# Analyticity with Respect to the Coupling Constant in Certain Two-dimensional Field Theoretic Models

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*Abstract.* SCHROER's model of a derivative coupling in two dimensions and THIRRING's model are analyzed with respect to the dependence on the coupling constant $g$. It is seen that SCHROER's model is analytic in the entire $g^2/2\pi$-plane, whereas the Thirring model presents singularities at $g = \pm 2\pi$ in the $g$-plane. The technique of analytic continuation to arbitrary complex values of the coupling constant, which involves the 'partie finie'-technique, is exhibited. The analyticity around $g = 0$ entails the absence of infrared divergences. For SCHROER's model and non-zero Fermion mass the perturbation series is given explicitly for the two-point function.

## 1. Introduction

In the last years several field-theoretic models in one-space dimension such as THIRRING's and SCHROER's models[1][2]) have been solved exactly. These models involve zero mass fields. Thus they are useful for the study of the infrared structure of fields. The possibility of creation of an unlimited number of virtual particles entails the disappearance of one-particle states in these models and the violation of the asymptotic condition.

An expansion of these exact solutions in powers of the coupling constant seems to yield infrared divergences[3]). However we show in this paper that this is not the case. Indeed the Wightman functions of SCHROER's model are distributions which are, with respect to the coupling constant $g^2/2\pi$, analytic in the entire $g^2/2\pi$-plane. Considered as functions of $g$, the Wightman functions of the Thirring model are analytic everywhere except at $g = \pm 2\pi$. Since in both models the solution is analytic at $g = 0$, a perturbation expansion will lead to finite terms.

The analytic continuation of the Wightman functions into the entire coupling constant plane requires the concept of the 'partie finie' or regularization. For the reader not acquainted with distribution theory, its properties are, as far as we need them described in the Appendices.

Chapter 2 presents briefly SCHROER's derivative coupling and its Wightman functions. In order to see by what techniques the analytic continuation has to be performed, we carry it through for the two-point function in Chapter 3.

In Chapter 4 we consider the neighbourhood of $g = 0$ and write down explicitly the Taylor series for the two-point function. In chapter 5 it is shown that, if the fermion mass is put equal to zero the method of chapter 3 and 4 fails, although the analyticity properties are preserved. The same is true for the Thirring model, which is discussed in chapter 6.

## 2. Derivative Coupling in Two Dimensions (Schroer's Model)[2])

The equations of motion of the model are:

$$\Box\, \varphi = 0\,, \qquad (i\,\gamma^\mu\,\partial_\mu - M)\,\psi = g\,\gamma^\mu\,\psi\,\partial_\mu\,\varphi\,. \tag{2.1}$$

The commutation relation for $\varphi$ is taken to be

$$[\varphi^{(-)}(x),\ \varphi^{(+)}(y)] = \tfrac{1}{i}\,D^-(x-y)\,, \tag{2.2}$$

$$D^-(x-y) = \frac{1}{4\,\pi\,i}\left\{\log\frac{-(x-y)^2 + i\,\varepsilon(x^0-y^0)}{4} + 2\,\gamma\right\}$$

$$= \frac{i}{2\,\pi}\int dp\; e^{-ipx}\,[(p^0+p^1)^{-1}_{+}\,\delta(p^0-p^1) + (p^0-p^1)^{-1}_{+}\,\delta(p^0+p^1)]\,. \tag{2.3}$$

The distributions $(p^0+p^1)^{-1}_{+}$, $(p^0-p^1)^{-1}_{+}$ are explained in Appendix 4. It has been shown[4]) that a free field which satisfies (2.2) and (2.3) can actually be constructed.

Guided by analogy with the unquantized case, one can consider as solution of (2.1) the operator

$$\psi(x) = \psi_0(x)\!:e^{-ig\,\varphi(x)}\!: \tag{2.4}$$

where $\psi_0(x)$ is a free spinor field of mass $M$, and $[\varphi(x), \psi_0(y)] = 0$. The two-point functions turns out to be

$$W_2(x-y) = \langle\, 0 \mid \psi(x)\,\overline{\psi}(y) \mid 0\,\rangle = \tfrac{1}{i}\,e^{-ig^2\,D^-(x-y)}\,S^-(x-y\,;M) \tag{2.5}$$

$$\longrightarrow \tfrac{1}{i}\,[-(x-y)^2 + i\,\varepsilon(x^0-y^0)]^{\,-g^2/4\pi}\;S^-(x-y;M)\,,$$

where in the last line, a trivial multiplicative factor has been omitted in the sense of a wave-function renormalization. The higher Wightman functions are (with the same renormalization)

$$W_{2n}(x_1\ldots x_n\,y_1\ldots y_n) = \langle\,0 \mid \psi(x_1)\ldots\psi(x_n)\,\overline{\psi}(y_1)\ldots\overline{\psi}(y_n)\mid 0\,\rangle$$

$$= \prod_{i<k}\,[-(x_i-x_k)^2 + i\,\varepsilon(x_i^0-x_k^0)]^{\frac{g^2}{4\pi}}\,[-(y_i-y_k)^2$$

$$+ i\,\varepsilon(y_i^0-y_k^0)]^{\frac{g^2}{4\pi}}\cdot\prod_{i,k}\,[-(x_i-y_k)^2 + i\,\varepsilon(x_i^0-y_k^0)]^{\frac{-g^2}{4\pi}}$$

$$\cdot\,\langle\,0 \mid \psi_0(x_1)\ldots\psi_0(x_n)\,\overline{\psi}_0(y_1)\ldots\overline{\psi}_0(y_n)\mid 0\,\rangle. \tag{2.6}$$

## 3. Analyticity in the Coupling Constant

Since the Wightman functions of Schroer's model are free-field functions multiplied by Fourier transforms of Riesz' distribution (see Appendix 4) they can be continued analytically to be entire functions in the complex $g^2/2\,\pi$-plane. We shall perform this continuation explicitly for the case of the two-point function. If the latter is represented in the form (we put $\lambda = g^2/2\,\pi$)

$$(-x^2 + i\,\varepsilon\,x^0)^{-\frac{\lambda}{2}}\,S^-(x;M) = (2\,\pi)^{-2\,\lambda}\int_{-\infty}^{+\infty} d\varkappa\;\varrho(\varkappa)\;S^-(x;\varkappa)\,, \tag{3.1}$$

then the problem is reduced to the analytic continuation of the spectral function $\varrho(\varkappa)$. (See Appendix 2.) In (3.1) a trivial factor has been separated for later convenience. The spectral function is computed to be [2]

$$\varrho(\varkappa) = \frac{(\varkappa - M)_+^{\lambda-1}}{\Gamma(\lambda)} \frac{\varkappa + M}{2\varkappa} F\left(-\frac{1}{2}, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \left(\frac{\varkappa - M}{\varkappa + M}\right)^2\right), \qquad (\varkappa > 0)$$

$$\varrho(\varkappa) = \frac{3\lambda}{2} \frac{(|\varkappa| - M)_+^{\lambda+1}}{(\lambda+2)} \frac{1}{|\varkappa|(|\varkappa| + M)} F\left(\frac{1}{2}, \frac{\lambda}{2}, \frac{\lambda+3}{2}; \left(\frac{|\varkappa| - M}{|\varkappa| + M}\right)^2\right), \qquad (\varkappa < 0) \tag{3.2}$$

where $F$ is the hypergeometric function.

For complex $\lambda$ with $\operatorname{Re}\lambda > 0$ the analytic continuation is still given by (3.2) whereas for $\operatorname{Re}\lambda < 0$, we have to apply the regularisation technique described in Appendix 3. To this end we write the spectral function in the form

$$\varrho(\varkappa) = \frac{(\varkappa - M)_+^{\lambda-1}}{\Gamma(\lambda)} f(\varkappa, \lambda) \qquad (\varkappa > 0), \tag{3.3}$$

$$\varrho(\varkappa) = \frac{(|\varkappa| - M)_+^{\lambda+1}}{\Gamma(\lambda+2)} f(\varkappa, \lambda). \qquad (\varkappa < 0).$$

Formula (A 5.2) of Appendix 5 leads us to the following form of the analytic continuation in the strip $-N < \operatorname{Re}\lambda < -N + 1$ ($N = 1, 2, 3 \ldots$):

$$\int_0^\infty d\varkappa\, \varrho(\varkappa)\, S^-(x; \varkappa) = \frac{1}{\Gamma(\lambda)} \int_M^\infty d\varkappa \frac{1}{(\varkappa - M)^{1-\lambda}} \Bigg\{ f(\varkappa, \lambda)\, S^-(x; \varkappa) - \sum_{m=0}^{N-1} \frac{(\varkappa - M)^m}{m!} \frac{\partial^m}{\partial \varkappa^m} [f(\varkappa, \lambda)\, S^-(x; \varkappa)]_{\varkappa = M} \Bigg\}$$

$$\int_{-\infty}^0 d\varkappa\, \varrho(\varkappa)\, S^-(x; \varkappa) = \frac{1}{\Gamma(\lambda+2)} \int_M^\infty d\varkappa \frac{1}{(\varkappa - M)^{-1-\lambda}} \Bigg\{ f(-\varkappa, \lambda)\, S^-(x; -\varkappa) - \sum_{m=0}^{N-3} \frac{(\varkappa - M)^m}{m!} \frac{\partial^m}{\partial \varkappa^m} [f(-\varkappa, \lambda)\, S^-(x; -\varkappa)]_{\varkappa = M} \Bigg\},$$

with (3.4)

$$\frac{\partial^m}{\partial \varkappa^m} [f(\varkappa, \lambda)\, S^-(x; \varkappa)]_{\varkappa = M} = \sum_{k=0}^m \binom{m}{k} A(m-k; M, \lambda)\, S^{-(k)}(x; M),$$

$$\frac{\partial^m}{\partial \varkappa^m} [f(-\varkappa, \lambda)\, S^-(x; -\varkappa)]_{\varkappa = M} = (-1)^m \sum_{k=0}^m \binom{m}{k} A(m-k; -M, \lambda)\, S^{-(k)}(x; -M). \tag{3.5}$$

The $A$'s are relatively complicated coefficients which we shall not write down explicitly. It is of some (perhaps academic) interest to consider the points $\lambda = 0, -2,$

$-4, \ldots$ where the spectral function happens to shrink to derivatives of delta-functions:

$$\lambda = 0 : \varrho(\varkappa) = \delta(\varkappa - M) ,$$

$$\lambda = -2N \quad (N = 1, 2, 3 \ldots):$$

$$\begin{aligned}\varrho(\varkappa) &= f(\varkappa, -2N)\, \delta^{2N}(\varkappa - M) + (-1)^{2N-2} f(\varkappa, -2N)\, \delta^{2N-2}(\varkappa + M) \\ &= (-1)^{2N} \sum_{k=0}^{2N} (-1)^k \binom{2N}{k} A(2N - k; M, -2N)\, \delta^{(k)}(\varkappa - M) \\ &+ \sum_{k=0}^{2N-2} (-1)^k \binom{2N-2}{k} A(2N - k - 2; -M, -2N + 2)\, \delta^{(k)}(\varkappa + M) . \end{aligned} \tag{3.6}$$

This means that for $\lambda = -2N$ $(N = 1, 2, 3 \ldots)$

$$\int_{-\infty}^{+\infty} d\varkappa\, \varrho(\varkappa)\, S^-(x; \varkappa) = (-1)^{2N} \sum_{k=0}^{2N} \binom{2N}{k} A(2N - k; M, -2N)\, S^{-(k)}(x; +M)$$

$$+ \sum_{k=0}^{2N-2} \binom{2N-2}{k} A(2N - k - 2; -M, -2N + 2)\, S^{-(k)}(x; -M) . \tag{3.7}$$

However the derivatives of the $S^-$-function with respect to the mass satisfy higher order Klein-Gordon equations:

$$(\Box - M^2)^{k+1} \frac{\partial^k}{\partial M^k} S^-(x; M) = (\Box - M^2)^{k+1} \frac{\partial^k}{\partial M^k} S^-(x; -M) = 0 . \tag{3.8}$$

Therefore in the unphysical points $\lambda = -2N$ the two-point function of our model satisfies

$$(\Box - M^2)^{2N+1}\, W_2(x; \lambda = -2N) = 0 . \tag{3.9}$$

For $\lambda = -2N + 1$ the hypergeometric function has single poles. This has the consequence that no point like support results. We shall not give the result here.

## 4. Perturbation Theory and the Question of Infrared Divergences

Since SCHROER's model describes the interaction of Fermions with massless Bosons, it is generally considered to be a model of the infrared structure of realistic theories such as quantum electrodynamics. Thus one would expect the model to exhibit infrared divergences. However, since all Wightman functions are analytic with respect to the coupling constant $\lambda$ at $\lambda = 0$ (and even in the entire $\lambda$-plane), a Taylor series expansion must lead to finite results. On the other hand, a naive attempt to develop the spectral function, say, would lead to a divergence. The correct expansion involves a distribution-theoretic subtlety which we will localize now. To this end let us consider the spectral function for $\varkappa > 0$,

$$\varrho(\varkappa) = \frac{(\varkappa - M)_+^{\lambda-1}}{\Gamma(\lambda)} f(\varkappa, \lambda) . \quad (\varkappa > 0) \tag{4.1}$$

From appendix 3, formula (A 3.5) we take the expansion

$$\frac{(\varkappa - M)_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{1}{\Gamma(\lambda+1)} \left[ \delta(\varkappa - M) + \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^{m+1} (\varkappa - M)_+^{-1} \log^m (\varkappa - M)_+ \right] . \quad (4.2)$$

If we insert this in (4.1) and also expand the factor $[f(x, \lambda)]/[\Gamma(\lambda + 1)]$, we get the desired expansion of $\varrho(\varkappa)$ for $\varkappa > 0$.

The occurrence of the regularization

$$(\varkappa - M)_+^{-1} \log^m (\varkappa - M)_+$$

instead of the corresponding 'ordinary function' gives to the two-point function a contribution (see Appendix 4 (4.6))

$$\int_M^{\infty} d\varkappa \frac{1}{\varkappa - M} \log^m (\varkappa - M) \left[ f^{(n)}(\varkappa, 0)\, S^-(x; \varkappa) - \theta(1 - \varkappa + M)\, f^{(n)}(M, 0)\, S^-(x; M) \right] ,$$

which is perfectly free from a divergence at $\varkappa = M$.

## 5. Schroer's Model with Zero Fermion Mass

If in Schroer's model the Fermion mass is set equal to zero, the Wightman functions are still entire functions in the coupling constant $\lambda$. However, the analytic continuation of the two-point function cannot be performed by means of the spectral function anymore. This is related to the fact that if the surface of the light cone is included in the support of a Lorentz-invariant distribution, then terms of the type

$$\Box^k \delta(p)$$

may occur [5][6]) which cannot be represented by a spectral function. We shall show that such a shrinking really occurs in the model under consideration. For $M = 0$, the two-point function is, in Fourier space

$$\begin{aligned} \int dx\, e^{ipx} (-x^2 + i\varepsilon x^0)^{-\lambda/2} S^-(x; 0) &= \frac{2\pi i}{\lambda\, 2^{\lambda-1}} \frac{1}{[\Gamma(\lambda/2)]^2} \gamma^\mu p_\mu\, \theta(p^0)\, \theta(p^2) \sqrt{p^2}^{\,-\lambda-2} \\ &= \frac{2\pi i}{\lambda} \gamma^\mu p_\mu Z_\lambda(p) . \end{aligned} \quad (5.1)$$

For $\lambda = -2N$ ($N = 1, 2, 3 \ldots$) this becomes, according to Appendix 4,

$$\frac{\pi}{iN} \gamma^\mu p_\mu \Box^N \delta(p) . \quad (5.2)$$

It is instructive to see how the limit $\lambda \to 0$ has to be performed in Fourier space. To this end we need the obvious

*Lemma:* Let $T_\lambda$ be a distribution which is analytic in $\lambda$ at $\lambda = 0$. Let $f$ be an ordinary function which has for $\lambda \to 0$, a simple pole with residue $R$. If $T_\lambda f_\lambda$ exists and is continuous in $\lambda$ at $\lambda = 0$ (this implies $T_0 R = 0$) then

$$\lim_{\lambda\to 0} T_\lambda f_\lambda = \lim_{\lambda\to 0} T_0 f_\lambda + \left.\frac{\partial T_\lambda}{\partial\lambda}\right|_{\lambda=0} R . \quad (5.3)$$

In our case, $T_\lambda = Z_\lambda(p)$, $T_0 = \delta(p)$, $R = 2\pi i \gamma^\mu p_\mu$ and, since $\delta(p)\gamma^\mu p_\mu = 0$,

$$\lim_{\lambda\to 0} \frac{2\pi i}{\lambda} \gamma^\mu p_\mu Z_\lambda(p) = \left.\frac{\partial Z_\lambda(p)}{\partial\lambda}\right|_{\lambda=0} 2\pi i \gamma^\mu p_\mu . \quad (5.4)$$

On the other hand, formula (A 4.3) of Appendix 4 tells us that

$$\left.\frac{\partial Z_\lambda(p)}{\partial \lambda}\right|_{\lambda=0} = \frac{1}{(2\pi)^2}\int dx\, e^{ipx}\, \frac{\partial}{\partial\lambda}\left(-x^2 + i\,\varepsilon\, x^0\right)^{-\lambda/2}\Big|_{\lambda=0}$$
$$= -\frac{1}{2}\,\frac{1}{(2\pi)^2}\int dx\, e^{ipx}\log\left(-x^2 + i\,\varepsilon\, x^0\right). \tag{5.5}$$

In view of (2.3) this means

$$\lim_{\lambda\to 0}\frac{2\pi i}{\lambda}\,\gamma^\mu\, p_\mu\, Z_\lambda(p) = 2\pi i\,\gamma^\mu\, p_\mu\,\theta(p^0)\,\delta(p^2)\,. \tag{5.6}$$

## 6. The Thirring Model

The Wightman functions resulting from the operator solution of the Thirring model[1]) are

$$\langle 0\,|\,\psi(x)\,\bar\psi(y)\,|\,0\rangle = e^{ig(a-\bar a)D^-(x-y)}\,\langle 0\,|\,\psi_0(x)\,\bar\psi_0(y)\,|\,0\rangle\,, \tag{6.1}$$

$$\langle 0\,|\,\psi(x_1)\ldots\psi(x_n)\,\bar\psi(y_1)\ldots\bar\psi(y_n)\,|\,0\rangle$$
$$= \prod_{i<k} e^{-ig(a-\bar a\gamma^5_{x_i}\gamma^5_{x_k})D^-(x_i-x_k)}\cdot\prod_{i<k} e^{-ig(a-\bar a\gamma^5_{y_i}\gamma^5_{y_k})D^-(y_i-y_k)}$$
$$\cdot\prod_{i,k} e^{ig(a+\bar a\gamma^5_{x_i}\lambda^5_{y_k})D^-(x_i-x_k)}\langle 0\,|\,\psi_0(x_1)\ldots\psi_0(x_n)\,\bar\psi_0(y_1)\ldots\bar\psi(y_n)\,|\,0\rangle\,, \tag{6.2}$$

where $\psi_0$ is a canonical free field, $D^-$ is defined by (2.3), $g$ is the coupling constant and

$$a = \frac{1}{1+(g/2\pi)}\,,\qquad \bar a = \frac{1}{1-(g/2\pi)}\,. \tag{6.3}$$

Since

$$e^{ig(a-\bar a)D^-(x-y)} = \text{const.}\left[-(x-y)^2 + i\,\varepsilon\,(x^0-y^0)\right]^{g(a-\bar a)/4\pi}, \tag{6.4}$$

the two-point function is analytic in $g$ except in the points $g = \pm 2\pi$. If we choose $\gamma^5$ to be diagonal

$$\gamma^5 = \begin{pmatrix}1 & \\ & -1\end{pmatrix},$$

we can represent $\gamma^5_{x_i}\gamma^5_{x_k}$ in the form

$$\gamma^5_{x_i}\gamma^5_{x_k} = \begin{pmatrix}+1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1\end{pmatrix}.$$

In this representation,

$$e^{-ig(a-\bar a\gamma^5_{x_i}\gamma^5_{x_k})D^-(x_i-x_k)} = \begin{pmatrix} e^{-ig(a-\bar a)D^-(x_i-x_k)} & & & \\ & e^{-ig(a+\bar a)D^-(x_i-x_k)} & & \\ & & \cdot\cdot\cdot\cdot\cdot & \\ & & & \cdot\cdot\cdot\cdot\cdot \end{pmatrix}$$

$$= \text{const.}\begin{pmatrix} \left[-(x-y)^2 + i\,\varepsilon\,(x^0-y^0)\right]^{-g(a-\bar a)/4\pi} & & & \\ & \left[-(x-y)^2 + i\,\varepsilon\,(x^0-y^0)\right]^{-g(a+\bar a)/4\pi} & & \\ & & \cdot\cdot\cdot\cdot\cdot & \\ & & & \cdot\cdot\cdot\cdot\cdot \end{pmatrix}. \tag{6.5}$$

If the other terms are represented in an analogous way, one sees that also the higher Wightman functions are analytic in $g$ up to singularities in the points $g = \pm 2\pi$.

In particular the analyticity around $g = 0$ implies that in the Thirring model infrared divergences do not occur either. However the analytic continuation of the two-point function can, again, not be performed by means of a spectral function.

## 7. Appendix

### 1. *Distributions Depending on a Parameter*

In the following $S'$ means the space of tempered distributions $T$, $S$ the space of the corresponding test functions $\varphi$, and $(T, \varphi)$ the value of the functional $T$ for the test function $\varphi$. We may consider a set of distributions $T_\lambda$ labelled by a parameter $\lambda$. If $\lambda$ is complex $T_\lambda$ is said to be analytic in $\lambda_0$, if

$$\lim_{\lambda \to \lambda_0} \frac{T_\lambda - T_{\lambda_0}}{\lambda - \lambda_0} \tag{A 1.1}$$

exists. This is equivalent to the statement that, at $\lambda = \lambda_0$, $(T_\lambda, \varphi)$ is analytic in $\lambda$ for all $\varphi$.

It can be shown that $S'$ is a complete space. This implies that if the limit (A 1.1) exists, it defines again a tempered distribution which is called the derivative of $T_\lambda$ with respect to $\lambda$:

$$\begin{aligned} \left(\frac{\partial T_\lambda}{\partial \lambda}, \varphi\right)_{\lambda=\lambda_0} &\equiv \lim_{\lambda \to \lambda_0} \left(\frac{T_\lambda - T_{\lambda_0}}{\lambda - \lambda_0}, \varphi\right) \\ &= \frac{\partial}{\partial \lambda} (T_\lambda, \varphi)_{\lambda=\lambda_0} . \end{aligned} \tag{A 1.2}$$

The following statement can easily be proved: Let $\varkappa$ be a real parameter. If $(T_\varkappa(x), \varphi(x))$ belong to $S(\varkappa)$ and if $\varrho(\varkappa)$ is a distribution $\epsilon$ $S'(\varkappa)$ then the prescription

$$\varphi(x) \to \big(\varrho(\varkappa), (T_\varkappa(x), \varphi(x))\big)$$

defines a distribution $\epsilon$ $S'(x)$ which we shall denote by

$$\int d\varkappa \, \varrho(\varkappa) \, T_\varkappa(x) \, .$$

### 2. *Spectral Representations*

Consider the two-point function of the free scalar field:

$$D^-(x; \varkappa^2) = \frac{1}{2\pi i} \int d\,p \; e^{-ipx} \, \theta(p^0) \, \delta(p^2 - \varkappa^2) \, . \tag{A 2.1}$$

Let $\varkappa > M$. The expression

$$\big(D^-(x; \varkappa^2), \varphi(x)\big)$$

is differentiable infinitely often and its derivatives drop faster than any power for $\varkappa \to \infty$. Therefore it can be extended to be a function belonging to $S(\varkappa)$. Thus for every $\varrho(\varkappa)$ $\epsilon$ $S'(\varkappa)$ whose support is contained in $\varkappa > M$ a distribution

$$\int d\varkappa \, \varrho(\varkappa) \, D^-(x; \varkappa^2) \in S'(x) \tag{A 2.2}$$

is defined uniquely. Conversely every distribution $\epsilon\, S'(x)$ whose Fourier transform has a support contained in $p^2 > M^2$ can be represented in the form (A 2.2) (see Reference [6])). Note that the point $\varkappa = 0$ ($p^2 = 0$) has to be excluded from the support.

Similar conclusions hold for $S^-(x;\varkappa) = (i\,\gamma^{\mu}\partial_{\mu} + \varkappa)\,D^-(x;\varkappa^2)$ however whereas in one space dimension $\lim_{\varkappa\to 0} D^-(x;\varkappa^2)$ does not exist, $\lim_{\varkappa\to 0} S^-(x;\varkappa)$ is well defined.

For a full discussion we refer to Reference [6]).

### 3. *The Distribution* $\xi_+^{\lambda}$

Let $\xi$ be a single variable. For $\operatorname{Re}\lambda > -1$ we can define a distribution $\xi_+^{\lambda}$ by

$$(\xi_+^{\lambda}, \varphi(\xi)) = \int_0^{\infty} d\xi\, \xi^{\lambda}\, \varphi(\xi)\,. \tag{A 3.1}$$

It is analytic in $\lambda$ and can be continued analytically to all values of $\lambda$. For $\operatorname{Re}\lambda \leqslant -1$ the resulting distribution turns out to be a regularisation of the (non-integrable) function $\xi^{\lambda}$, i.e. (4.1) still holds for all $\varphi(\xi)$ which vanish in some neighbourhood of $\xi = 0$. The results are [7]):

In the strip $-N-1 < \operatorname{Re}\lambda < -N$ ($N = 1, 2, 3 \ldots$) we have

$$(\xi_+^{\lambda}, \varphi) = \int_0^{\infty} d\xi\, \xi^{\lambda} \left[\varphi(\xi) - \sum_{k=0}^{N-1} \frac{\xi^K \varphi^{(K)}(0)}{k!}\right]. \tag{A 3.2}$$

In the points $\lambda = -N$ ($N = 1, 2, 3 \ldots$) $\xi_+^{\lambda}$ has simple poles with residue

$$\frac{(-1)^{N-1}}{(N-1)!}\, \delta^{(N-1)}(\xi)\,. \tag{A 3.3}$$

The normalized function

$$\frac{\xi_+^{\lambda}}{\Gamma(\lambda+1)}$$

is an entire function in $\lambda$ and we have

$$\left.\frac{\xi_+^{\lambda}}{\Gamma(\lambda+1)}\right|_{\lambda=-N} = \delta^{(N-1)}(\xi)\,, \qquad (N = 1, 2, 3 \ldots). \tag{A 3.4}$$

The Taylor series around the point $\lambda = -1$ reads

$$\frac{\xi_+^{\lambda}}{\Gamma(\lambda+1)} = \frac{\delta(\xi)}{\Gamma(\lambda+2)} + \frac{1}{\Gamma(\lambda+2)} \sum_{m=0}^{\infty} \frac{1}{m!}\,(\lambda+1)^{m+1}\, \xi_+^{-1} \log^m \xi_+\,, \tag{A 3.5}$$

where $\xi_+^{-1} \log^m \xi_+$ has to be defined as follows:

$$(\xi_+^{-1} \log^m \xi_+, \varphi) \equiv \int_0^{\infty} d\xi\, \xi^{-1} \log^m \xi\, [\varphi(\xi) - \theta(1-\xi)\,\varphi(0)]\,. \tag{A 3.6}$$

(Note that $\xi_+^{-1}$ is not the value of $\xi_+^{\lambda}$ for $\lambda = -1$ which would not exist.)

## 4. Riesz' *Distribution*[8]

Let $p = (p^0, p^1)$. For $\operatorname{Re}\lambda > 0$ Riesz' distribution $Z_\lambda(p)$ is defined by

$$Z_\lambda(p) = \frac{1}{2^{\lambda-1}\left[\Gamma\left(\frac{\lambda}{2}\right)\right]^2}\,\theta(p^0)\,\theta(p^2)\,\sqrt{p^2}^{\,\lambda-2} \tag{A 4.1}$$

and for $\operatorname{Re}\lambda < 0$ as the analytic continuation of (A 4.1). In the 'singular' points $\lambda = -2N$ $(N = 0, 1, 2 \ldots)$ we have

$$Z_{-2N} = \Box^N\,\delta(p). \tag{A 4.2}$$

The Fourier transform is given by

$$\int dp\, e^{-ipx}\, Z_\lambda(p) = (-x^2 + i\,\varepsilon\,x^0)^{-\lambda/2}. \tag{A 4.3}$$

$Z_\lambda$ is an entire function of $\lambda$.

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