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The Lee Model as Scattering System

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1. Introduction

The mathematical properties of a scattering system were first discussed in two papers[1]) by JAUCH. There has since been a continued interest in obtaining conditions on the Hamiltonian sufficient to ensure the existence of the wave operator[2]) Ω formally given by $\Omega_{\pm} = \lim e^{-iHt} e^{+iH_0t}$. The existence of the limit is essential for a mathematically satisfactory treatment of a scattering problem. However, most of the results obtained so far[3]) refer to the case of potential scattering. In this paper the mathematical structure of the Lee model as scattering system will be investigated. The Lee model is a simple field theory, exactly soluble[4]) in one of its invariant subspaces. In order to establish the operator limit it is essential to have a topology, i. e. we shall have to avoid the use of non-normalizable or improper eigenfunctions. The first two chapters are devoted to setting up the mathematical framework and obtaining the solution of the eigenvalue problem in terms of a family of projection operators, rather than improper eigenstates. As examples of this kind are rare in the literature, this is of some interest in itself. Next there is a brief general discussion on the asymptotic limit of the time development operator $e^{-iH_0t} e^{iHt}$ including a simple but useful theorem which provides a necessary and sufficient condition for the existence of the strong limit in terms of the weak limit. The following chapter contains a detailed discussion of the asymptotic limit of $e^{-iH_0t} e^{iHt}$ in the Lee model and it is proved that the strong limit exists on a domain which coincides with the continuum part of H (the whole Hilbert space if there is no bound state). The limit is explicitly evaluated and it is shown to define an isometric wave operator $\Omega_{\pm}$ which maps the proton one-meson states into the continuum part of H. The usual definition of the scattering operator $S = \Omega_-^{\dagger} \Omega_+$ leads to a unitary operator in the proton-meson subspace which in the Dirac δ limit agrees with the scattering matrix previously given in the literature[5]).

2. Preliminary Digression

We shall base our discussion on the following Hamiltonian:

$$\left.\begin{aligned} H = \frac{1-\tau_3(t)}{2}\,\mathscr{E}_0 &+ \int d^3\,\boldsymbol{r}\,\{\nabla\,\psi^\dagger\cdot\nabla\,\psi + \mu\,\psi^\dagger\,\psi\} \\ &- \int d^3\,\boldsymbol{r}\,\{g\,\varrho(r)\,\psi\,\tau_-(t) + g\,\varrho(r)\,\psi^\dagger\,\tau_+(t)\} - \Delta M\,\frac{1-\tau^3(t)}{2}\,, \end{aligned}\right\} \tag{1}$$

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which at $t = 0$ in the Schrodinger picture can be written in the Fourier transformed form

$$\left.\begin{aligned} H^V = \frac{1-\tau_3}{2}\,\mathscr{E}_0 + \sum_{\boldsymbol{k}} w_{\boldsymbol{k}}\, a_{\boldsymbol{k}}^{\dagger}\, a_{\boldsymbol{k}} \\ - \frac{g}{\sqrt{V}} \sum_{\boldsymbol{k}} \varrho(k)\, [a_{\boldsymbol{k}}\, \tau_- + a_{\boldsymbol{k}}^{\dagger}\, \tau_+] - \Delta M\, \frac{1-\tau_3}{2}\,, \end{aligned}\right\} \tag{2}$$

where the operators $a_{\boldsymbol{k}}$ obey the usual commutation rules $[a_{\boldsymbol{k}}, a_{\boldsymbol{b}}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}$, and the τ operators $[\tau_+, \tau_-] = \tau_3$, $\frac{1}{2}[\tau_3, \tau_\pm] = \pm\,\tau_\pm$. The τ operators are defined in terms of the bare nucleon states $\tau_3 \mid p\rangle = + \mid p\rangle$, $\tau_3 \mid n\rangle = - \mid n\rangle$, $\tau_+ \mid p\rangle = \tau_- \mid n\rangle = 0$, $\tau_- \mid p\rangle = \mid n\rangle$ and $\tau_+ \mid n\rangle = \mid p\rangle$. The quantity ΔM is the mass renormalization term for the neutron[6]). We shall be using the unrenormalized coupling constant g and square integrable cut-off function $\varrho(\boldsymbol{k})$ and will not be concerned with the difficulties of indefinite metric[7]). In the case of stable neutron particle the existence of the wave operator necessitates mass renormalization, whereas for the unstable case ΔM can be adjusted to have the peak of the decay spectrum at $\mathscr{E}_0$.

The superscript V in H^V refers to the dependence on the volume of the box. Eventually we shall let $V \to \infty$. For a finite volume[8]) V, the eigenvectors of H in the invariant subspace spanned by the vectors $\tau_- \mid p\rangle$ and $a_{\boldsymbol{k}}^{+} \mid p\rangle$ are given[9]) by

$$\mid i\rangle = \frac{1}{\sqrt{\dot{\phi}^V(E_i)}} \left[\tau_- \mid p\rangle + \frac{g^2}{V} \sum_{\boldsymbol{k}} \frac{\varrho(k)}{w_{\boldsymbol{k}} - E_i}\, a_{\boldsymbol{k}}^{\dagger} \mid p\rangle\right], \tag{3}$$

$$\phi^V(E_i) = 0\,, \tag{4}$$

where

$$\phi^V(z) = z - (\mathscr{E}_0 - \Delta M) + \frac{g^2}{V} \sum_{\boldsymbol{k}} \frac{\varrho(k)^2}{w_{\boldsymbol{k}} - z} \tag{5}$$

and $\dot{\phi}^V z = \phi^V(z)\, d/dz$. For finite V the spectrum of H^V is discrete and one can easily write down the resolution of identity[10]) P_λ^V belonging to H^V;

$$\left.\begin{aligned} P_\lambda^V = \sum_{E_i < \lambda} \frac{1}{\dot{\phi}^V(E_i)} \Big\{\mid \tau_-\, p\rangle \langle \tau_-\, p \mid + \frac{g}{\sqrt{V}} \sum_{\boldsymbol{k}} \frac{\varrho(k)}{w_{\boldsymbol{k}} - E_i} [\mid a_{\boldsymbol{k}}^{\dagger}\, p\rangle \langle \tau_-\, p \mid \\ + \mid \tau_-\, p\rangle \langle a_{\boldsymbol{k}}^{\dagger}\, p \mid] + \frac{g^2}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{\varrho(k)\,\varrho(k')}{(w_{\boldsymbol{k}} - E_i)\,(w_{\boldsymbol{k}'} - E_i)} \mid a_{\boldsymbol{k}'}^{\dagger}\, p\rangle \langle a_{\boldsymbol{k}}^{\dagger}\, p \mid \Big\}. \end{aligned}\right\} \tag{6}$$

This expression is manifestly unsuitable for studying the limit $V \to \infty$. However, since $\phi^V(z)$ is a meromorphic function with simple zeros[11]) the first terms e.g. in expression (6) can be written by the residue theorem as

$$\sum_{E_i < \lambda} \frac{1}{\dot{\phi}^V(E_i)} \mid \tau_-\, p\rangle \langle \tau_-\, p \mid = \frac{1}{2\,\pi\, i} \oint_\lambda \frac{1}{\phi^V(z)}\, dz \mid \tau_-\, p\rangle \langle \tau_-\, p \mid\,, \tag{7}$$

where the path of integration can be taken to be a circle, with centre to the left of the lowest eigenvalue E_0, and going through λ if $\phi^V(\lambda) \neq 0$ and if $\phi^V(\lambda) = 0$

slightly to the left of λ. Similarly, the second term in (6) can be put into the form

$$\left.\begin{aligned} &\sum_{E_i<\lambda} \frac{1}{\dot{\phi}^V(E_i)} \frac{g}{\sqrt{V}} \sum_{\boldsymbol{k}} \frac{\varrho(k)}{w_{\boldsymbol{k}} - E_i} [| a_{\boldsymbol{k}}^\dagger p\rangle \langle \tau_- p | + | \tau_- p\rangle \langle a_{\boldsymbol{k}}^\dagger p |] \\ &= \frac{g}{\sqrt{V}} \sum_{\boldsymbol{k}} \frac{1}{2\pi i} \oint_\lambda \frac{\varrho(k)}{\phi^V(z)\,(w_{\boldsymbol{k}} - z)}\, dz\, [| a_{\boldsymbol{k}}^\dagger p\rangle \langle \tau_- p | + | \tau_- p\rangle \langle a_{\boldsymbol{k}}^\dagger p |]\,. \end{aligned}\right\} \quad (8)$$

Finally for the last term

$$\left.\begin{aligned} &\sum_{E_i<\lambda} \frac{1}{\dot{\phi}^V(E_i)} \frac{g^2}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{\varrho(k)\,\varrho(k')}{(w_{\boldsymbol{k}} - E_i)\,(w_{\boldsymbol{k}'} - E_i)} | a_{\boldsymbol{k}'}^\dagger p\rangle \langle a_{\boldsymbol{k}}^\dagger p | \\ &\qquad = \frac{g^2}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{1}{2\pi i} \oint_\lambda \frac{\varrho(k)\,\varrho(k')}{\phi^V(z)\,(w_{\boldsymbol{k}} - z)\,(w_{\boldsymbol{k}'} - z)}\, dz\, | a_{\boldsymbol{k}'}^\dagger p\rangle \langle a_{\boldsymbol{k}}^\dagger p | \\ &\qquad\quad + \sum_{\boldsymbol{k}} \theta(\lambda - w)\, | a_{\boldsymbol{k}}^\dagger p\rangle \langle a_{\boldsymbol{k}}^\dagger p |\,. \end{aligned}\right\} \quad (9)$$

The substitution of Equations (7), (8) and (9) into (6) gives

$$\left.\begin{aligned} P_\lambda^V &= \frac{1}{2\pi i} \oint_\lambda \frac{1}{\phi^V(z)}\, dz\, | \tau_- p\rangle \langle \tau_- p | \\ &+ \frac{g}{\sqrt{V}} \sum_{\boldsymbol{k}} \frac{1}{2\pi i} \oint_\lambda \frac{\varrho(k)}{\phi^V(z)\,(w_{\boldsymbol{k}} - z)}\, dz\, [| a_{\boldsymbol{k}}^\dagger p\rangle \langle \tau_- p | + | \tau_- p\rangle \langle a_{\boldsymbol{k}}^\dagger p |] \\ &+ \sum_{\boldsymbol{k},\boldsymbol{k}'} \left[\frac{g^2}{V} \frac{1}{2\pi i} \oint_\lambda \frac{\varrho(k)\,\varrho(k')}{\phi^V(z)\,(w_{\boldsymbol{k}} - z)\,(w_{\boldsymbol{k}'} - z)}\, dz + \theta(\lambda - w_{\boldsymbol{k}})\, \delta_{\boldsymbol{k}\boldsymbol{k}'}\right] \\ &\times | a_{\boldsymbol{k}'}^\dagger p\rangle \langle a_{\boldsymbol{k}}^\dagger p |\,. \end{aligned}\right\} \quad (10)$$

This expression tends formally to a limit as $V \to \infty$.

3. The Limit of Infinite Volume

As the volume V tends to infinity the Hamiltonian given by Equation (2) becomes

$$\left.\begin{aligned} H = \frac{1-\tau_3}{2} (\mathscr{E}_0 - \Delta M) &+ \int w(k)\, a(\boldsymbol{k})^\dagger\, a(\boldsymbol{k})\, d\boldsymbol{k} \\ &- g \int \varrho(k)\, [a(\boldsymbol{k})\, \tau_- + a(\boldsymbol{k})^\dagger\, \tau_+]\, d\boldsymbol{k}\,, \end{aligned}\right\} \quad (11)$$

with the commutation relations

$$[a(\boldsymbol{k}), a(\boldsymbol{k}')^\dagger] = \delta(\boldsymbol{k} - \boldsymbol{k}')\,,\ [a(\boldsymbol{k}), a(\boldsymbol{k}')] = 0\,,$$

the others remaining unchanged. We assume the cut-off function $\varrho(k)$ to be square integrable and for convenience continuous and non-vanishing. The subspace spanned by the states $\tau_- | p\rangle$ and $a(\boldsymbol{k})^\dagger | p\rangle$ is isomorphic to the Hilbert space $l = l_0 \oplus l_1$, where l_0 is the space of complex numbers[12]) and l_1 is the space of square integrable functions in three variables, i.e. the space of all $f(\boldsymbol{k}) \mathscr{E} L^2$. If P_0 and P_1 are the projection operators in l projecting onto l_0 and l_1, any operator A in l can be decomposed[13]) into

$$A = (P_0 + P_1)\, A\, (P_0 + P_1) = A^{00} + A^{10} + A^{01} + A^{11} . \tag{12}$$

In particular $H = H^{00} + H^{10} + H^{01} + H^{11}$, where

$$\left.\begin{aligned} &H^{00}: \; l_0 \to (\mathscr{E}_0 - \Delta M)\, l_0 , \\ &H^{10}: \; l_0 \to -\, l_0\, g\, \varrho(k) , \\ &H^{01}: \; f(\boldsymbol{k}) \to -\, g \int \varrho(k)\, f(\boldsymbol{k})\, d\boldsymbol{k} , \\ &H^{11}: \; f(\boldsymbol{k}) \to w(k)\, f(\boldsymbol{k}) . \end{aligned}\right\} \tag{13}$$

The operator H is clearly self-adjoint[14]), its domain being the set of all vectors $l_0 \oplus f(\boldsymbol{k})$ such that $w(k)\, f(\boldsymbol{k}) \in L^2$. This would not be so had not we taken a square integrable cut-off $\varrho(k)$.

The function $\phi^V(z)$, as $V \to \infty$, goes over into a function $\phi(z)$, where

$$\phi(z) = z - (\mathscr{E}_0 - \Delta M) + g^2 \int \frac{\varrho(k)^2}{w(k) - z}\, d\boldsymbol{k} . \tag{14}$$

This is a regular function of z, cut along the real axis $\mu < x < \infty$, where μ is the mass of the meson. The boundary value of $\phi(z)$ exists[15]) for $x > \mu$ and is given by

$$\phi^{\pm}(\lambda) = \lambda - (\mathscr{E}_0 - \Delta M) + g^2\, \mathscr{P} \int \frac{\varrho(k)^2}{w(k) - \lambda}\, d\boldsymbol{k} \pm i\, \pi\, g^2\, \varrho(\lambda)\, \alpha(\lambda) , \tag{15}$$

where

$$\varrho(\lambda) = \varrho(k) \quad \text{at} \quad w(k) = \lambda , \quad \alpha(\lambda) = 4\, \pi\, k^2 \left.\frac{dk}{dw}\right|_{w = \lambda} ,$$

and the $\pm$ sign refers to whether the cut is approached from above or below. The mass renormalization term ΔM is chosen to be

$$\Delta M = -\, g^2\, \mathscr{P} \int \frac{\varrho(k)^2}{w(k) - \mathscr{E}_0}\, d\boldsymbol{k} . \tag{16}$$

With this choice of ΔM, $\phi(\mathscr{E}_0) = 0$ if $\mathscr{E}_0 < \mu$, this being the only zero $\phi(z)$ we can have.

As $V \to \infty$ the family of projection operators P_λ^V also tend formally to a limit P_λ

$$\left.\begin{aligned} P_\lambda = {} & \frac{1}{2\pi i} \oint_\lambda \frac{1}{\phi(z)}\, dz\, |\, \tau_-\, p\rangle \langle \tau_-\, p\, | + g \int d\boldsymbol{k}\, \frac{1}{2\pi i} \oint_\lambda \frac{\varrho(k)}{\phi(z)\, (w - z)}\, dz \\ & \times [\, |\, a(\boldsymbol{k})^\dagger\, p\rangle \langle \tau_-\, p\, | + |\, \tau_-\, p\rangle \langle a(\boldsymbol{k})^\dagger\, p\, |\,] \\ & + g^2 \int d\boldsymbol{k} \int d\boldsymbol{k}' \oint_\lambda \frac{\varrho(k)\, \varrho(k')}{\phi(z)\, (w - z)\, (w' - z)}\, dz\, |\, a(\boldsymbol{k}')^\dagger\, p\rangle \langle a(\boldsymbol{k})^\dagger\, p\, | \\ & + \int d\boldsymbol{k}\, \theta\, (\lambda - w)\, |\, a(\boldsymbol{k})^\dagger\, p\rangle \langle a(\boldsymbol{k})^\dagger\, p\, | , \end{aligned}\right\} \tag{17}$$

where the path of the z integration is taken along an open circle as indicated by Fig. 1. (The improper integral exists, except for $\lambda = \mathscr{E}_0$ if $\mathscr{E}_0 < \mu$, since $1/[\phi(z)]$ is regular on the open circle and is bounded at the endpoints.) The four 'components' of P_λ (see equation (12)) are:

$$
\left.\begin{aligned}
P_\lambda^{00}&: l_0 \to \frac{1}{2\pi i} \oint_\lambda \frac{1}{\phi(z)}\, dz\, l_0\,,\\
P_\lambda^{10}&: l_0 \to \frac{l_0\, g}{2\pi i} \oint_\lambda \frac{\varrho(k)}{\phi(z)\,(w-z)}\, dz\,,\\
P_\lambda^{01}&: f(\boldsymbol{k}) \to \frac{g}{2\pi i} \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\lambda \frac{\varrho(k)}{\phi(z)\,(w-z)}\, dz\,,\\
P_\lambda^{11}&: f(\boldsymbol{k}) \to g\,\varrho(k) \int d\boldsymbol{q}\, f(\boldsymbol{q})\, \frac{1}{2\pi i} \oint_\lambda \frac{g\,\varrho(q)}{\phi(z)\,(w(k)-z)\,(w(q)-z)}\, dz\\
&\qquad + \theta\big(\lambda - w(k)\big)\, f(\boldsymbol{k})\,.
\end{aligned}\right\} \tag{18}
$$

The operators P_λ are clearly hermitian and it is shown in the Appendix that they satisfy $P_\lambda P_\nu = P_\varkappa$, where $\varkappa = \min(\lambda, \nu)$, and also $P_\infty = I$, $\int_{-\infty}^{+\infty} \lambda\, dP_\lambda = H$. Therefore we conclude that the operators P_λ as defined by equation (18) constitute the resolution of the identity associated with the operator H defined by Equation (13). Some properties of P_λ can easily be read-off; $P_\lambda = 0$ if $\lambda < \min(\mathscr{E}_0, \mu)$, P_λ is discontinuous at $\lambda = \mathscr{E}_0$ if $\mathscr{E}_0 < \mu$, corresponding to the stable neutron, and it changes continuously in the interval $\mu < \lambda < \infty$. This expresses the well known fact that H has a continuous spectrum starting from μ and a bound state at $\mathscr{E}_0$ if $\mathscr{E}_0 < \mu$. If $\mathscr{E}_0 < \mu$ the projection operator $P_{\mathscr{E}_0}$ which projects onto the bound state is given by

$$
P_{\mathscr{E}_0} = \lim_{\delta\to 0} (P_{\mathscr{E}_0+\delta} - P_{\mathscr{E}_0-\delta}) = \lim_{\delta\to 0} P_{\mathscr{E}_0+\delta} \tag{19}
$$

and its components are

$$
\left.\begin{aligned}
P_{\mathscr{E}_0}^{00}&: l_0 \to \Gamma\, \theta(\mu - \mathscr{E}_0)\, l_0\,,\\
P_{\mathscr{E}_0}^{10}&: l_0 \to l_0\, \Gamma\, g\, \frac{\varrho(k)}{w(k) - \mathscr{E}_0}\, \theta(\mu - \mathscr{E}_0)\,,\\
P_{\mathscr{E}_0}^{01}&: f(\boldsymbol{k}) \to \Gamma\, \theta(\mu - \mathscr{E}_0) \int d\boldsymbol{k}\, \frac{g\,\varrho(k)\, f(\boldsymbol{k})}{w(k) - \mathscr{E}_0}\,,\\
P_{\mathscr{E}_0}^{11}&: f(\boldsymbol{k}) \to \Gamma\, \theta(\mu - \mathscr{E}_0)\, \frac{g\,\varrho(k)}{w(k) - \mathscr{E}_0} \int d\boldsymbol{q}\, \frac{g\,\varrho(q)\, f(\boldsymbol{q})}{w(q) - \mathscr{E}_0}\,,
\end{aligned}\right\} \tag{20}
$$

where

$$
\Gamma = \frac{1}{1 + g^2 \int \frac{\varrho(k)^2}{(w(k) - \mathscr{E}_0)^2}\, d\boldsymbol{k}}\,.
$$

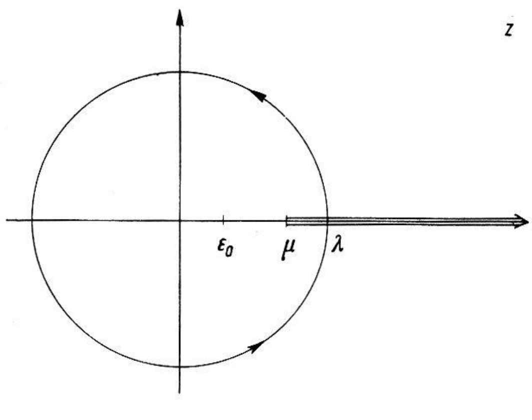

Fig. 1
Path of integration in Equations (17) and (18)

4. Some General Properties of the Limit of the Time Development Operator

If the Hamiltonian H is to describe a quantum mechanical scattering system it is necessary that $\lim_{t \to -\infty} e^{-iH_0 t} e^{iHt}$ should exist in the strong sense at least on a subset of l. (Time reversal symmetry then assures the existence of $\lim_{t \to +\infty} e^{-iH_0 t} e^{iHt}$ on the same subset.) Here H_0 is the free Hamiltonian, in our case $H_0 = [(1 - \tau_3)/2]\, \mathscr{E}_0 + \int d\boldsymbol{k}\, w(k)\, a(k)^\dagger\, a(k)$. Let R denote the set of all $x \mathscr{E} l$ such that $\lim_{t \to -\infty} e^{-iH_0 t} e^{iH_0 t} x$ exists in the strong sense and let Q denote the set of all $y \mathscr{E} l$ such that $y = \lim_{t \to -\infty} e^{-iH_0 t} e^{iHt} x$ for some $x \mathscr{E} l$. There are a number of general theorems on these operator limits[17]. The subsets R and Q are closed linear vector spaces. These two limits define a pair of adjoint partial isometries which map $R \to Q$ and $Q \to R$ respectively. The wave operator Ω_+ is defined by

$$\left.\begin{aligned} \Omega_+\, y &= \lim_{t \to -\infty} e^{-iHt} e^{iH_0 t}\, y \quad \text{if} \quad y \in Q\,, \\ \Omega_+\, y &= 0 \quad \text{if} \quad y \perp Q\,. \end{aligned}\right\} \tag{21}$$

Similarly

$$\left.\begin{aligned} \Omega_+^\dagger\, x &= \lim_{t \to -\infty} e^{-iH_0 t} e^{iHt}\, x \quad \text{if} \quad x \in R\,, \\ \Omega_+^\dagger\, x &= 0 \quad \text{if} \quad x \perp R\,. \end{aligned}\right\} \tag{22}$$

The operator Ω_+ and its adjoint $\Omega_+^\dagger$ satisfy the following relations

$$\left.\begin{aligned} (\Omega_+\, x, y) &= (x, \Omega_+^\dagger\, y)\,, \\ \Omega_+^\dagger\, \Omega_+ &= E_Q\,, \\ \Omega_+\, \Omega_+^\dagger &= E_R\,, \end{aligned}\right\} \tag{23}$$

where E_Q and E_R are the projection operators projecting onto Q and R. In the same way one defines Ω_- and $\Omega_-^\dagger$ through the limits as $t \to +\infty$. The scattering operator S is defined as

$$S = \Omega_-^\dagger\, \Omega_+ \tag{24}$$

and it satisfies

$$S^\dagger S = S\, S^\dagger = E_Q\,, \tag{25}$$

i.e. S is unitary in the subspace Q.

In the following we shall frequently make use of a simple lemma.

Lemma. Let $U(t)$ be a one parameter family of unitary operators and let

$$U(t)\, f \xrightarrow[\text{weakly}]{t = -\infty} f'\,.$$

The vector f' is the strong limit of $U(t)\, f$ as $t \to -\infty$, if and only if $\| f' \| = \| f \|$. If $\| f' \| \neq \| f \|$ the strong limit does not exist.

Proof: The necessity of this condition is well known and the sufficiency follows from

$$\begin{aligned} \| U(t)\, f - f' \|^2 &= \| U(t)\, f \|^2 - (U(t)\, f, f') - (f', U(t)\, f) + \| f' \|^2 \\ &= 2\, \| f' \|^2 - (U(t)\, f, f') - (f', U(t)\, f) \end{aligned}$$

by assumption. The assumed weak convergence gives that

$$\| U(t) f - f' \|^2 \to 2 \| f' \|^2 - 2(f', f) = 0 . \quad \text{Q.E.D.}$$

The lemma provides a criterion for the existence of the strong limit in terms of the weak limit and it will greatly facilitate the evaluation of the wave operator in the next chapter.

5. The Wave Operator

In this section the strong limit of $e^{-iH_0 t} e^{iHt}$, as $t \to -\infty$, will be investigated. In view of the previous chapter we shall start by considering the weak limit first. A general vector in l is of the form $l_0 \oplus f(\boldsymbol{k})$, where l_0 is a complex number and $f(\boldsymbol{k}) \mathscr{E} L^2$. It is therefore sufficient to look at the weak limits of $e^{-iH_0 t} e^{iHt} l_0$ and $e^{-iH_0 t} e^{iHt} f(\boldsymbol{k})$ separately. To obtain the weak limit of $e^{-iH_0 t} e^{iHt} l_0$ we have to consider the limit

$$\lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, m_0 \oplus h(\boldsymbol{k}) = \lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, m_0) + \lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, h(\boldsymbol{k}) \tag{26}$$

for all $m_0 \oplus h(\boldsymbol{k})$. The first term in the right hand side can easily be evaluated[18]):

$$\begin{aligned} \lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, m_0) &= \lim_{t \to -\infty} e^{-i\mathscr{E}_0 t} (e^{iHt} l_0, m_0) \\ &= \lim_{t \to -\infty} e^{-i\mathscr{E}_0 t} \left(\Gamma \theta (\mu - \mathscr{E}_0)\, e^{i\mathscr{E}_0 t}\, l_0\, m_0^{\times} + \int_{\mu}^{\infty} e^{i\lambda t}\, d_{\lambda} (P_{\lambda}^{00}\, l_0, m_0) \right) \\ &= \Gamma \theta (\mu - \mathscr{E}_0)\, (l_0, m_0) + \lim_{t \to -\infty} e^{-i\mathscr{E}_0 t} \int_{\mu}^{\infty} e^{i\lambda t}\, d_{\lambda} \left(\frac{1}{2\pi i} \oint_{\lambda} \frac{1}{\phi(z)}\, dz\, (l_0, m_0) \right). \end{aligned}$$

Since the cut-off function $\varrho(k)$ is continuous the derivative

$$\frac{d}{d\lambda} \left(\frac{1}{2\pi i} \oint_{\lambda} \frac{1}{\phi(z)}\, dz \right) = \frac{-1}{2\pi i} \left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)} \right), \tag{27}$$

where ϕ^+ and ϕ^- are the boundary values of $\phi(z)$ defined by equation (15). This allows us to write

$$\left. \begin{aligned} &\lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, m_0) \\ &= \Gamma \theta (\mu - \mathscr{E}_0)\, (l_0, m_0) + (l_0, m_0) \lim_{t \to -\infty} e^{-i\mathscr{E}_0 t} \int_{\mu}^{\infty} e^{i\lambda t} \frac{d}{d\lambda} \left(\frac{1}{2\pi i} \oint_{\lambda} \frac{1}{\phi(z)}\, dz \right) d\lambda \\ &= \Gamma \theta (\mu - \mathscr{E}_0)\, (l_0, m_0) - (l_0, m_0) \lim_{t \to -\infty} e^{-i\mathscr{E}_0 t} \frac{1}{2\pi i} \int_{\mu}^{\infty} e^{i\lambda t} \left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)}\, d\lambda \right). \end{aligned} \right\} \tag{28}$$

As $| \phi^+(\lambda) |^2$ is bounded from below by some positive number, the function

$$\frac{1}{2\pi i} \left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)} \right) = \frac{1}{\pi} \frac{\operatorname{Im} \phi^+(\lambda)}{| \phi^+(\lambda) |^2} = \frac{g^2\, \varrho(\lambda)^2\, \alpha(\lambda)}{| \phi^+(\lambda) |^2}$$

is integrable in $\mu \leqslant \lambda \leqslant \infty$ and therefore its Fourier transform tends to zero as $t \to -\infty$.

The second term in expression (28) thus vanishes, i.e.

$$\lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, m_0) = \Gamma \theta(\mu - \mathscr{E}_0) (l_0, m_0) . \tag{29}$$

Returning to the second term in the right hand side of equation (26)

$$\lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, h) = \lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} P_{\mathscr{E}_0} l_0, h) + \lim_{t \to -\infty} \big(e^{-iH_0 t} e^{iHt} (1 - P_{\mathscr{E}_0}) l_0, h\big) ,$$

where the first term can be shown to vanish in the limit. Consequently

$$\left.\begin{aligned} \lim_{t \to -\infty} (e^{-iH_0 t} e^{iHt} l_0, h) &= \lim_{t \to -\infty} \big(e^{-iH_0 t} e^{iHt} (1 - P_{\mathscr{E}_0}) l_0, h\big) \\ &= -\int_{-\infty}^{0} \frac{d}{dt} \big(e^{-iH_0 t} e^{iHt} (1 - P_{\mathscr{E}_0}) l_0, h\big)\, dt - (P_{\mathscr{E}_0} l_0, h) \end{aligned}\right\} \tag{30}$$

at least for all $h \mathscr{E} D_{H_0} = D_H$.[19]). On the other hand

$$\begin{aligned} &\frac{d}{dt} \big(e^{-iH_0 t} e^{iHt} (1 - P_{\mathscr{E}_0}) l_0, h\big) \\ &\quad = i \big(e^{iHt} H (1 - P_{\mathscr{E}_0}) l_0, e^{iH_0 t} h\big) - i \big(e^{iHt} (1 - P_{\mathscr{E}_0}) l_0, H_0 e^{iH_0 t} h\big) \\ &\quad = i\, l_0 \int_{\mu}^{\infty} \lambda\, e^{i\lambda t}\, d_\lambda \left(\frac{1}{2\pi i} \int d\boldsymbol{k} \oint_{\lambda} \frac{g\, \varrho(k)}{\phi(z)\, (w - z)}\, dz\, e^{-iwt}\, h^{\times}(\boldsymbol{k}) \right) \\ &\qquad - i\, l_0 \int_{\mu}^{\infty} e^{i\lambda t}\, d_\lambda \left(\frac{1}{2\pi i} \int d\boldsymbol{k} \oint_{\lambda} \frac{g\, \varrho(k)}{\phi(z)\, (w - z)}\, dz\, e^{-iwt}\, w\, h^{\times}(\boldsymbol{k}) \right) \\ &\quad = i\, l_0 \int_{\mu}^{\infty} \lambda\, e^{i\lambda t}\, d_\lambda \left(\frac{1}{2\pi i} \oint_{\lambda} dz \frac{1}{\phi(z)} \int d\boldsymbol{k}\, \frac{g\, \varrho(k)\, e^{-iwt}\, h^{\times}(\boldsymbol{k})}{(w - z)} \right) \\ &\qquad - i\, l_0 \int_{\mu}^{\infty} e^{i\lambda t}\, d_\lambda \left(\frac{1}{2\pi i} \oint_{\lambda} dz \frac{1}{\phi(z)} \int d\boldsymbol{k}\, \frac{g\, \varrho(k)\, e^{-iwt}\, w\, h^{\times}(\boldsymbol{k})}{w - z} \right) \\ &\quad = -\frac{l_0}{2\pi} \int_{\mu}^{\infty} \lambda\, e^{i\lambda t} \lim_{\mathscr{E} \to 0} \left(\frac{F(\lambda + i\mathscr{E}, t)}{\phi(\lambda + i\mathscr{E})} - \frac{F(\lambda - i\mathscr{E}, t)}{\phi(\lambda - i\mathscr{E})} \right) d\lambda \\ &\qquad + \frac{l_0}{2\pi} \int_{\mu}^{\infty} e^{i\lambda t} \lim_{\mathscr{E} \to 0} \left(\frac{F'(\lambda + i\mathscr{E}, t)}{\phi(\lambda + i\mathscr{E})} - \frac{F'(\lambda - i\mathscr{E}, t)}{\phi(\lambda - i\mathscr{E})} \right) d\lambda , \end{aligned}$$

where

$$F(z, t) = \int d\boldsymbol{k}\, \frac{g\, \varrho(k)\, e^{-iwt}\, h^{\times}(\boldsymbol{k})}{w - z} \quad \text{and} \quad F'(z, t) = \int d\boldsymbol{k}\, \frac{g\, \varrho(k)\, e^{-iwt}\, w\, h^{\times}(\boldsymbol{k})}{w - z} .$$

In one of the intermediate steps we changed the order of integratiln with respect to $\boldsymbol{k}$ and z. This is justified because the conditions of FUBINI's theorem[20]) on successive integration are here satisfied. Moreover

$$\lim_{\mathscr{E}\to 0}\left(\lambda\, F(\lambda \pm i\,\mathscr{E}, t) - F'(\lambda \pm i\,\mathscr{E}, t)\right) = -\,g \int d\boldsymbol{k}\, e^{-iwt}\, \varrho(k)\, h^{\times}(\boldsymbol{k})\,,$$

and therefore

$$\left.\begin{aligned}&\frac{d}{dt}\left(e^{-iH_0 t}\, e^{iHt}\,(1 - P_{\mathscr{E}_0})\, l_0\,,\, h\right)\\ &\quad = \frac{l_0}{2\,\pi}\int_{\mu}^{\infty} e^{i\lambda t}\left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)}\right) d\lambda \int_{\mu}^{\infty} e^{-iwt}\, g\, \varrho(k)\, \bar{h}^{\times}(\boldsymbol{k})\, \alpha(w)\, dw\,,\end{aligned}\right\} \tag{31}$$

where $\bar{h}(\boldsymbol{k})$ is the spherically symmetric component of $h(\boldsymbol{k})$. Substitution into Equation (30) gives

$$\begin{aligned}&\lim_{t\to-\infty}\left(e^{-iH_0 t}\, e^{iHt}\, l_0\,,\, h\right)\\ &= -\,\frac{l_0}{2\,\pi}\int_{-\infty}^{0} dt\left[\int_{\mu}^{\infty} e^{i\lambda t}\left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)}\right) d\lambda \int_{\mu}^{\infty} e^{-iwt}\, g\, \varrho(k)\, \bar{h}^{\times}(k)\, \alpha(w)\, d(w)\right] - (P_{\mathscr{E}_0}\, l_0\,,\, h)\\ &= \frac{l_0}{2\,\pi}\int_{-\infty}^{+\infty} dt\left[\frac{1}{2}\left(1 - \mathscr{E}(t)\right)\int_{\mu}^{\infty} e^{i\lambda t}\left(\frac{1}{\phi^+(\lambda)} - \frac{1}{\phi^-(\lambda)}\right) d\lambda \int_{\mu}^{\infty} e^{-iwt}\, g\, \varrho(k)\, \bar{h}^{\times}(k)\, \alpha(w)\, dw\right]\\ &\quad - \Gamma\,\theta\,(\mu - \mathscr{E}_0)\, l_0 \int_{\mu}^{\infty}\frac{g\,\varrho(k)}{w - \mathscr{E}_0}\, h^{\times}(\boldsymbol{k})\, d\boldsymbol{k}\,,\end{aligned}$$

where we used Equation (20) for $P_{\mathscr{E}_0}$. The function $\mathscr{E}(t)$ is the step function

$$\mathscr{E}(t) = \begin{cases}\;\;\,1 \text{ if } t > 0\\ -1 \text{ if } t < 0\end{cases}.$$

We now take the Fourier Plancharel transform[21]) in this expression to obtain

$$\left.\begin{aligned}&\lim_{t\to-\infty}\left(e^{-iH_0 t}\, e^{iHt}\, l_0\,,\, h\right) = \frac{l_0}{2}\int_{\mu}^{\infty}\left(\frac{1}{\phi^-(w)} - \frac{1}{\phi^+(w)}\right) g\, \varrho(k)\, \bar{h}^{\times}(k)\, \alpha(w)\, dw\\ &\quad - \frac{1}{2}\,\frac{l_0}{\sqrt{2\,\pi}}\int_{\mu}^{\infty}\left(\left(\mathscr{F}^{-1}\,\mathscr{E}(t)\,\mathscr{F}\right)\left(\frac{1}{\phi^-(w)} - \frac{1}{\phi^+(w)}\right)\right) g\, \varrho(k)\, \bar{h}^{\times}(k)\, \alpha(w)\, dw\\ &\quad - \Gamma\,\theta\,(\mu - \mathscr{E}_0)\, l_0 \int_{\mu}^{\infty}\frac{g\,\varrho(k)}{w - \mathscr{E}_0}\, \bar{h}^{\times}(k)\, \alpha(w)\, dw\,,\end{aligned}\right\} \tag{32}$$

where $F^{-1}\,\mathscr{E}(t)\,F$ is the Fourier transform of the multiplicative operator $\mathscr{E}(t)$ and is given by

$$\frac{1}{\sqrt{2\pi}}\left(\mathscr{F}^{-1}\,\mathscr{E}(t)\,\mathscr{F}\right)\left(\frac{1}{\phi^-(w)}-\frac{1}{\phi^+(w)}\right)=\frac{i}{\pi}\,\mathscr{P}\int_\mu^\infty\frac{1}{\lambda-w}\left(\frac{1}{\phi^-(\lambda)}-\frac{1}{\phi^+(\lambda)}\right)d\lambda$$

$$=\frac{2}{\pi}\,\mathscr{P}\int_\mu^\infty\frac{1}{\lambda-w}\,\mathrm{Im}\,\frac{1}{\phi^+(\lambda)}\,d\lambda\,.$$

The function $1/\phi(z)-\Gamma\,\theta(\mu-\mathscr{E}_0)\,1/(z-\mathscr{E}_0)$ is analytic in the cut plane and the real and imaginary parts of its boundary value satisfy the conditions of the Hilbert relation[22]):

$$\frac{1}{\sqrt{2\pi}}\left(\mathscr{F}^{-1}\,\mathscr{E}(t)\,\mathscr{F}\right)\left(\frac{1}{\phi^-(w)}-\frac{1}{\phi^+(w)}\right)=\frac{2}{\pi}\,\mathscr{P}\int_\mu^\infty\frac{1}{\lambda-w}\,\mathrm{Im}\,\frac{1}{\phi^+(\lambda)}\,d\lambda$$

$$=2\,\mathrm{Re}\,\frac{1}{\phi^+(w)}-2\,\Gamma\,\theta(\mu-\mathscr{E}_0)\,\frac{1}{w-\mathscr{E}_0}=\frac{1}{\phi^+(w)}+\frac{1}{\phi^-(w)}-2\,\Gamma\,\theta(\mu-\mathscr{E}_0)\,\frac{1}{w-\mathscr{E}_0}\,.$$

Substitution back into Equation (32) gives

$$\lim_{t\to-\infty}(e^{-iH_0t}\,e^{iHt}\,l_0,\,h)=-\,l_0\int d\boldsymbol{k}\,g\,\varrho(k)\,\frac{1}{\phi^+(w)}\,h^\times(\boldsymbol{k})\,. \tag{33}$$

Equations (29) and (33) are equivalent to the statement

$$e^{-iH_0t}\,e^{iHt}\,l_0\xrightarrow[\text{weakly}]{t=-\infty}\Gamma\,\theta(\mu-\mathscr{E}_0)\,l_0\oplus-\,l_0\,g\,\varrho(k)\,\frac{1}{\phi^+(w)}\,. \tag{34}$$

Using the same technique one can also evaluate

$$\lim_{t\to-\infty}(e^{-iH_0t}\,e^{iHt}\,f,\,m_0)\quad\text{and}\quad\lim_{t\to-\infty}(e^{-iH_0t},\,e^{iHt}\,f,\,h)$$

with the result

$$e^{-iH_0t}\,e^{iHt}\,f(\boldsymbol{k})\xrightarrow[\text{weakly}]{t=-\infty}\Gamma\,\theta(\mu-\mathscr{E}_0)\,\chi(\mathscr{E}_0)\oplus\left[f(\boldsymbol{k})-g\,\varrho(k)\,\frac{\chi^+(w)}{\phi^+(w)}\right], \tag{35}$$

where

$$\chi(z)=g\int d\boldsymbol{q}\,\frac{\varrho(q)\,f(\boldsymbol{q})}{w(q)-z}\quad\text{and}\quad\chi^+(w)=\lim_{\mathscr{E}\to0}\chi(w+i\,\mathscr{E})\,.$$

Thus the weak limit of $e^{-iH_0t}\,e^{iHt}$ exists on the whole Hilbert space l. To obtain the domain of strong convergence we apply the lemma of the previous chapter. First of all, the norm of the right hand side of Equation (34) is seen to be

$$\left\|\Gamma\,\theta(\mu-\mathscr{E}_0)\,l_0\oplus-\,l_0\,g\,\varrho(k)\,\frac{1}{\phi^+(w)}\right\|^2=\Gamma^2\,\theta(\mu-\mathscr{E}_0)\,|\,l_0\,|^2+|\,l_0\,|^2\int d\boldsymbol{k}\,\frac{g^2\,\varrho(k)^2}{\phi^+(w)\,\phi^-(w)}$$

$$=\Gamma^2\,\theta(\mu-\mathscr{E}_0)\,|\,l_0\,|^2+|\,l_0\,|^2\,\frac{1}{2\pi i}\int_\mu^\infty dw\left(\frac{\phi^+(w)-\phi^-(w)}{\phi^+(w)\,\phi^-(w)}\right)$$

$$=|\,l_0\,|^2\left[\Gamma^2\,\theta(\mu-\mathscr{E}_0)-\frac{1}{2\pi i}\int_\mu^\infty dw\left(\frac{1}{\phi^+(w)}-\frac{1}{\phi^-(w)}\right)\right],$$

by equation (15). The second term in this expression may be evaluated[23]) by opening up the contour of integration and one obtains the equation

$$\left\| \Gamma\,\theta(\mu - \mathscr{E}_0)\, l_0 \oplus - l_0\, g\, \varrho(k)\, \frac{1}{\phi^+(w)} \right\|^2 = |\, l_0\,|^2\, [\theta(\mu - \mathscr{E}_0)\, \Gamma(\Gamma - 1) + 1]$$

which shows, since $\Gamma(\Gamma - 1) \neq 0$ for $g \neq 0$, that $e^{-iH_0 t}\, e^{iHt}\, l_0$ converges strongly, as $t \to -\infty$, if and only if $\theta(\mu - \mathscr{E}_0) = 0$, i.e. if and only if there is no neutron bound state. Similarly for Equation (35)

$$\left\| \Gamma\,\theta(\mu - \mathscr{E}_0)\, \chi(\mathscr{E}_0) \oplus \left[f(\boldsymbol{k}) - g\, \varrho(k)\, \frac{\chi^+(w)}{\phi^+(w)} \right] \right\|^2$$

$$= \Gamma^2\, \theta(\mu - \mathscr{E}_0)\, |\, \chi(\mathscr{E}_0)\,|^2 + \|\, f\,\|^2 - \int_\mu^\infty \bar{f}(k)\, g\, \varrho(k) \left(\frac{\chi^+(w)}{\phi^+(w)}\right)^\times \alpha(w)\, dw$$

$$- \int_\mu^\infty \frac{\chi^+(w)}{\phi^+(w)}\, g\, \varrho(k)\, \bar{f}^\times(k)\, \alpha(w)\, dw + \int_\mu^\infty g^2\, \varrho(k)^2 \left| \frac{\chi^+(w)}{\phi^+(w)} \right|^2 \alpha(w)\, dw$$

$$= \Gamma^2\, \theta(\mu - \mathscr{E}_0)\, |\, \chi(\mathscr{E}_0)\,|^2 + \|\, f\,\|^2 - \frac{1}{2\pi i} \int_\mu^\infty (\chi^+(w) - \chi^-(w)) \left(\frac{\chi^+(w)}{\phi^+(w)}\right)^\times dw$$

$$- \frac{1}{2\pi i} \int_\mu^\infty (\chi^+(w) - \chi^-(w))^\times \frac{\chi^+(w)}{\phi^+(w)}\, dw + \frac{1}{2\pi i} \int_\mu^\infty (\phi^+(w) - \phi^-(w)) \left| \frac{\chi^+(w)}{\phi^+(w)} \right|^2 dw$$

$$= \Gamma^2\, \theta(\mu - \mathscr{E}_0)\, |\, \chi(\mathscr{E}_0)\,|^2 + \|\, f\,\|^2 - \frac{1}{2\pi i} \int_\mu^\infty \left(\frac{\chi^+(w)\, (\chi^\times(w))^+}{\phi^+(w)} - \frac{\chi^-(w)\, (\chi^\times(w))^-}{\phi^-(w)} \right) dw$$

$$= \|\, f\,\|^2 + \theta(\mu - \mathscr{E}_0)\, \Gamma(\Gamma - 1)\, |\, \chi(\mathscr{E}_0)\,|^2\,,$$

which shows that again the strong limit of $e^{-iH_0 t}\, e^{iHt}\, f(\boldsymbol{k})$ exists for all $f(\boldsymbol{k})$ if and only if H has no bound state, i.e. $\theta(\mu - \mathscr{E}_0) = 0$.

It will now be proved that even in the case of stable neutrons the operator $e^{-iH_0 t}\, e^{iHt}$ converges strongly, as $t \to -\infty$, on a domain R which coincides with the continuum part of H. In fact, from equations (20), (34) and (35) it follows that

$$e^{-iH_0 t}\, e^{iHt}\, (1 - P_{\mathscr{E}_0})\, l_0 \xrightarrow[\text{weakly}]{t = -\infty} 0 \oplus \Big[- l_0 \Big(\Gamma\,\theta(\mu - \mathscr{E}_0)\, g\, \frac{\varrho(k)}{w - \mathscr{E}_0}$$

$$- \lim_{\mathscr{E} \to 0} \Gamma\,\theta(\mu - \mathscr{E}_0)\, g\, \frac{\varrho(k)}{\phi^+(w + i\mathscr{E})} \int d\boldsymbol{q}\, \frac{g^2\, \varrho(q)^2}{(w(q) - \mathscr{E}_0)\, (w(q) - w(k) + i\mathscr{E})}$$

$$+ \lim_{\mathscr{E} \to 0} \big(1 - \Gamma\,\theta(\mu - \mathscr{E}_0)\big)\, g\, \frac{\varrho(k)}{\phi^+(w + i\mathscr{E})} \Big) \Big]\,.$$

As

$$\int d\boldsymbol{q}\, \frac{g^2\, \varrho(q)^2}{(w(q) - \mathscr{E}_0)\, (w(q) - z)} = \frac{\phi(z)}{z - 1} - 1\,,$$

this expression can be simplified to

$$e^{-iH_0t}\, e^{iHt}\,(1 - P_{\mathscr{E}_0})\, l_0 \underset{\text{weakly}}{\overset{t=-\infty}{\longrightarrow}} 0 \oplus - l_0\, g\, \frac{\varrho(k)}{\phi^+(w)}\,. \tag{36}$$

Similarly one shows that

$$e^{-iH_0t}\, e^{iHt}\,(1 - P_{\mathscr{E}_0})\, f(\boldsymbol{k}) \underset{\text{weakly}}{\overset{t=-\infty}{\longrightarrow}} 0 \oplus \left[f(\boldsymbol{k}) - g\,\varrho(k)\,\frac{\chi^+(w)}{\phi^+(w)}\right]. \tag{37}$$

In order to prove strong convergence one has to calculate the norm of the right hand sides of equations (36) and (37):

$$\left\| - l_0\, g\, \frac{\varrho(k)}{\phi^+(w)} \right\|^2 = |\, l_0\,|^2 \int \frac{g^2\,\varrho(k)^2}{|\,\phi^+(w)\,|^2}\, d\boldsymbol{k} = - \frac{|\, l_0\,|^2}{2\,\pi\, i} \int_\mu^\infty \left(\frac{1}{\phi^+(w)} - \frac{1}{\phi^-(w)}\right) dw$$

$$= |\, l_0\,|^2\,(1 - \Gamma\,\theta\,(\mu - \mathscr{E}_0)) = \|\,(1 - P_{\mathscr{E}_0})\, l_0\,\|^2\,,$$

$$\left\| f(\boldsymbol{k}) - g\,\varrho(k)\,\frac{\chi^+(w)}{\phi^+(w)} \right\|^2 = \|\, f\,\|^2 - \Gamma\,\theta\,(\mu - \mathscr{E}_0)\,|\,\chi(\mathscr{E}_0)\,|^2 = \|\,(1 - P_{\mathscr{E}_0})\, f\,\|^2\,,$$

which proves the strong convergence on the domain $R = (1 - P_{\mathscr{E}_0})\, l$ i.e. on the continuum part H. In other words, it has been shown that

$$\left.\begin{aligned} &e^{-iH_0t}\, e^{iHt}\,(1 - P_{\mathscr{E}_0})\, l_0 \underset{\text{strongly}}{\overset{t=-\infty}{\longrightarrow}} 0 \oplus - l_0\, g\,\varrho(k)\,\frac{1}{\phi^+(w)}\,,\\ &e^{-iH_0t}\, e^{iHt}\,(1 - P_{\mathscr{E}_0})\, f(\boldsymbol{k}) \underset{\text{strongly}}{\overset{t=-\infty}{\longrightarrow}} 0 \oplus \left[f(\boldsymbol{k}) - g\,\varrho(k)\,\frac{\chi^+(w)}{\phi^+(w)}\right]. \end{aligned}\right\} \tag{38}$$

We are now in the position to define the adjoint of the wave operator Ω_+ for both the stable and the unstable neutron case:

$$\left.\begin{aligned} &\Omega_+^\dagger\, x = \lim_{t=-\infty} e^{-iH_0t}\, e^{iHt}\, x\,, \quad \text{if} \quad x \in R\,,\\ &\Omega_+^\dagger\, x = 0\,, \quad \text{if} \quad x \perp R\,. \end{aligned}\right\} \tag{39}$$

The wave operator Ω_+ is the adjoint of $\Omega_+^\dagger$ and one can readily show that it is given by

$$\left.\begin{aligned} &\Omega_+\, m_0 = 0\,,\\ &\Omega_+\, h(\boldsymbol{k}) = - \int \frac{g\,\varrho(k)}{\phi^+(w)}\, h(k)\, d\boldsymbol{k}\\ &\qquad \oplus \left[h(\boldsymbol{k}) + g\,\varrho(k) \lim_{\mathscr{E}\to 0} \int d\boldsymbol{q}\, \frac{g\,\varrho(q)}{(w(q) - w(k) + i\,\mathscr{E})\;\phi(w + i\,\mathscr{E})}\, h(\boldsymbol{q})\right]. \end{aligned}\right\} \tag{40}$$

In agreement with the general considerations of the previous chapter one has

$$\left.\begin{aligned} &\Omega_+\,\Omega_+^\dagger = (1 - P_{\mathscr{E}_0})\,,\\ &\Omega_+^\dagger\,\Omega_+ = P_1\,, \end{aligned}\right\} \tag{41}$$

where P_1 projects onto the proton, one-meson subspace, i.e. $P_1(l_0 \oplus f(k)) = 0 \oplus f(\boldsymbol{k})$. Equations (41) can be verified by straightforward computation.

The limit $t \to +\infty$ can also be investigated. Not surprisingly one finds that equations (38), (39), (40) and (41) remain valid if one substitutes Ω_- for Ω_+ provided one also makes the substitution $\mathscr{E} \to -\mathscr{E}$ in the right hand side of the equations which changes $\phi^+(w)$ into $\phi^-(w)$ and $\chi^+(w)$ into $\chi^-(w)$. The scattering operator is defined as usual by

$$S = \Omega_-^\dagger \, \Omega_+$$

and it can be shown that

$$S\, f(\boldsymbol{k}) = f(\boldsymbol{k}) + 2\,\pi\, i\, \frac{g^2\, \varrho(k)^2\, \alpha(w)}{\phi^+(w)}\, \bar{f}(\boldsymbol{k})', \tag{42}$$

where $\bar{f}(k)$ is the spherically symmetric component of $f(\boldsymbol{k})$. The scattering operator is unitary in the proton, one-meson subspace, i.e.

$$S^\dagger\, S\, f(\boldsymbol{k}) = S\, S^\dagger\, f(\boldsymbol{k}) = f(\boldsymbol{k})\;.$$

In the limit $f(\boldsymbol{k}) \to \delta(\boldsymbol{k} - \boldsymbol{k}')$ formula (42) agrees with the S-matrix given in the literature.

6. Summary and Conclusion

It has been proved that the strong limit of $e^{-iH_0 t}\, e^{iHt}$ as $t \to \pm\infty$ exists on the continuum part of H which if $\mu < \mathscr{E}_0$ coincides with the total Hilbert space l. This led to the definition of the wave operator Ω and the scattering operator S, unitary in the proton-meson subspace. There were two essential requirements for the proof to go through; the cut-off function $\varrho(k)$ had to be square integrable and the discrete eigenvalues of H and H_0 had to be the same, i.e. mass renormalization. These two conditions fulfilled, there is no need, at least as far as the mathematics goes, for coupling constant renormalization.

On the other hand if one wants to have a non-trivial scattering operator in the relativistic point particle limit, coupling constant renormalization becomes necessary. The unrenormalized coupling constant g must then tend to zero through imaginary values[25]), implying a Hamiltonian H which is no longer self-adjoint. The operator H would probably still possess a spectral resolution, the spectrum now containing points of the complex plane. It would be interesting to carry out this kind of analysis for such a Hamiltonian and see what the asymptotic limit of $e^{-iH_0 t}\, e^{iHt}$ looks like when e^{iHt} is no longer unitary.

It is customary to interprete the resonance peak in the cross-section by saying that the matrix element $\langle n \mid (H - z)^{-1} \mid n \rangle$ has a complex pole[26]) on the second Riemann sheet, the position of the pole being related to the energy and life time of the unstable particle $\mid n\rangle$. In the Lee model with $\mathscr{E}_0 > \mu$ $\langle n \mid (H - z)^{-1} \mid n \rangle = 1/\phi(z)$ and whether this can be analytically continued across the cut depends entirely whether $\varrho(k)$ can be analytically continued off the real axes. This seems to some extent irrelevant as it is always possible to choose two cut off functions $\varrho(k)$ and $\varrho'(k)$ which are for all practical purposes indistinguishable; they give the same resonance peak and the same cross-section, such that say $\varrho'(k)$ permits the analytic continuation of $\langle n \mid (H - z)^{-1} \mid n \rangle$ whereas $\varrho(k)$ does not.

It is tempting to speculate how much of this sort of analysis can be carried over to more realistic field theories. It is clear that many of the equations used in the proof of the existence of the strong limit $\lim_{t=-\infty} e^{-iH_0 t} e^{iHt}$ are more general. There are for example the well known relations between the family of projection operators P_λ and the resolvent operators $(H - z)^{-1}$. However, for a proof of this kind it is essential to have a well defined, preferably self-adjoint Hamiltonian. Unfortunately this is not so for, e.g. quantum electrodynamics. The Hamiltonian there is extremely ill-defined, its domain contains only the zero vector. Even so one may try to introduce a cut-off into the Hamiltonian which makes it properly self-adjoint and show that the wave operator exists in each channel for a square integrable cut off. This approach may lead for example to a proof of renormalizability not based on perturbation theory.

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Appendix

It will be shown here that the operators P_λ defined by Equation (18) satisfy

$$P_\lambda P_\nu = P_\varkappa , \quad \varkappa = \min(\lambda, \nu) \tag{A.1}$$

and

$$H = \mathscr{E}_0 P_{\mathscr{E}_0} + \int_\mu^\infty \lambda \, dP_\lambda , \quad \lim_{\lambda \to \infty} P_\lambda = I . \tag{A.2}$$

Equation (A.1) is equivalent to the four equations

$$\left.\begin{aligned} &\text{I} \quad P_\lambda^{00} P_\nu^{00} + P_\lambda^{01} P_\nu^{10} = P_\varkappa^{00} , \\ &\text{II} \quad P_\lambda^{00} P_\nu^{01} + P_\lambda^{01} P_\nu^{11} = P_\varkappa^{01} , \\ &\text{III} \quad P_\lambda^{11} P_\nu^{10} + P_\lambda^{10} P_\nu^{00} = P_\varkappa^{10} , \\ &\text{IV} \quad P_\lambda^{11} P_\nu^{11} + P_\lambda^{10} P_\nu^{01} = P_\varkappa^{11} , \end{aligned}\right\} \tag{A.3}$$

where $\varkappa = \min(\lambda, \nu)$. We can prove these by straightforward computation:

$$\text{I} \quad (P_\lambda^{00} P_\nu^{00} + P_\lambda^{01} P_\nu^{10})\, l_0 = \frac{l_0}{2\pi i} \oint_\lambda \frac{1}{\phi(z)} dz \frac{1}{2\pi i} \oint_\nu \frac{1}{\phi(z')} dz'$$
$$+ \int d\boldsymbol{k} \left(\frac{1}{2\pi i} \oint_\lambda \frac{g\,\varrho(k)}{\phi(z)\,(w - z)} dz \frac{1}{2\pi i} \oint_\nu \frac{g\,\varrho(k)}{\phi(z')\,(w - z')} dz' \right) l_0 .$$

The order of integration can be changed in the second term:

$$(P_\lambda^{00} P_\nu^{00} + P_\lambda^{01} P_\nu^{10})\, l_0 = l_0 \left(\frac{1}{2\pi i}\right)^2 \oint_\lambda \frac{1}{\phi(z)} dz \oint_\nu \frac{1}{\phi(z')} dz'$$
$$+ l_0 \left(\frac{1}{2\pi i}\right)^2 \oint_\lambda dz \oint_\nu dz' \frac{1}{\phi(z)\,\phi(z')} \int d\boldsymbol{k} \frac{g^2\,\varrho(k)^2}{(w - z)\,(w - z')} .$$

From equation (14)

$$\int d\boldsymbol{k}\, \frac{g^2\, \varrho(k)^2}{(w-z)\,(w-z')} = \frac{\phi(z)-\phi(z')}{z-z'} - 1 \tag{A.4}$$

and so

$$(P_\lambda^{00}\, P_\nu^{00} + P_\lambda^{01}\, P_\nu^{10})\, l_0 = \left(\frac{1}{2\,\pi\, i}\right)^2 \oint_\lambda dz \oint_\nu dz'\, \frac{1}{z-z'} \left(\frac{1}{\phi(z)} - \frac{1}{\phi(z')}\right) = P_\varkappa^{00}\, l_0\,,$$

where $\varkappa = \min(\lambda, \nu)$ proving equation I.

II $(P_\lambda^{00}\, P_\nu^{01} + P_\lambda^{01}\, P_\nu^{11})\, f(\boldsymbol{k})$

$$= \left(\frac{1}{2\,\pi\, i}\right)^2 \oint_\lambda \frac{1}{\phi(z)}\, dz \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\nu \frac{g\, \varrho(k)}{\phi(z')\,(w-z')}\, dz' + \int d\boldsymbol{k}\, \frac{1}{2\,\pi\, i} \oint_\lambda \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz$$

$$\times \left[g\, \varrho(k)\, \frac{1}{2\,\pi\, i} \int d\boldsymbol{q}\, f(q) \oint_\nu \frac{g\, \varrho(q)}{\phi(z')\,(w-z')\,(w'-z')}\, dz' + \theta\,(\nu - w)\, f(\boldsymbol{k})\right],$$

where $w = w(k)$ and $w' = w(q)$.

Change of the order of integration and equation (A.4) allows us to write

$$(P_\lambda^{00}\, P_\nu^{01} + P_\lambda^{01}\, P_\nu^{11})\, f(\boldsymbol{k}) = \left(\frac{1}{2\,\pi\, i}\right)^2 \oint_\lambda \frac{1}{\phi(z)}\, dz \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\nu \frac{g\, \varrho(k)}{\phi(z')\,(w-z')}\, dz'$$

$$+ \left(\frac{1}{2\,\pi\, i}\right)^2 \int d\boldsymbol{k}\, g\, \varrho(k)\, f(\boldsymbol{k}) \oint_\lambda dz \oint_\nu dz'\, \frac{1}{\phi(z)\, \phi(z')\,(w-z)} \left(\frac{\phi(z)-\phi(z')}{z-z'} - 1\right)$$

$$+ \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, \theta\,(\nu - w)\, f(\boldsymbol{k}) \oint_\lambda \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz\,,$$

which simplifies to

$$(P_\lambda^{00}\, P_\nu^{01} + P_\lambda^{01}\, P_\nu^{11})\, f(\boldsymbol{k}) = \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, \theta\,(\nu - w)\, f(\boldsymbol{k}) \oint_\lambda \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz$$

$$+ \left(\frac{1}{2\,\pi\, i}\right)^2 \int d\boldsymbol{k}\, g\, \varrho(k)\, f(\boldsymbol{k}) \oint_\lambda dz \oint_\nu dz' \left(\frac{1}{\phi(z')\,(w-z')\,(z-z')} - \frac{1}{\phi(z)\,(w-z')\,(z-z')}\right)$$

$$= \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, \theta\,(\nu - w)\, f(\boldsymbol{k}) \oint_\lambda \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz + \theta\,(\lambda - \nu)$$

$$\times \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\nu \frac{g\, \varrho(k)}{\phi(z')\,(w-z')}\, dz'$$

$$- \left(\frac{1}{2\,\pi\, i}\right)^2 \int d\boldsymbol{k}\, g\, \varrho(k)\, f(\boldsymbol{k}) \oint_\lambda dz\, \frac{1}{\phi(z)} \oint_\nu \frac{1}{(w-z')\,(z-z')}\, dz'$$

$$= \theta\,(\lambda - \nu)\, \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\nu \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz + \theta\,(\nu - \lambda)$$

$$\times \frac{1}{2\,\pi\, i} \int d\boldsymbol{k}\, f(\boldsymbol{k}) \oint_\lambda \frac{g\, \varrho(k)}{\phi(z)\,(w-z)}\, dz\,.$$

which verifies equation II. Equation III follows by taking the adjoint of equation II.

IV. The proof goes along the same lines. The details are somewhat lengthy, only the main steps in the argument will be reproduced here:

$$\left.\begin{aligned}
&(P^{10}_\lambda P^{01}_\nu + P^{11}_\lambda P^{11}_\nu) = \left(\frac{1}{2\pi i}\right)^2 \oint_\lambda \frac{g\,\varrho(k)}{\phi(z)\,(w-z)}\,dz \int d\boldsymbol{q}\, f(\boldsymbol{q}) \\
&\times \oint_\nu \frac{g\,\varrho(q)}{\phi(z')\,(w'-z')}\,dz' + g\,\varrho(k) \int d\boldsymbol{q}\,\frac{1}{2\pi i}\oint_\lambda \frac{g\,\varrho(q)}{\phi(z)\,(w-z)\,(w'-z)}\,dz \\
&\times \left[g\,\varrho(q)\,\frac{1}{2\pi i}\int d\boldsymbol{q}'\, f(\boldsymbol{q}')\oint_\nu \frac{g\,\varrho(q')}{\phi(z')\,(w'-z')\,(w''-z')}\,dz' + \theta(\nu-w')\,f(\boldsymbol{q})\right] \\
&+ \theta(\lambda-w)\,g\,\varrho(k)\,\frac{1}{2\pi i}\int d\boldsymbol{q}\, f(\boldsymbol{q})\oint_\nu \frac{g\,\varrho(q)}{\phi(z)\,(w-z)\,(w'-z)}\,dz \\
&+ \theta(\lambda-w)\,\theta(\nu-w)\,f(\boldsymbol{k})\,,
\end{aligned}\right\} \qquad (A.5)$$

where $w = w(k)$, $w' = w(q)$ and $w'' = w(q')$. Change in the order of integration and use of equation (A.4) gives

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2 \int d\boldsymbol{q} \oint_\lambda \frac{g\,\varrho(q)}{\phi(z)\,(w-z)\,(w'-z)}\,dz\; g\,\varrho(q)\int d\boldsymbol{q}'\, f(\boldsymbol{q}')\oint_\nu \frac{g\,\varrho(q')}{\phi(z')\,(w'-z')\,(w''-z')}\,dz' \\
&= \left(\frac{1}{2\pi i}\right)^2 \int d\boldsymbol{q}'\, g\,\varrho(q')\, f(\boldsymbol{q}') \oint_\lambda dz \oint_\nu dz'\, \frac{1}{\phi(z)\,\phi(z')\,(w-z)\,(w''-z')} \\
&\qquad \times \int d\boldsymbol{q}\, \frac{g^2\,\varrho(q)^2}{(w'-z)\,(w'-z')} \\
&= -\left(\frac{1}{2\pi i}\right)^2 \oint_\lambda \frac{1}{\phi(z)\,(w-z)}\,dz \int d\boldsymbol{q}\, f(\boldsymbol{q}) \oint_\nu \frac{g\,\varrho(q')}{\phi(z')\,(w'-z')}\,dz' \\
&\qquad + \left(\frac{1}{2\pi i}\right)^2 \int d\boldsymbol{q}\, g\,\varrho(q)\, f(\boldsymbol{q}) \oint_\lambda dz \oint_\nu dz'\, \frac{\phi(z)-\phi(z')}{\phi(z)\,\phi(z')\,(w-z)\,(w'-z')\,(z'-z)} \\
&= -\left(\frac{1}{2\pi i}\right)^2 \oint_\lambda \frac{1}{\phi(z)\,(w-z)}\,dz \int d\boldsymbol{q}\, f(\boldsymbol{q}) \oint_\nu \frac{g\,\varrho(q)}{\phi(z')\,(w'-z')}\,dz' \\
&\qquad + \frac{1}{2\pi i}\,\theta(\lambda-\nu) \int d\boldsymbol{q}\, g\,\varrho(q)\, f(\boldsymbol{q}) \oint_\nu \frac{g\,\varrho(q)}{\phi(z')\,(w'-z')\,(w-z')}\,dz' \\
&\qquad - \frac{1}{2\pi i}\,\theta(\lambda-w) \int d\boldsymbol{q}\, g\,\varrho(q)\, f(\boldsymbol{q}) \oint_\nu \frac{1}{\phi(z)\,(w-z)\,(w'-z)}\,dz \\
&\qquad + \frac{1}{2\pi i}\,\theta(\nu-\lambda) \int d\boldsymbol{q}\, g\,\varrho(q)\, f(\boldsymbol{q}) \oint_\lambda \frac{1}{\phi(z)\,(w-z)\,(w'-z)}\,dz \\
&\qquad - \frac{1}{2\pi i} \int d\boldsymbol{q}\,\theta(\nu-w')\, g\,\varrho(q)\, f(\boldsymbol{q}) \oint_\lambda \frac{1}{\phi(z)\,(w-z)\,(w'-z)}\,dz\,.
\end{aligned}$$

Substitution back into equation (A.5) gives

$$(P_\lambda^{10} P_\nu^{01} + P_\lambda^{11} P_\nu^{11}) f(\boldsymbol{k}) = g\,\varrho(k) \frac{1}{2\pi i} \left[\theta(\lambda - \nu) \int d\boldsymbol{q} \oint_\nu \frac{g\,\varrho(q)}{\phi(z)\,(w - z)\,(w' - z)}\,dz \right.$$

$$\left. + \theta(\nu - \lambda) \int d\boldsymbol{q}\, f(\boldsymbol{q}) \oint_\lambda \frac{g\,\varrho(q)}{\phi(z)\,(w - z)\,(w' - z)}\,dz\right] + \theta(\lambda - w)\,\theta(\nu - w)\,f(\boldsymbol{k})\,.$$

Q. E. D.

In addition to equation (A.3) we have to prove

$$\lim_{\lambda\to\infty} P_\lambda = I \quad \text{and} \quad \mathcal{E}_0 P_{\mathcal{E}_0} + \int_\mu^\infty \lambda\, dP_\lambda = H\,.$$

The first statement follows easily from the asymptotic behaviour of $1/\phi(z)$:

$$\left| \frac{1}{\phi(z)} - \frac{1}{z - \mathcal{E}_0 + \Delta M} \right| = 0\left(\frac{1}{R^3}\right)$$

for $z = R\,e^{i\phi}$ and large R. The proof of

$$\mathcal{E}_0 P_{\mathcal{E}_0} + \int_\mu^\infty \lambda\, dP_\lambda = H \tag{A.6}$$

follows from equations (18) and (19).

$$\left.\begin{aligned}
&(\mathcal{E}_0 P_{\mathcal{E}_0} + \int_\mu^\infty \lambda\, P_\lambda)\, l_0 \\
&= \left[l_0\, \mathcal{E}_0\, \Gamma\, \theta(\mu - \mathcal{E}_0) - l_0 \frac{1}{2\pi i} \int_\mu^\infty \lambda \lim_{\mathcal{E}\to 0} \left(\frac{1}{\phi(\lambda + i\,\mathcal{E})} - \frac{1}{\phi(\lambda - i\,\mathcal{E})}\right) d\lambda\right] \\
&\quad \oplus \left[l_0\, \Gamma\, \theta(\mu - \mathcal{E}_0) \frac{g\,\varrho(k)}{w - \mathcal{E}_0} - l_0\, g\,\varrho(k) \frac{1}{2\pi i}\right. \\
&\quad \left.\times \lim_{\mathcal{E}\to 0} \int_\mu^\infty \left(\frac{\lambda}{\phi(\lambda + i\,\mathcal{E})\,(w - \lambda - i\,\mathcal{E})} - \frac{\lambda}{\phi(\lambda - i\,\mathcal{E})\,(w - \lambda + i\,\mathcal{E})}\right) d\lambda\right] \\
&= l_0 \frac{1}{2\pi i} \lim_{R\to\infty} \oint_R \frac{z}{\phi(z)}\,dz \oplus l_0\, g\,\varrho(k) \frac{1}{2\pi i} \lim_{R\to\infty} \oint_R \frac{z}{\phi(z)\,(w - z)}\,dz \\
&= l_0 \frac{1}{2\pi i} \lim_{R\to\infty} \oint_R \frac{z}{z - \mathcal{E}_0 + \Delta M}\,dz \\
&\quad \oplus l_0\, g\,\varrho(k) \frac{1}{2\pi i} \lim_{R\to\infty} \oint_R \frac{z}{(z - \mathcal{E}_0 + \Delta M)\,(w - z)}\,dz \\
&= (\mathcal{E}_0 - \Delta M)\, l_0 \oplus \left[l_0\, g\,\varrho(k) \left(\frac{\mathcal{E}_0 - \Delta M}{w - \mathcal{E}_0 + \Delta M} - \frac{w}{w - \mathcal{E}_0 + \Delta M}\right)\right] = H\, l_0\,.
\end{aligned}\right\} \tag{A.7}$$

Similarly one proves

$$\left(\mathcal{E}_0 P_{\mathcal{E}_0} + \int_\mu^\infty \lambda\, dP_\lambda\right) f(\boldsymbol{k}) = -\,g \int \varrho(k)\, f(\boldsymbol{k})\, d\boldsymbol{k} \oplus w\, f(\boldsymbol{k}) = H\, f(\boldsymbol{k})$$

which taken with equation (A.7) is equivalent to equation (A.6).

This completes the proof that the family of operators P_λ is the resolution of the identity associated with the self-adjoint operator H.

Literature

1) J. M. Jauch, Helv. Phys. Acta *31*, 127 (1958), Helv. Phys. Acta *31*, 661 (1958).

2) C. Møller, Dan. Mat. Fys. Medd. *23*, No. 1 (1945).

3) See e.g. J. M. Cook, J. Math. Phys. *36*, 82 (1957); S. T. Kuroda, Nuovo Cim. *12*, 431 (1959); J. M. Jauch and I. I. Zinnes, Nuovo Cim. *11*, 553 (1959); R. T. Prosser, J. Math. Phys. *4*, 1048 (1963).

4) T. D. Lee, Phys. Rev. *95*, 1329 (1954), and also G. Källén and W. Pauli, Dan. Mat. Fys. Medd. No. 7 (1955); V. Glaser and G. Källén, Nucl. Phys. *2*, 706 (1957); E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill, 1962).

5) See e.g. Henley and Thirring, loc. cit.

6) For further explanations of the notations used here see Henley and Thirring, loc. cit.

7) Glaser and Källén, loc. cit.

8) For convenience we assume the lattice to be non degenerate, i.e. $w_{\boldsymbol{k}} = w_{\boldsymbol{k}'} \rightarrow \boldsymbol{k} = \boldsymbol{k}'$.

9) Källén and Pauli, loc. cit.

10) A family of projection operators P_λ is said to be a resolution of the identity belonging to a self-adjoint operator H if $P_{-\infty} = 0$, $P_{+\infty} = I$, $P_\lambda P_\nu = P_\varkappa$, $\varkappa = \min(\lambda, \nu)$, $\lim_{\mathscr{E} \to 0} P_{\lambda - \mathscr{E}} = P_\lambda$, and $\int_{-\infty}^{+\infty} \lambda \, dP = H$. See e.g. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Berlin 1932); N. I. Akhiezer and I. M. Glasmann, *Theorie der linearen Operatoren im Hilbert-Raum* (Berlin 1954).

11) See e.g. Glaser and Källén, loc. cit.

12) The symbol l_0 will be used to denote an element of l of the form $l_0 = l_0 \oplus 0$ and also to denote the complex number l_0 (see e.g. the right hand side of equation (13)). We hope this will not be confusing for the reader.

13) If the operator A is unbounded with a domain $D_A \subset l$, the decomposition is valid only if $P_0 D_A \subset D_A$ and $P_1 D_A \subset D_A$. This is, however, trivially satisfied in all the cases we are going to consider.

14) For the mathematical definition of a self-adjoint operator see the references given in footnote 10.

15) E. C. Titchwarsh, Theory of Fouricr Integrals, Chapter V (Oxford, 1948).

16) The strong and weak limits are defined as
$x_n \underset{\text{strongly}}{\longrightarrow} x \Rightarrow \| x_n - x \| \rightarrow 0$ and $x_n \underset{\text{weakly}}{\longrightarrow} x \Rightarrow (x_n, y) \rightarrow (x, y)$ for all $y \mathscr{E} l$.
See, e.g., Akhiezer and Glasmann, loc. cit.

17) See the papers by Jauch, loc. cit.

18) It is here that the significance of mass renormalization can clearly be seen. Without it even the weak limit would not exist.

19) It is sufficient to consider the dense domain D_{H_0} in order to establish the weak limit.

20) F. Riesz and B. Sz. Nagy, *Functional Analysis* (New York 1955).

21) Akhiezer ans Glasmann, loc. cit.

22) Titchmarsh, loc. cit.

23) See e.g. Källén and Pauli, loc. cit.

24) In the case of unstable neutron $P_{\mathscr{E}_0} = 0$, i.e. $R = l$.

25) See e.g. Källén and Pauli and also Henley and Thirring, loc. cit.

26) For a review see e.g. the articles by Zumino and Levy in lectures on *Field Theory and the Many-Body Problem* (Academic Press, London 1961, edited by E. R. Caianiello).

After the completion of this work the author's attention was brought to a recent paper by Ezawa (Ann. Phys. *24*, 46, 1963) in which the weak limit of $e^{iHt} e^{-iH_0 t}$ is evaluated. Ezawa integrates the equations of motion with a δ-function initial condition and from this he derives the weak limit. It seems, however, instructive to calculate the weak limit using projection operators instead, which avoids the use of non-normalizable initial conditions.