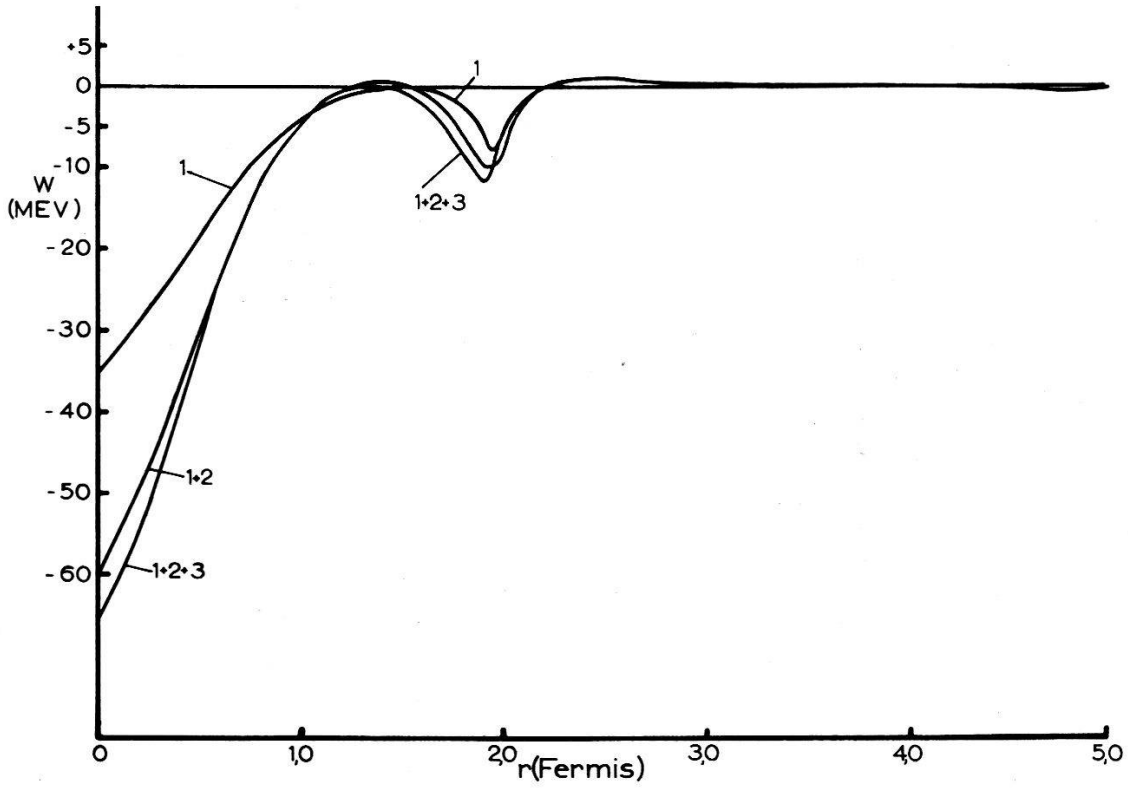

Fig. 5

The imaginary part of the optical potential for various numbers of inelastic channels. The numbers labeling the plots have the same significance as in fig. 3. All curves are for $E_0 = 5{,}4$ MeV, and $l = 0$

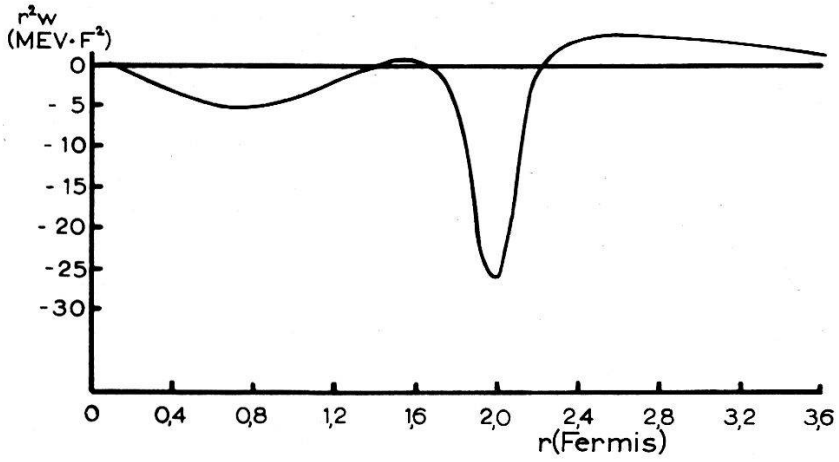

Fig. 6

A plot of r^2W showing the relative importance of absorption from the elastic channel as a function of r

Figure 5 shows the effective imaginary potential as computed from equation (9) for $l = 0$. It is interesting to note that it has a "volume" and

a "surface" part, although the latter comes considerably inside of the surface. The volume portion is not as effective in absorbing particles as is the surface part as may be seen from the plot of r^2W in figure 6. The positive part of r^2W above $r = 2{,}2\ f$ should be disregarded since it comes from a very small contribution to W. This very small contribution is within the error of the calculation which is about 1%. The depth of W depends on V_0^2 and is thus very sensitive to the choice of this parameter. Since the choice of $V_0 = -500\ \mathrm{MeV} \cdot \mathrm{F}^3$ satisfied the relation among V_s, r_0, and a so well, we should expect it to give the right answer here too. We see that it predicts the correct magnitude for the depth of the imaginary potential. It is also interesting to see that W is almost all negative which is necessary physically but is not obvious from the mathematical formalism. Indeed the real part of the second term of expression (9) is oscillatory. The energy dependence of W is shown in figure 7. It is seen that the place where most of the reaction takes place moves toward the center of the nucleus as the energy increases. This may be understood in a quantitative manner as follows.

The solution to the elastic equation in first approximation (i.e. no coupling and square well potential) will be given by

$$U_0^l = r\, A\, j_l(k\, r)\,, \qquad k = \sqrt{\frac{2\,m}{\hbar^2}\,(V_s + E)}\,.$$

Since this appears in the denominator of the expression for W, there will be a singularity at $j_l(k\,r) = 0$, except for $k\,r = 0$ in which case the numerator also goes to zero. When coupling is introduced a small imaginary part is added to U_0^l and W no longer goes to infinity but a maximum remains. For $l = 0$ the prediction

$$R_p = \frac{j\,\pi}{\sqrt{(2\,m/\hbar^2)\,(V_s + E)}}$$

may be compared with figure 7 for $j = 1$. For $E \lesssim 25$ MeV and light nuclei there will be only one such peak inside of the nucleus, the second lying far outside since the wave number becomes much smaller outside of the nucleus. The decaying exponentials appearing in the numerator of the expression for W will make these peaks occurring outside of the nucleus negligible. For low energies but heavy nuclei there will be as many as three peaks inside of the nucleus so that a volume absorption becomes a good approximation.

For protons at low energies the same considerations may be applied except that Coulomb effects must be taken into account. The Coulomb repulsion will cause the first zero to come near the nuclear radius for all nuclei.

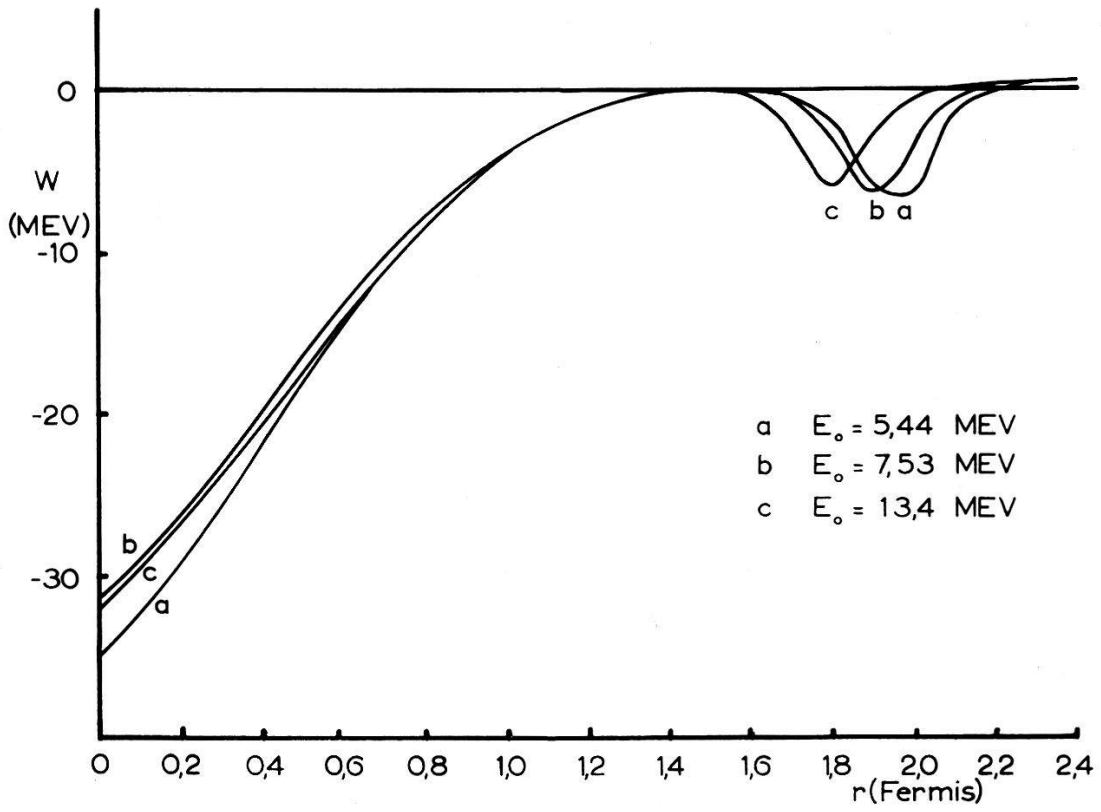

Fig. 7

The imaginary part of the optical potential for various incident energies. All curves were calculated with one inelastic channel coupled

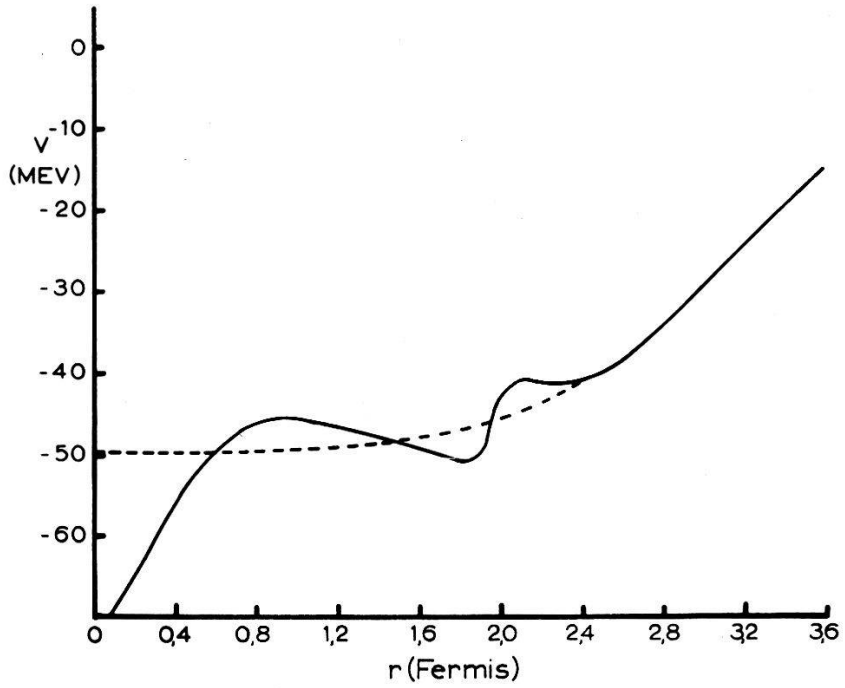

Fig. 8

The real part of the optical potential. The dotted curve shows the Woods-Saxon potential used in the calculation

The real part of the optical potential is shown in figure 8. It may be seen that the contribution of the coupling well potential to the saxon potential is indeed small. Since it is oscillatory its integral will be still smaller ($< 1\%$ of the integral of the Saxon potential in this case). Thus we are justified in droping this term to get eq. (10a).

VII. Conclusions

We have seen that this formalism is capable of describing the general features of the optical model and that it is self consistent (e.g. the correct magnitude of the imaginary part of the optical potential is consistent with the correct magnitude of the real part). A relation has been derived between three optical model parameters which seems to hold very well. This means that R and a are the only parameters left and these describe the average properties of the nuclear wave function.

The final evaluation of the utility of this method as a means of predicting nuclear cross sections awaits more detailed calculations in which some of the approximations used here are relaxed and comparison with data is made.

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