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A Generalization of Borchers Theorem

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Abstract. BORCHERS' theorem on causal dependence of rings of operators in quantum field theory is generalized by a new method of proof, based on the uniqueness theorem for hyperbolic partial differential equations.

1. Introduction

Recently, the rings of operators associated with fields in a restricted region of space-time have attracted some attention in axiomatic quantum field theory. One way of defining such an object, in the framework of the Wightman axioms, is to consider the set $R'(\mathfrak{B})$ of all bounded operators C, satisfying

$$(C\,\Psi_1, A(f)\,\Psi_2) = (A(f)^*\,\Psi_1, C^*\,\Psi_2) \tag{1.1}$$

where $\mathfrak{B}$ is some open set in space-time, $A(f)$ is the smeared-out field, the support of the test function $f(x)$ is restricted to $\mathfrak{B}$, Ψ_1, and Ψ_2 are arbitrary vectors in the common dense domain D of all the smeared-out fields and their adjoints and the equation (1.1) is to be satisfied by C for all such f, Ψ_1 and Ψ_2. The von Neumann algebra $R(\mathfrak{B})$ associated with the region $\mathfrak{B}$ is then defined as the commutant of $R'(\mathfrak{B})$.[1][2]

BORCHERS[3]) has given a theorem of the type $R(\mathfrak{B}) = R(\hat{\mathfrak{B}})$, where $\mathfrak{B}$ is a timelike cylinder, $\{x;\, |\, x^0\, | < T,\, |\, \boldsymbol{x}\, | < \varepsilon\}$, and $\hat{\mathfrak{B}}$ is «causally dependent» region of $\mathfrak{B}$, defined by $\hat{\mathfrak{B}} = \{(t, \boldsymbol{x});\, |\, t\, | < T,\, |\, \boldsymbol{x}\, | < \varepsilon + T - |\, t\, |\}$. The main tool of his proof is the technique of analytic completion.

In this paper, we give an alternative proof of Borchers' theorem, based on the uniqueness theorem for hyperbolic partial differential equation. This technique enables us to generalize Borchers' theorem. For example, if

$$\mathfrak{B} = \{x;\, |\, x^0\, | < |\, x^1\, | + \varepsilon,\, |\, x^2\, | + |\, x^3\, | < \delta\} \tag{1.2}$$

then $R(\mathfrak{B}) = R(M)$ where M is the whole Minkowski space.

We discuss the generalized definition and some properties of $\hat{\mathfrak{B}}$ for an open set $\mathfrak{B}$ in section 2 and then prove the main theorem, $R(\mathfrak{B}) = R(\hat{\mathfrak{B}})$ in section 3. Some examples will be discussed in section 4. The main part of the proof in section 3 is similar to that used in Garding's proof of the Jost-Lehmann-Dyson representation.[4])

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2. Definition and properties of $\hat{\mathfrak{B}}$

Consider the space of points (x, s) with the metric

$$(x, x) - s^2 = (x^0)^2 - \boldsymbol{x}^2 - s^2 .$$

For any time like segment T in (x, s) space, open at both ends, $K(T)$ is defined as the open double light cone in (x, s) space spanned by T. For any sphere S on a space-like hyperplane in (x, s) space, open at its surface, $K(S)$ is defined as the open double light cone with the base S. For example, if

$$T = \{(x, s);\ | x^0 | < t, \boldsymbol{x} = 0, s = 0\},$$

then

$$K(T) = \{(x, s);\ | x^0 | + (\boldsymbol{x}^2 + s^2)^{1/2} < t\}.$$

If

$$S = \{(x, s);\ \boldsymbol{x}^2 + s^2 < r^2, x^0 = 0\},$$

then

$$K(S) = \{(x, s);\ | x^0 | + (\boldsymbol{x}^2 + s^2)^{1/2} < r\}.$$

For any set B, $K(B)$ is the smallest set containing B such that $T \subset K(B)$ implies $K(T) \subset K(B)$ and $S \subset K(B)$ implies $K(S) \subset K(B)$, i.e. $K(B)$ is the intersection of all sets satisfying this condition. For any set $\mathfrak{B}$ in the Minkowski space of x, we define $\varphi\, \mathfrak{B} \equiv \{(x, 0);\ x \in \mathfrak{B}\}$ and for any set B in (x, s) space, we define $\varphi^{-1} B \equiv \{x;\ (x, 0) \in B\}$. Finally, for any set $\mathfrak{B}$ in x space

$$\hat{\mathfrak{B}} = \varphi^{-1} K(\varphi\, \mathfrak{B}) . \tag{2.1}$$

Lemma 1. If $\mathfrak{B}$ is an open set in x space, then $K(\varphi\, \mathfrak{B})$ is an open set in (x, s) space.

Proof: Let $K_\alpha(\varphi\, \mathfrak{B})$ be a set in (x, s) space, defined inductively for any ordinal number α by

$$K_1(\varphi\, \mathfrak{B}) = \varphi\, \mathfrak{B}$$

$$K_\alpha(\varphi\, \mathfrak{B}) = \bigcup_{\alpha > \alpha'} \left\{ \left[\bigcup_{T \subset K_{\alpha'}(\mathfrak{B})} K(T) \right] \cup \left[\bigcup_{S \subset K_{\alpha'}(\mathfrak{B})} K(S) \right] \right\} \tag{2.2}$$

where T and S are as described before. For $\alpha > 1$, $K_\alpha(\varphi\, \mathfrak{B})$ is open, because it is a union of open sets. Take any point $(x, s) \in K_\alpha(\varphi\, \mathfrak{B})$. Since $K_\alpha(\varphi\, \mathfrak{B})$ is open for $\alpha > 1$, and $K_1(\varphi\, \mathfrak{B})$ is a relatively open set on the $s = 0$ plane, there always exists $T \subset K_\alpha(\varphi\, \mathfrak{B})$ such that $(x, s) \in T$. Since $T \subset K(T)$, we have $K_{\alpha'}(\varphi\, \mathfrak{B}) \subset K_\alpha(\varphi\, \mathfrak{B})$ if $\alpha' \leq \alpha$. Furthermore if $K_{\alpha'}(\varphi\, \mathfrak{B}) = K_\alpha(\varphi\, \mathfrak{B})$ for some $\alpha > \alpha'$, then the same is true for any $\alpha > \alpha'$. Hence*) it has the limit $K_\alpha(\varphi\, \mathfrak{B}) \equiv \bigcup_\alpha K_\alpha(\varphi\, \mathfrak{B})$. Since $K_0(\varphi\, \mathfrak{B})$ is the same as any $K_\alpha(\varphi\, \mathfrak{B})$ for sufficiently large ordinal number α, $K_0(\varphi\, \mathfrak{B}) \supset K_1(\varphi\, \mathfrak{B}) = \varphi(\mathfrak{B})$, $T \subset K_0(\varphi\, \mathfrak{B})$ implies $K(T) \subset K_0(\varphi\, \mathfrak{B})$ and $S \subset K_0(\varphi\, \mathfrak{B})$ implies $K(S) \subset K_0(\varphi\, \mathfrak{B})$. It is also obvious that any set having this property contains any $K_\alpha(\varphi\, \mathfrak{B})$ and hence $K_0(\varphi\, \mathfrak{B}) = K(\varphi\, \mathfrak{B})$. Thus $K(\varphi\, \mathfrak{B})$ is open.

Lemma 2. If $\mathfrak{B}_n$ are open sets in x-space, $\mathfrak{B}_n \supset \mathfrak{B}_{n'}$ for $n > n'$, and $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ then $\bigcup_n K(\varphi\, \mathfrak{B}_n) = K(\varphi\, \mathfrak{B})$.

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*) It has been pointed out by Dr. RUELLE that $K_\omega(\varphi\, \mathfrak{B})$ is already the limit, namely

$$K(\varphi\, \mathfrak{B}) = \bigcup_{n=1}^{\infty} K_n(\varphi \mathfrak{B}) .$$

Proof: Let $K_\alpha(\varphi\,\mathfrak{B})$ be as in the proof of Lemma 1. It is enough to prove $\bigcup_n K_\alpha(\varphi\,\mathfrak{B}_n) = K_\alpha(\varphi\,\mathfrak{B})$ assuming $\bigcup_n K_{\alpha'}(\varphi\,\mathfrak{B}_n) = K_{\alpha'}(\varphi\,\mathfrak{B})$ for any $\alpha' < \alpha$. $K_{\alpha'}(\varphi\,\mathfrak{B}) \supset K_{\alpha'}(\varphi\,\mathfrak{B}_n)$ obviously implies $K_\alpha(\varphi\,\mathfrak{B}) \supset K_\alpha(\varphi\,\mathfrak{B}_n)$. Now take any $T \subset K_{\alpha'}(\varphi\,\mathfrak{B})$. Then there exists a sequence T_μ, $\mu = 1, 2, \ldots$ such that $T_\mu \subset T_{\mu'}$ for $\mu < \mu'$, $\bar{T}_\mu \subset T$ and $\bigcup_n K(T_\mu) = K(T)$. Since $\bar{T}_\mu$ is compact and is covered by the union of open sets $K_{\alpha'}(\varphi\,\mathfrak{B}_n)$ and since $K_{\alpha'}(\varphi\,\mathfrak{B}_n) \supset K_{\alpha'}(\varphi\,\mathfrak{B}_m)$ for $n > m$, there exists an $n(\mu)$ such that $\bar{T}_\mu \subset K_{\alpha'}(\varphi\,\mathfrak{B}_{n(\mu)})$. Hence $K(T_\mu) \subset K_\alpha(\varphi\,\mathfrak{B}_{n(\mu)})$. This implies $K(T) = \bigcup_\mu K(T_\mu) \subset \bigcup_n K_\alpha(\varphi\,\mathfrak{B}_n)$. In a similar manner, $S \subset K_{\alpha'}(\varphi\,\mathfrak{B})$ implies $K(S) \subset \bigcup_n K_\alpha(\varphi\,\mathfrak{B}_n)$. Therefore $K_\alpha(\varphi\,\mathfrak{B}) \subset \bigcup_n K_\alpha(\varphi\,\mathfrak{B}_n)$.

It should be noted here that $K(\varphi\,\mathfrak{B})$ is independent of whether partially infinite T and S are allowed in its definition or not, because of the above Lemma.

We now introduce a further notation $\mathfrak{B}_\lambda$ for any open set $\mathfrak{B}$ in x-space by

$$\mathfrak{B}_\lambda = \{(x^0, \lambda^{-1}\,\boldsymbol{x});\, x \in \mathfrak{B}\}, \quad \lambda > 0\,. \tag{2.3}$$

Lemma 3. For any open set $\mathfrak{B}$ in x-space,

$$\bigcup_{0<\lambda<1} (\hat{\mathfrak{B}}_\lambda)_{\lambda^{-1}} \supset \hat{\mathfrak{B}}\,. \tag{2.4}$$

Proof: We define, for any set K in (x, s) space,

$$p(\lambda)\,K = \{(x^0, \lambda^{-1}\,\boldsymbol{x}, \lambda^{-1}\,s);\, (x, s) \in K\}$$

and we prove, by induction

$$\bigcup_{0<\lambda<1} p(\lambda^{-1})\,K_\alpha(p(\lambda)\,\varphi\,\mathfrak{B}) \supset K_\alpha(\varphi\,\mathfrak{B})\,.$$

Any finite T can be written as

$$T = \{(x, s);\, x^\mu = a^\mu + \varrho\,b^\mu,\, s = a^4 + \varrho\,b^4,\, |\varrho| < 1\}$$

where $b^0 > \sqrt{\boldsymbol{b}^2 + (b^4)^2}$. Using this explicit form, we can easily verify that $p(\lambda)\,T$ is time-like for sufficiently small $1-\lambda$, and $\bigcup_{0<\lambda<1} p(\lambda^{-1})\,K(p(\lambda)\,T) = K(T)$. (For space like T, $K(T) = T$). Similarly we have $\bigcup_{0<\lambda<1} p(\lambda^{-1})\,K(p(\lambda)\,S) \supset K(S)$ for

$$S = \{(x, s);\, x^0 = a^0 + \boldsymbol{b}\,\boldsymbol{x} + b^4\,s;\, (\boldsymbol{x} - \boldsymbol{a})^2 + (s - a^4)^2 < r\}, \quad \boldsymbol{b}^2 + (b^4)^2 < 1\,.$$

We now use the definition

$$\bigcup_\lambda p(\lambda^{-1})\,K_\alpha(p(\lambda)\,\varphi\,\mathfrak{B}) = \bigcup_\lambda p(\lambda^{-1}) \bigcup_{\alpha'<\alpha} \Big[\bigcup_{T,\,S \subset K_{\alpha'}(p(\lambda)\varphi\mathfrak{B})} (K(T) \cup K(S))\Big]\,.$$

Since $K_{\alpha'}(p(\lambda)\,\varphi\,\mathfrak{B}) \supset p(\lambda)\,K_{\alpha'}(\varphi\,\mathfrak{B})$, $K(p(\lambda)\,T)$ and $K(p(\lambda)\,S)$, for any $T, S \subset K_{\alpha'}(\varphi\,\mathfrak{B})$, are contained in the last union. Hence

$$\bigcup_\lambda p(\lambda^{-1})\,K_\alpha(p(\lambda)\,\varphi\,\mathfrak{B})$$
$$\supset \bigcup_{\alpha'<\alpha} \cdot \bigcup_{T,\,S \subset K_{\alpha'}(\varphi\mathfrak{B})} \Big\{\bigcup_\lambda [p(\lambda^{-1})\,K(p(\lambda)\,T) \cup p(\lambda^{-1})\,K(p(\lambda)\,S)]\Big\} = K_\alpha(\varphi\,\mathfrak{B})\,.$$

It is also easy to prove the equality in the formula (2.4) but this is not needed in the following.

3. Proof of the Main Theorem

(a) Uniqueness theorem for solutions of the wave equations.

We quote some classical theorems, which will be used in the proof.

Lemma 4. Let $u(t, \boldsymbol{y})$ be twice continuously differentiable solution of

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial y^i}\right)^2 u .$$

Let G be a closed set on a smooth space-like surface H and $C(T)$ be the set of points causally dependent*) on G. If u and all its first derivatives vanish on G, then $u = 0$ in $C(G)$.

The proof will be found, for example, in reference 5 p.p. 379–380.

Lemma 5. (Asgeirsson) Let $u(\boldsymbol{y}, \boldsymbol{z})$ be twice continuously differentiable solution of

$$\sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i}\right)^2 u = \sum_{i=1}^{n} \left(\frac{\partial}{\partial y^i}\right)^2 u .$$

Then
$$\int_{(e,e)=1} \cdots \int u\,(y + r\,e;\, z)\, d\,\Omega(e) = \int_{(e,e)=1} \cdots \int u\,(y;\, z + r\,e)\, d\,\Omega(e)$$

where $d\Omega(e)$ is an invariant measure on the sphere $(e, e) = 1$.

The proof is to be found in reference 5. p.p. 417–419. It is important in our application that the proofs of Lemma 4 and 5 can be carried out with u in a restricted region and the behaviour of u at infinity does not matter.

Lemma 6. Let $u(t, \boldsymbol{y})$ be an infinitely continuously differentiable solution of

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial y^i}\right)^2 u .$$

Let G be a segment of a time like straight line and $C(G)$ be the double light cone spanned by G. If u and all its derivatives vanish on G then $u = 0$ on $C(G)$.

The proof will be found in reference 4, pp. 294–296. (Substitute Lemma 5 into the corresponding statement for tempered distribution there.)

(b) We now present the main part of the proof as two Lemmas.

Lemma 7. Let $\hat{f}(p)$ be a tempered distribution, such that**)

$$\text{Supp. } \hat{f}(p) \subset \overline{V}_+ \cup \overline{V}_- \cup E$$

where E is a compact set. Furthermore, the Fourier transform

$$f(x) = \int \hat{f}(p)\, e^{i(p,x)}\, d^4 p$$

vanishes on an open set $\mathfrak{B}$. Then $f(x)$ vanishes in $\hat{\mathfrak{B}}$.

Proof: Let $h(t) \in \mathscr{S}_{(t)}$ such that $h(t) = 1$ if $t \geqslant 0$ and $h(t) = 0$ if $t \leqslant -1$. Let $(e, e) = 1$ and let $\chi(t) \in \mathscr{D}_{(t)}$.

*) Let $\overline{V}_\pm = \{(t, \boldsymbol{y});\, |\,\boldsymbol{y}\,| \leqq \pm\, t\}$. Then $C(G) = C_+ \cup C_-$, $C_\pm = \{(t, \boldsymbol{y});\, ((t, \boldsymbol{y}) + \overline{V}_\mp) \cap H \subset G\}$.

**) Supp. is the abbreviation for 'the support of'.

Define
$$F_{\chi e}(x, s) = \int \psi_{\chi e}(x, s; p)\, \hat{f}(p)\, d^4 p$$
$$\psi_{\chi e}(x, s; p) = \tilde{\chi}((p, e))\, h\,((p, p) + M)\, e^{i(p, x)} \cos\sqrt{(p, p)}\, s$$
$$\tilde{\chi}(\alpha) = \int_{-\infty}^{\infty} \chi(t)\, e^{i\alpha t}\, dt$$
where M is taken such that $p \in E$ implies $(p, p) \geqslant -M$. Since $(p, p) \geqslant -M$ and $|(p, e)| \leqslant k_0$ define a compact set in p-space, and since $e^{i(p, x)} \cos\sqrt{(p, p)}\, s$ is holomorphic for (x, s) in a compact set and for $(p, p) \geqslant -M$, $\psi_{\chi e}(x, s; p)$ as well as all its derivatives with respect to x and s belong to $\mathcal{S}_{(p)}$ and all its derivatives with respect to x and s exist in the topology of $\mathcal{S}_{(p)}$. Hence $F_{\chi e}$ is an infinitely continuously differentiable function of (x, s).

It is easy to verify the following equations,
$$\left(\square_x - \left(\frac{\partial}{\partial s}\right)^2\right) F_{\chi e}(x, s) = 0$$
$$\left(\frac{\partial}{\partial s}\right)^n F_{\chi e}(x, 0) = \begin{cases} 0 & n\text{: odd} \\ \square_x^{n/2} F_{\chi e}(x, 0) & n\text{: even}. \end{cases}$$
Furthermore, for any $g \in \mathcal{D}_{(x)}$
$$\int g(x)\, F_{\chi e}(x, 0)\, dx = \int g_{\chi e}(x)\, f(x)\, dx$$
where
$$g_{\chi e}(x) = \int_{-\infty}^{\infty} g(x - e\, t)\, \chi(t)\, dt.$$
Hence, if $\overline{\mathfrak{B}_1 - (\text{supp. } \chi)\, e} \subset \mathfrak{B}$, then
$$\left(\frac{\partial}{\partial s}\right)^n F_{\chi e}(x, 0) = 0 \ \text{ if } \ x \in \mathfrak{B}_1, \ \ n = 0, 1, \ldots .$$
By Lemma 4 and Lemma 6
$$F_{\chi e}(x, s) = 0 \ \text{ if } \ (x, s) \in K(\varphi\, \mathfrak{B}_1).$$
Hence, if $\mathfrak{B}_2 + (\text{Supp. } \chi)\, e \subset \varphi^{-1} K(\varphi\, \mathfrak{B}_1)$,
$$f(x) = 0 \ \text{ if } \ x \in \mathfrak{B}_2.$$
Since the support of χ can be as small as one likes,
$$f(x) = 0 \ \text{ if } \ x \in \varphi^{-1} K(\varphi\, \mathfrak{B}).$$
(Here we use Lemma 2.)

Lemma 8. Let $\tilde{f}(p)$ be in $\mathcal{S}'_{(p)}$ such that
$$\text{Supp. } \tilde{f}(p) \subset (-p_a + \overline{V}_+) \cup (p_b + \overline{V}_-)$$
where $p_a, p_b \in V_+$. If
$$f(x) = \int \tilde{f}(p)\, e^{i(p, x)}\, d^4 p$$
vanishes in an open set $\mathfrak{B}$, then $f(x) = 0$ in $\hat{\mathfrak{B}}$.

Proof: We define

$$\tilde{f}_\lambda(g) = \tilde{f}(g^0, \lambda^{-1}\,\boldsymbol{g}), \quad f_\lambda(x) = \lambda^3 f(x^0, \lambda\,\boldsymbol{x})$$

where $0 < \lambda < 1$. Now $(g, g) \geqslant 0$ corresponds to $|g^0| \geqslant \lambda\, |\lambda^{-1}\,\boldsymbol{g}|$ and hence Supp. $\tilde{f}_\lambda(g) \subset \overline{V}_+ \cup \overline{V}_- \cup E$ for some compact E. By Lemma 7 $f_\lambda(x) = 0$ if $x \in (\hat{\mathfrak{B}}_\lambda)$. Hence $f(x) = 0$ if $x \in (\hat{\mathfrak{B}}_\lambda)_{\lambda^{-1}}$.

By Lemma 3, we have the Lemma 8.

(c) Assumptions on the theory.

We now list those parts of the Wightman axioms which are necessary for our theorem. We assume the existence of the unitary representation $T(a)$ of the translation group. The corresponding spectral measure E is defined by

$$T(a) = \int e^{i(p,a)}\, dE(p)\,.$$

The support of E is assumed to be in the forward light cone $\overline{V}_+$ and there exists a unique translationally invariant state Ω. The fields $A_i(x)$ are assumed to be tempered operator valued distributions, defined on a common dense domain D. (The adjoint of each $A_i(x)$, restricted to the domain D, is also to be considered as a field.) D consists of a finite linear combination of vectors of the type $A_1(f_1)\ldots A_n(f_n)\,\Omega$. Fields have the translation property $T(a)\,A_i(x)\,T(-a) = A_i\,(x + a)$ and commute with each other at space-like separation.

It is known[6]) that the vector

$$[A,\ldots A_n](\varphi)\,\Phi = \int\cdots\int A_1(x_1)\,\ldots\,A_n(x_n)\,\varphi\,(x_1\ldots x_n)\,dx_1\ldots dx_n\,\Phi$$

can be defined by linearity and continuity from the vector with the product function $\varphi(x_1\ldots x_n) = \varphi_1(x_1)\ldots\varphi_n(x_n)$ and that this vector is strongly continuous with the topology of $\mathcal{S}$ for φ.

Now consider the partition of unity $\check{\varphi}_n(p)$, such that each $\check{\varphi}_n(p)$ is in $\mathcal{D}$ and $\sum_{n=1}^{\infty} \check{\varphi}_n(p) = 1$. The convergence shall be such that for each $\check{\varphi} \in \mathcal{S}$, $\sum_{n=1}^{\infty} \check{\varphi}(p)\,\check{\varphi}_n(p) = \check{\varphi}(p)$ converges in $\mathcal{S}$.

We define the operator

$$\check{\varphi}_n(P) = \int \check{\varphi}_n(p)\, dE(p)\,.$$

Then,

$$\check{\varphi}_n(P)\,[A_1\ldots A_n]\,(\varphi)\,\Omega = [A_1\ldots A_n]\,(\varphi')\,\Omega\,,$$

$$\tilde{\varphi}'(p_1\ldots p_n) = \tilde{\varphi}(p_1\ldots p_n)\,\check{\varphi}_n\,(p_1 + \ldots + p_n)$$

where $\tilde{\varphi}'$ and $\tilde{\varphi}$ denotes the Fourier transform of φ' and φ.

Hence for any vector Φ in D, and $f \in \mathcal{S}$, we have

$$\sum_{\alpha=1}^{\infty} \varphi_n(P)\,\Phi = \Phi, \tag{3.1}$$

$$\sum_{n=1}^{\infty} A(f)\,\varphi_n(P)\,\Phi = A(f)\,\Phi \tag{3.2}$$

where the sum converges in the strong topology of Hilbert space vectors.

(d) Main Theorem

Theorem: If $\mathfrak{B}$ is any open set in x-space,

$$R(\mathfrak{B}) = R(\hat{\mathfrak{B}})$$

where $R(\mathfrak{B})$ is defined in the introduction and $\hat{\mathfrak{B}}$ is defined at the beginning of section 2.

Proof: For a bounded operator C, the field A, and vectors $\Psi_1, \Psi_2 \in D$ we define the tempered distribution

$$f(x) = (C\,\Phi_1, A(x)\,\Phi_2) - (A(x)^*\,\Phi_1, C^*\,\Phi_2)\,.$$

It is enough to prove that if $f(x) = 0$ in $x \in \mathfrak{B}$ for any choice of Ψ_1, Ψ_2 from D, then $f(x) = 0$ for $x \in \hat{\mathfrak{B}}$. By using (3.1) and (3.2), we have

$$f(x) = \sum_{n,\,m} f_{n,\,m}(x)$$

$$f_{n,\,m}(x) = \big(C\,\varphi_n(P)\,\Phi_1, A(x)\,\varphi_m(P)\,\Phi_2\big) - \big(A(x)^*\,\varphi_n(P)\,\Phi_1, C^*\,\varphi_m(P)\,\Phi_2\big)\,.$$

Hence, it is enough to prove that $f_{n\,m}(x) = 0$ for $x \in \mathfrak{B}$ implies $f_{n\,m}(x) = 0$ for $x \in \hat{\mathfrak{B}}$.

The Fourier transform of $f_{n\,m}$ has its support in $(\overline{V}_+ - \text{Supp.}\,\varphi_m) \cup (\overline{V}_- + \text{Supp.}\,\varphi_n)$, where Supp. φ_n and Supp. φ_m are compact. Hence there exists sufficiently large p_a and p_b such that the support of the Fourier transform of $f_{n\,m}$ is in $(\overline{V}_+ - p_a) \cup (\overline{V}_- + p_b)$. Hence, by Lemma 8, $f_{n\,m}(x) = 0$ for $x \in \mathfrak{B}$ implies $f_{n\,m}(x) = 0$ for $x \in \hat{\mathfrak{B}}$.

4. Example and Discussion

If $\mathfrak{B}$ is the timelike cylinder $\{x;\, |\, x^0\,| < T,\, |\,\boldsymbol{x}\,| < \varepsilon\}$, then $\hat{\mathfrak{B}}$ is the double cone spanned by $\mathfrak{B}$ and hence Borchers' theorem is a special case of our main theorem. The following example illustrates the case, where our theorem is stronger than Borchers' theorem.

Example: Consider $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$ where

$$\mathfrak{B}_i = \{x;\, |\,\boldsymbol{x} - \boldsymbol{a}_i\,| + |\,x^0\,| < r_i\},\quad |\,\boldsymbol{a}_1 - \boldsymbol{a}_2\,| < r_1 + r_2\,.$$

$K_2(\varphi\,\mathfrak{B})$ is obviously given by $K_2(\varphi\,\mathfrak{B}) = K_2(\varphi\,\mathfrak{B}_1) \cup K_2(\varphi\,\mathfrak{B}_2)$ where $K_2(\varphi\,\mathfrak{B}_i)$ are the double light cone in (x, s) space spanned by two vertices $(r_i, \boldsymbol{a}_i, 0)$ and $(-\,r_i, \boldsymbol{a}_i, 0)$. Then it is easy to see that $K_3(\varphi\,\mathfrak{B})$ is given by the causally dependent region of the union of two spheres

$$S_i = \{(t, s);\, x^0 = 0,\, |\,\boldsymbol{x} - \boldsymbol{a}\,|^2 + s^2 \leqslant r_i^2\}$$

on $x^0 = 0$ plane, namely

$$K_3(\varphi\,\mathfrak{B}) = C_+\,(S_1 \cup S_2) \cup C_-\,(S_1 \cup S_2)$$

$$C_\pm(S_1 \cup S_2) = \{(x, s);\, \big((x, s) + \overline{V}_\mp\big) \cap \{x^0 = 0\} \subset S_1 \cup S_2\}\,.$$

If one writes the nearest distance between a point P in $S_1 \cup S_2$ and the boundary of $S_1 \cup S_2$ on $x^0 = 0$ plane, by $d_s(p)$, then $C_\pm(S_1 \cup S_2)$ is bounded by two surfaces $x^0 = d_s(P)$ and $x^0 = -\,d_s(P)$. No further extension is possible and $K(\varphi\,\mathfrak{B}) = K_3(\varphi\,\mathfrak{B})$.

It is rather easy to see that $\hat{\mathfrak{B}} = \varphi^{-1} K(\varphi \mathfrak{B})$ is given by $C(\Sigma_1 \cup \Sigma_2)$ where Σ_i is the sphere with center $\boldsymbol{a}_i$ and radius r_i on $x^0 = 0$ plane, C is the same as in Lemma 4. $\hat{\mathfrak{B}}$ is bounded by a part of the original double light cones and the surface of the form

$$(x_{\|} - \alpha)^2 + (x_{\perp} - \beta)^2 = (x^0)^2$$

where

$$x_{\|} = (\boldsymbol{x} - \boldsymbol{a}_1)(\boldsymbol{a}_2 - \boldsymbol{a}_1)/d, \quad d = |\boldsymbol{a}_2 - \boldsymbol{a}_1|, \quad x_{\perp} = ((\boldsymbol{x} - \boldsymbol{a}_1)^2 - x_{\|}^2)^{1/2},$$

$$\alpha = (d^2 + r_1^2 - r_2^2)/2d, \quad \beta = (-\lambda(d^2 r_1^2 r_2^2))^{1/2}/2d,$$

$$\lambda(a\,b\,c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca.$$

The example of (1.2) is equivalent, due to the Borchers theorem, to the limiting case of this example as the radii r_i and centers a_i tends to infinity.

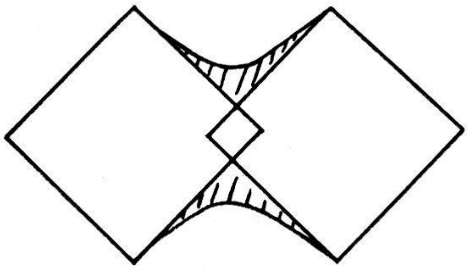

Fig. Example

Our main theorem is not of the most general form. For example if $\mathfrak{B}$ is the union of the set $\{x; |x^0| < |x^1|\}$ and some other open sets, there is a possibility of extending $\hat{\mathfrak{B}}$ farther than in our theorem.

More generally, as can be seen from the proof, if one has any theorem of the type of Lemma 4 and 6 about the vanishing region of the solution of the wave equation, one has the corresponding generalization of our main theorem.

For an example, if l is a snake-like line, connecting two mutually time like points P_1 and P_2 but everywhere spacelike, and $\mathfrak{B}$ is a tube around l, of a small diameter, then it is rather likely that any solution of the wave equation vanishing in $\mathfrak{B}$ might always vanish in the double light cone spanned by P_1 and P_2. If such a conjecture turns out to be true, then we immediately have the corresponding theorem for $R(\mathfrak{B})$.

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