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Autor(en): **Hepp, Klaus**

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On the Analyticity Properties of the Scattering Amplitude in Relativistic Quantum Field Theory

by **Klaus Hepp**

Seminar für theoretische Physik, Eidgenössische Technische Hochschule, Zürich*)

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Abstract. The well-known analyticity properties of the 2-particle scattering amplitude are rigorously derived in the Wightman framework of a local relativistic field theory. No use is made of the LSZ asymptotic condition and of implicit 'technical assumptions'.

§ 0. Introduction

Dispersion relations[1]) have brought a fruitful new approach to the physics of strongly interacting particles. Since these relations are to a high degree model-independent, much work has been done in proving the necessary analyticity properties from a minimal set of mathematically well-defined postulates. The framework of LSZ[15]) or Bogoliubov[4]) has usually been the starting point for these quite involved investigations.

Recently a physically satisfying relativistic scattering theory has been given by Haag[8]) and Ruelle[16]), which is based on the general set of axioms of Wightman[21]) for a local relativistic quantum field theory. On the other hand there has been some doubt[6]) as to whether a rigorous proof of dispersion relations is possible or not. The aim of this investigation is to show that, if one-particle states are created from the vacuum by Wightman fields and if certain mass-spectrum conditions are satisfied, the well-known analyticity properties of the 2-particle scattering amplitude can be rigorously proved. Therefore no further assumptions on the asymptotic behaviour of matrix elements of the interacting fields[15]) or on the existence and regularity of Green's functions (see [19])) are necessary.

Needless to say our considerations are mainly technical, which is also reflected in the choice of a theory of only one kind of neutral scalar massive particles in self-interaction. Although the main idea of the proof is known to many workers in the field, it seemed desirable to fit this mosaic together, in order to clarify the interplay of locality, relativistic invariance and mass-spectrum conditions leading to dispersion relations in relativistic quantum field theory.

I am greatly indebted to many physicists in Zürich and Paris for stimulating discussions, especially to Professors R. Jost and A. S. Wightman and to Drs. M. Froissart, J. Lascoux, and R. Stora. I further wish to thank Dr. L. Motchane for extending to me his kind hospitality at the Institut des Hautes Etudes Scientifiques.

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*) Present address: Institute for Advanced Study, Princeton.

§ 1. Asymptotic Condition and Reduction Formulae

In this section we shall clarify the relation between the Haag-Ruelle asymptotic condition and the LSZ reduction formulae for the 2-particle scattering amplitude. For completeness we start by stating the general assumptions, which characterise the theory of a neutral scalar field in the axiomatic framework of WIGHTMAN[21]).

(A) A neutral scalar field $A(x)$ is a tempered operator-valued distribution[17]). For all $\varphi \in \mathfrak{S}(R^4)$ the linear operators $A(\varphi) \equiv \int A(x)\,\varphi(x)\,dx$ are defined on a common invariant dense linear manifold D in a Hilbert space $\mathfrak{H}$. D and $A(x)$ transform covariantly under a continuous unitary representation $U(a, \Lambda)$ of the inhomogeneous Lorentz group $i\,L_+^\uparrow$:

$$U(a, \Lambda)\, A(x)\, U^{-1}(a, \Lambda) = A(\Lambda\, x + a),\ U(a, \Lambda)\, D \subset D\,. \tag{1.1}$$

The spectrum of the energy-momentum operator P^μ is assumed to lie in the foreward light-cone V_+, except for a one-dimensional eigenspace spanned by the vacuum state Ω, corresponding to the eigenvalue 0. Ω is cyclic with respect to the algebra generated by $\{A(\varphi) : \varphi \in \mathfrak{S}(R^4)\}$. Finally the theory is local:

$$[A(x), A(y)] = 0 \ \text{for}\ (x - y)^2 < 0\,. \tag{1.2}$$

In our investigation $A(x)$ is to be the interpolating field for a relativistic scattering theory of particles of mass m and spin 0. The most natural way is to postulate that the one-particle states of the discrete irreducible representation $[m, 0]$ of $i\,L_+^\uparrow$ are generated by the application of $A(x)$ to the vacuum, which we express by:

$$\text{(B)}\qquad \langle A(x)\, A(y)\rangle_0 = i\,\Delta_m^+\,(x-y) + i\int_M^\infty d\varrho(\mu)\,\Delta_\mu^+\,(x-y),\ (M > m)\,. \tag{1.3}$$

Let $\tilde{A}(p)$ be the Fourier transform of $A(x)$. We consider for $\tilde{f} \in \mathfrak{S}(R^4)$ and $\omega_p \equiv \sqrt{\boldsymbol{p}^2 + m^2}$ the well-defined operator:

$$A(f, t) = \int dx\, A(x)\, f^*(x, t)$$

$$f(x, t) = (2\,\pi)^{-5/2}\int dp\,\tilde{f}(p)\left(\frac{p_0+\omega_p}{2\,\omega_p}\right) e^{i(p_0-\omega_p)t}\, e^{-i(p,x)}\,. \tag{1.4}$$

Let $D_M \equiv \{p : (p, p) < M^2\}$. Then if follows from (1.3) for $f \in \mathfrak{S}(D_M)$ that $A(f, t)\,\Omega = 0$ and that $A^*(f, t)\,\Omega$ is a one-particle state $\Phi_{\hat{f}} = |\hat{f}) \in \mathfrak{H}_{[m,o]}$ with a wave function of the form:

$$f(x) = (2\,\pi)^{-3/2}\int\frac{d^3\boldsymbol{p}}{2\,\omega_p}\,\hat{f}(\boldsymbol{p})\, e^{-i(\omega_p x^0 - \boldsymbol{p}\cdot\boldsymbol{x})},\ \hat{f}(\boldsymbol{p}) = \tilde{f}(\omega_p, \boldsymbol{p}) \in \mathfrak{S}(R^3)\,. \tag{1.5}$$

Under the assumptions (A) and (B) relativistic scattering states can be constructed, as has been shown by HAAG[8]) and RUELLE[16]) in a much more general version of the following theorem:

Theorem 1.1: Under the assumptions (A), (B) let $\mathfrak{P}\,\{A^{(*)}\,(f_i, t)\}$ be an arbitrary polynomial in the $A(f_i, t)$, $A^*(f_j, t)$, $\tilde{f}_i, \tilde{f}_j \in \mathfrak{S}(D_M)$. Then one has strong convergence in $\mathfrak{H}$ for $t \to \pm\infty$ (ex = out, in):

$$s - \lim_{t \to \pm\infty} \mathfrak{P} \{A^{(*)}(f_i, t)\} \, \Omega = \Phi_{\mathfrak{P}}^{ex} \,. \tag{1.6}$$

In particular $s - \lim_{t \to \pm\infty} \prod_{i=1}^{m} A^*(f_i, t) \, \Omega = | \hat{f}_1, \ldots \hat{f}_m^{ex})$ is an asymptotic m-particle state with wave functions (1.5) $\hat{f}_i(\boldsymbol{p}) = \tilde{f}_i(\omega_p, \boldsymbol{p})$.

This theorem can be proved by following closely RUELLE[16])[11]). Yet the use of the local expressions $A(f, t)$ instead of the quasi-local operators of HAAG and RUELLE has to be paid for by a more careful study of the asymptotic behaviour of the truncated vacuum expectation values (TVEV) $\langle \prod_{i=1}^{k} A^{(*)}(f_i, t) \rangle_0^T$ for $k \geqslant 3$. This is contained in the following:

Lemma 1.1: Under the assumption (A)

$$(1 + | t |)^{3/2 \, (k-2)} \langle \prod_{i=1}^{k} A^{(*)}(f_i, t) \rangle_0^T \tag{1.7}$$

is bounded for all t, if $\tilde{f}_i \in \mathfrak{S}(R^4)$ for $1 \leqslant i \leqslant k$, $k \geqslant 3$.

Proof: Using (1.4) the TVEV (1.7) can be written as

$$\int d^4x_1 \ldots d^4x_k \, d^3\boldsymbol{p}_1 \ldots d^3\boldsymbol{p}_k \langle A(-x_1^0, x_1) \ldots A(-x_k^0, x_k) \rangle_0^T \times$$
$$f_1'(x_1^0, \boldsymbol{p}_1) \ldots f_k'(x_k^0, \boldsymbol{p}_k) \exp i \left(\pm (\omega_{p1} t - \boldsymbol{p}_1 \boldsymbol{x}_1) \ldots \pm (\omega_{pk} t - \boldsymbol{p}_k x_k)\right), \tag{1.8}$$

with $f_i' \in \mathfrak{S}(R^4)$. From the asymptotic behaviour of $\langle A(x_1) \ldots A(x_k) \rangle_0^T$ for large space-like separation of the arguments[16])[3]), it follows that for any integer $M \geqslant 0$ there exists an integer $N = N(M) \geqslant 0$ such that

$$\prod_{i=1}^{k} (1 + (x_i^0)^2)^{-N/2} \prod_{l=1}^{k-1} \prod_{j=1}^{3} (1 + (x_{l+1}^j - x_l^j)^2)^{M/2} \langle A(-x_1^0, \boldsymbol{x}_1) \ldots A(-x_k^0, \boldsymbol{x}_k) \rangle_0^T \tag{1.9}$$

is a bounded distribution in x_i^0, $\xi_l^j \equiv x_{l+1}^j - x_l^j$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant 3$, $1 \leqslant l \leqslant k-1$. According to[17]) (vol. II, p. 57) (1.9) can be represented as $\sum_{q=1}^{Q} D_q F_q (x_1^0, \ldots x_k^0, \xi_1, \ldots \xi_{k-1})$ with functions $F_q \in L^\infty (x_1^0, \ldots x_k^0, \xi_1, \ldots \xi_{k-1})$ and monomials D_q in the derivatives $\partial/\partial x_i^0$, $\partial/\partial \xi_l^j$. After partial integrations one can therefore transform (1.8) into

$$\sum_{r=1}^{R} \int d^4x_1 \ldots d^4x_k \, \hat{F}_r (x_1^0, \ldots x_k^0, \xi_1, \ldots \xi_{k-1}) \prod_{l=1}^{k-1} \prod_{j=1}^{3} (1 + (\xi_l^j)^2)^{-1} \times$$
$$\prod_{i=1}^{k} \int d\boldsymbol{p}_i \, e^{\pm i[\omega_{p_i} t - \boldsymbol{p}_i \boldsymbol{x}_i]} f_i^r(x_i^0, \boldsymbol{p}_i) \,, \tag{1.10}$$

with continuous bounded functions $\hat{F}_r$ and with $f_i^r (x_i^0, \boldsymbol{p}_i) \in \mathfrak{S}(R^4)$. Then we can majorize (1.8) by:

$$\sum_{r=1}^{R} c_r \int dx_1^0 \ldots dx_k^0 \, d\boldsymbol{x}_k \max_{\boldsymbol{x}_1 \ldots \boldsymbol{x}_{k-1}} \prod_{i=1}^{k} \left| \int d\boldsymbol{p}_i \, f_i^r (x_i^0, \boldsymbol{p}_i) \, e^{\pm i [\omega_{p_i} t - \boldsymbol{p}_i \boldsymbol{x}_i]} \right| . \tag{1.11}$$

From the asymptotic behaviour of the solutions (1.5) of the Klein-Gordon equation[16])[2]), it follows that (1.11) is bounded for all t when multiplied by $(1 + | t |)^{3/2 \, (k-2)}$, $k \geqslant 3$. This proves theorem 1.1 as in [8]), [16]).

For a pure scattering theory of one kind of neutral scalar particles (see [16]) for the general case) it is reasonable to assume that the asymptotic states are dense in $\mathfrak{H}$, that is:

$$\text{(C)} \qquad \mathfrak{H} = \mathfrak{H}_{ex} = \overline{L}\,\{|\,\hat{f}_1, \ldots \hat{f}_m^{ex}):\ m = 0, 1, \ldots \hat{f}_i \in \mathfrak{S}(R^3)\}, \tag{1.12}$$

where $\overline{L}\,\{\ldots\}$ denotes the closed linear hull of the vectors $\{|\hat{f}_1, \ldots \hat{f}_m^{ex})\}$ and 'ex' stands for 'in' or 'out' (equivalent due to the TCP theorem). Then the following weak convergence theorem can be proved:

Corollary 1.1: Under the assumptions (A), (B), (C) one has in the sense of weak convergence in $\mathfrak{H}$:

$$\text{w} - \lim_{t \to \pm\infty} \prod_{i=1}^{m} A^*(f_i, t)\, \Omega = |\,\hat{f}_1, \ldots \hat{f}_m^{ex})\,, \tag{1.13}$$

$$\text{w} - \lim_{t \to \pm\infty} \prod_{i=1}^{m} A(f_i, t) \prod_{j=1}^{n} A^*(g_j, t)\, \Omega = \prod_{i=1}^{m} A_{ex}(\hat{f}_i)\, |\hat{g}_1, \ldots \hat{g}_n^{ex})\,, \tag{1.14}$$

$$\text{for } \tilde{f}_i \in \mathfrak{S}(R^4)\,,\ \tilde{g}_j \in \mathfrak{S}(D_{2m}) \text{ with } \hat{f}_i(\boldsymbol{p}) = \tilde{f}_i(\omega_p, \boldsymbol{p})\,,\ \hat{g}_j(\boldsymbol{p}) = \tilde{g}_j(\omega_p, \boldsymbol{p})\,.$$

The proof of H. Araki[2]) for local rings of observables can be immediately translated into the Wightman framework[9]) using lemma 1.1 for the majorization of the higher TVEV.

Let $\overline{D} \equiv \bigcap_{B \in \mathfrak{P}} D_{\overline{B}}$ be the intersection of the domains of the closures $\overline{B} = B^{**}$ of all quasi-local operators $B \in \mathfrak{P}$ (polynomials in the smeared-out fields). Then one can prove the following version of the LSZ asymptotic condition[15]):

Corollary 1.2: Under (A) and (B) one has

(a) $|\,\hat{g}_1 \ldots \hat{g}_n^{ex}) \in \overline{D}$ for non-overlapping wave-packets $\hat{g}_i \in \mathfrak{D}(R^3)$ (supp $\hat{g}_i \cap$ supp $\hat{g}_j = \phi$ for $i \neq j$).

(b) $\overline{B}\,|\,\hat{g}_1 \ldots \hat{g}_n^{ex}) = s - \lim_{t \to \pm\infty} B \prod_{i=1}^{n} A^*(g_i, t)\, \Omega \in \overline{D}$ *)

$\overline{B}_1\, (\overline{B}_2\, |\hat{g}_1 \ldots \hat{g}_n^{ex})) = \overline{B_1\, B_2}\, |\hat{g}_1 \ldots \hat{g}_n^{ex})$ for $B, B_1, B_2 \in \mathfrak{P}$.

(c) $s - \lim_{t \to \pm\infty} A^{(*)}(f, t)\,|\,\hat{g}_1 \ldots \hat{g}_n^{\substack{out\\in}}) = A^{(*)}_{\substack{out\\in}}(\hat{f})\,|\,\hat{g}_1 \ldots \hat{g}_n^{\substack{out\\in}})$ for $\tilde{f} \in \mathfrak{S}(D_M)$ (and for $\tilde{f} \in \mathfrak{S}(R^4)$ in the weak topology in $\mathfrak{H}_{ex}$

For $n = 0, 1$ the proof follows immediately from theorem 1.1 and corollary 1.1. All the reduction formulae necessary for the proof of the analyticity properties of the 2-particle scattering amplitude can be derived using only this information. For $n > 1$ and $B \in \mathfrak{P}$ one uses the Schwarz inequality:

$$\|\, B \frac{d}{dt} \prod_{i=1}^{n} A^*(g_i, t)\, \Omega \,\|^2 \leqslant \|\, \frac{d}{dt} \prod_{i=1}^{n} A^*(g_i, t)\, \Omega \,\|\; \|\, B^*\, B \frac{d}{dt} \prod_{i=1}^{n} A^*(g_i, t)\, \Omega \,\|\,. \tag{1.15}$$

*) One can choose e.g. $\tilde{g}_i(p) = \hat{g}_i(\boldsymbol{p})\, \hat{\alpha}_i\, (\sqrt{\boldsymbol{p}^2 + m^2} - p^0)$ with $\hat{\alpha}_i \in \mathfrak{D}(\langle -\varepsilon, +\varepsilon\rangle)$, $\varepsilon > 0$ sufficiently small, $\sqrt{2\,\pi}\, \hat{\alpha}(0) = 1$.

For non-overlapping wave-packets $\{\hat{g}_i\}$ the first factor decreases stronger than any power of $(1 + |t|)^{-1}$ for $t \to \pm\infty$ (see [9]), whilst the second factor increases only polynomially in $|t|$, due to the temperedness of the WIGHTMAN distributions. This proves the convergence of (b) and similarly of (c). One can see that on the smooth non-overlapping states $|\hat{g}_1 \dots \hat{g}_n^{ex})$ the mapping $\varphi \to \overline{A(\varphi)}\,|\hat{g}_1 \dots \hat{g}_n^{ex})$ is a vector-valued tempered distribution. Finally corollary 1.2 can be extended to the general Haag-Ruelle scattering theory *).

Under the assumptions (A), (B), (C) the Fock spaces $\mathfrak{H}_{in}$ and $\mathfrak{H}_{out}$ are related by a unitary S-matrix, which is defined in terms of its matrix elements:

$$S_{mn}(\tilde{f}_1^*, \dots \tilde{f}_m^*, \tilde{g}_1, \dots \tilde{g}_n) \equiv (\hat{f}_1, \dots \hat{f}_m^{out} \mid \hat{g}_1, \dots \hat{g}_n^{in}) \,. \tag{1.16}$$

The tempered distributions

$$S_{mn}(p_1, \dots p_m, -p_{m+1}, \dots -p_{m+n}) = \prod_{i=1}^{m+n} \theta_0(p_i)\, \delta(p_i^2 - m^2) \langle \hat{a}_{out}(\mathbf{p}_1) \dots \hat{a}_{in}^*(\mathbf{p}_{m+n}) \rangle_0 \tag{1.17}$$

are $L_+^\uparrow$- and TCP-invariant, symmetric in the $\{p_1, \dots p_m\}$ and $\{p_{m+1}, \dots p_{m+n}\}$ and have their supports on the mass shell $\{p_i^2 = m^2, p_i^0 > 0, 1 \leqslant i \leqslant m+n, \sum_{i=1}^{m} p_i = \sum_{j=m+1}^{m+n} p_j\}$ as well as the scattering amplitudes T_{mn}, defined in (1.16) by $S = 1 + i\,T$.

The aim of our investigation is to derive analyticity properties of the 2-particle scattering amplitude T_{22}. For that purpose we express T_{22} and its 'imaginary part' by certain matrix elements of retarded or time-ordered products:

$$R(x, x_1, \dots x_n) \equiv (i)^n \sum_{p \in \gamma^n} \theta(x - x_{p(1)}) \dots \theta(x_{p(n-1)} - x_{p(n)}) \, [[A(x), A(x_{p(1)})], \dots A(x_{p(n)})]$$

$$T(x_1, \dots x_n) \equiv \sum_{p \in \gamma^n} \theta(x_{p(1)} - x_{p(2)}) \dots \theta(x_{p(n-1)} - x_{p(n)}) \, A(x_{p(1)}) \dots A(x_{p(n)}) \,. \tag{1.18}$$

R and T are well-defined for C^∞-functions θ with supp $(\theta - \theta_0)$ compact (θ_0: Heaviside step function). Let $\delta_{ab}^m(p)$ be the $O_M(R^4)$-function:

$$\delta_{ab}^m(p) \equiv [2\pi i(p_0 - \omega_p)\, 2\,\omega_p]^{-1} \{e^{i(p_0 - \omega_p)a} - e^{-i(p_0 - \omega_p)b}\}, \tag{1.19}$$

which converges to $\theta_0(p)\,\delta(p^2 - m^2)$ for a, b tending independently to ∞. In this notation one can prove the following 'reduction formula'[15])[4]):

Theorem 1.2: Under the assumptions (A) and (B) one has in the strong topology for $p_1^2, q_1^2 < M^2$:

$$T_{22}(p_1, p_2, -q_1, -q_2) =$$

$$2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta_{ab}^m(p_1)^*\, \delta_{cd}^m(q_1)\, [(p_1^2 - m^2)(q_1^2 - m^2) \langle p_2 \mid \tilde{R}(p_1, -q_1) \mid q_2 \rangle] \,, \tag{1.20}$$

independently of the order of $\lim_{a,b\to\infty}$ and $\lim_{c,d\to\infty}$. If (C) holds in addition, then (1.20) is true for all p_1, q_1.

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*) Professor R. HAAG has kindly informed me that results similar to those in corollary 1.2 have been obtained by him and D. W. ROBINSON.

Proof: We choose $\tilde{f}_1, \tilde{g}_1 \in \mathfrak{S}(D_M)$ ($\in \mathfrak{S}(R^4)$, if (C) holds), $\hat{f}_2, \hat{g}_2 \in \mathfrak{S}(R^3)$, and a C^∞-function θ with supp $(\theta - \theta_0)$ compact. Then one proves that for fixed s the limit

$$\lim_{t \to -\infty} \int dx\, dy\, f_1^*(x, s)\, g_1(y, t)\, \theta(x - y) \langle \hat{f}_2 \mid A(x)\, A(y) \mid \hat{g}_2 \rangle \tag{1.21}$$

is equal to ($\hat{g}_1(\boldsymbol{p}) = \tilde{g}_1(\omega_p, \boldsymbol{p})$ and using theorem 1.1 or corollary 1.1):

$$\begin{aligned} &= \lim_{t \to -\infty} \int dx\, dy\, f_1^*(x, s)\, g_1(y, t) \langle \hat{f}_2 \mid A(x)\, A(y) \mid \hat{g}_2 \rangle = \\ &= \lim_{t \to -\infty} (A^*(f_1, s)\, \Phi_{\hat{f}_2}, A^*(g_1, t)\, \Phi_{\hat{g}_2}) = (A^*(f_1, s)\, \Phi_{\hat{f}_2}, \Phi^{in}_{\hat{g}_1 \hat{g}_2}) \,. \end{aligned} \tag{1.22}$$

This follows from the fact that for supp $(\theta - \theta_0)$ compact there exists an $R > 0$, such that the tempered distribution $(1 - \theta(x - y)) \langle \hat{f}_2 \mid A(x)\, A(y) \mid \hat{g}_2 \rangle$ has support in $G_R \equiv \{(x, y) \in R^8,\ y_0 > x_0 - R\}$. Therefore (1.21) is equal to (1.22) in the limit $t \to -\infty$, if for fixed s

$$\lim_{t \to -\infty} f_1^*(s, x)\, g_1(t, y) = 0 \tag{1.23}$$

in the topology of $\mathfrak{S}(G_R)$. For any monomial $\mathfrak{P}_1(y)$ in the y_i and $\mathfrak{P}_2(\partial/\partial y)$ in the $\partial/\partial y_i$ one has

$$\begin{aligned} &\mathfrak{P}_1(y)\, \mathfrak{P}_2\left(\frac{\partial}{\partial y}\right) g_1(y, t) = \\ &\quad = (2\pi)^{-5/2} \int dp\, e^{-i(p, y)} \left[\mathfrak{P}_1\left(-i \frac{\partial}{\partial p}\right) \mathfrak{P}_2(-i\, p)\, \tilde{g}_1(p) \left(\frac{\omega_p + p_0}{2\, \omega_p}\right) e^{i(p_0 - \omega_p)t} \right] \\ &\quad = \sum_{\mu=0}^{M} t^\mu \int dp\, e^{-i(p, y) + i(p_0 - \omega_p)t}\, \tilde{g}_{1,\mu}(p) \,, \end{aligned} \tag{1.24}$$

with $\tilde{g}_{1,\mu} \in \mathfrak{S}(R^4)$ and $M = M(\mathfrak{P}_1)$. Therefore:

$$\begin{aligned} &| y_0 - t |^L \, | \mathfrak{P}_1(y)\, \mathfrak{P}_2\left(\frac{\partial}{\partial y}\right) g_1(y, t) | \leqslant \\ &\leqslant \sum_{\mu=0}^{M} | t |^\mu \left| \int dp\, e^{-i(p, y) + i(p_0 - \omega_p)t} \left(\frac{\partial}{\partial p_0}\right)^L \tilde{g}_{1,\mu}(p) \right| \leqslant \sum_{\mu=0}^{M} c_\mu^L\, | t |^\mu \,, \end{aligned} \tag{1.25}$$

for all L and t. Finally one has for fixed s and $K, L > M$ and for sufficiently large $|t|$:

$$\begin{aligned} &\sup_{y_0 > x_0 - R} \left| \mathfrak{P}_1(x, y)\, \mathfrak{P}_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f_1^*(s, x)\, g_1(t, y) \right| \leqslant \\ &\leqslant \sup_{y_0 > x_0 - R} c\, | t |^M (| x_0 - s |^K + 1)^{-1} (| y_0 - t |^L + 1)^{-1} \to 0, \end{aligned} \tag{1.26}$$

for $t \to -\infty$. This proves (1.23).

Applying theorem 1.1 or corollary 1.1 again to (1.21) one obtains:

$$\begin{aligned} &\lim_{s \to \pm\infty} \lim_{t \to -\infty} \int dx\, dy\, f_1^*(x, s)\, g_1(y, t)\, \theta(x - y) \langle \hat{f}_2 \mid A(x)\, A(y) \mid \hat{g}_2 \rangle \\ &\qquad = \langle \hat{f}_1 \hat{f}_2 {}^{out}_{in} \mid \hat{g}_1 \hat{g}_2{}^{in} \rangle \,. \end{aligned} \tag{1.27}$$

By the same argument one can show that:

$$\lim_{t\to+\infty} \int dx\, dy\, f_1^*(s,x)\, g_1(y,t)\, \theta(x-y) \langle \hat{f}_2 \mid A(x)\, A(y) \mid \hat{g}_2\rangle = 0$$

$$= \Big(\lim_{s\to+\infty} - \lim_{s\to-\infty}\Big) \lim_{t\to\pm\infty} \int dx\, dy\, f_1^*(x,s)\, g_1(y,t)\, \theta(x-y) \mid \langle \hat{f}_2 \mid A(y)\, A(x) \mid \hat{g}_2\rangle . \tag{1.28}$$

Collecting these terms one obtains in momentum space:

$$\langle \hat{f}_1 \hat{f}_2^{out} \mid \hat{g}_1 \hat{g}_2^{in} \rangle - \langle \hat{f}_1 \hat{f}_2^{out} \mid \hat{g}_1 \hat{g}_2^{out} \rangle = 2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty}$$

$$\times \int dp\, dq\, \tilde{f}_1^*(p)\, \tilde{g}_1(q)\, \delta^m_{ab}(p)^*\, \delta^m_{cd}(q)\, [(p^2-m^2)(q^2-m^2) \langle \hat{f}_2 \mid \tilde{R}(p,-q) \mid \hat{g}_2\rangle . \tag{1.29}$$

In (1.29) a, b and c, d, respectively, tend independently to ∞ and the order of $\lim_{a,b\to\infty}$ and $\lim_{c,d\to\infty}$ is immaterial. The majorizations leading to (1.27) and (1.28) can be carried through uniformly in $\tilde{f}_1 \in \tilde{B}_1$, $\tilde{g}_1 \in \tilde{B}_1'$, $\hat{f}_2 \in \hat{B}_2$, $\hat{g}_2 \in \hat{B}_2'$ for bounded sets $\tilde{B}_1, \tilde{B}_1' \subset \mathfrak{S}(D_M)$, $\hat{B}_2, \hat{B}_2' \subset \mathfrak{S}(R^3)$. As a consequence of the nuclear theorem[7]) the limit (1.29) equally holds for any $\Psi \in \mathfrak{S}(R^6 \times D_M \times D_M)$. Since weak and strong sequential limits are identical in the topology of $\mathfrak{S}'$ [17]), theorem 1.2 is proved.

Remark: One sees that the reduction formula (1.20) can be derived for $p_1^2, q_1^2 < M^2$ without using the postulate (C) of asymptotic completeness. In such a framework T_{22} is again defined by (1.16) but might lose its physical significance as scattering amplitude. For the proof of dispersion relations for T_{22} one has then to make assumptions on the mass spectrum of the theory (see postulate (D) in section 2). Another reduction formula will be necessary for the proof of the analyticity properties of T_{22}:

Theorem 1.3: Under (A), (B) one has (independently of the order of $\lim_{a,b\to\infty}$ and $\lim_{c,d\to\infty}$) for $p_1^2, p_2^2 < M^2$:

$$T_{22}(p_1, p_2, -q_1, -q_2)$$

$$= 2\pi \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta^m_{ab}(p_1)^*\, \delta^m_{cd}(p_2)^*\, [(p_1^2-m^2)(p_2^2-m^2) \langle \tilde{R}(p_1,p_2) \mid q_1 q_2^{in} \rangle] \tag{1.30}$$

and for $p_1^2, p_2^2, q_1^2, q_2^2 < M^2$ (with $a, b, c, d \to \infty$ independently):

$$(p_1^2-m^2)(q_1^2-m^2) \langle p_2 \mid \tilde{A}(p_1)\, \tilde{A}(-q_1) \mid q_2\rangle$$

$$= 2\pi \lim_{a,b,c,d\to\infty} \delta^m_{ab}(p_2)^*\, \delta^m_{cd}(q_2)\, [(p_1^2-m^2)(p_2^2-m^2)(q_1^2-m^2)(q_2^2-m^2) \tag{1.31}$$

$$\times \langle \tilde{R}(p_1, p_2 \mid \tilde{R}(-q_1, -q_2)\rangle] .$$

Proof: One chooses $\tilde{f}_1, \tilde{f}_2 \in \mathfrak{S}(D_M)$ with $\tilde{f}_i(\omega_p, \boldsymbol{p}) = \hat{f}_i(\boldsymbol{p})$. Then theorem 1.1 gives for all t:

$$\mid \hat{f}_1 \hat{f}_2{}^{out}_{in} \rangle = \lim_{s\to\pm\infty} A^*(f_1, s)\, A^*(f_2, t)\, \Omega . \tag{1.32}$$

Furthermore one has as in theorem 1.2 for fixed s:

$$A^*(f_1, s)\, A^*(f_2, t)\, \Omega = \lim_{t\to-\infty} \int dx\, dy\, f_1^*(x,s)\, f_2^*(y,t)\, \theta(x-y)\, A(x)\, A(y)\, \Omega \tag{1.33}$$

for any C^∞-function θ with supp $(\theta - \theta_0)$ compact. Since the other terms in (1.30) and (1.31) can be treated in the same way, theorem 1.3 is proved in the weak and therefore in the strong topology.

We finally remark that T_{22} can be expressed (essentially as in [15]) by the VEV of a 4-fold retarded or time-ordered product (1.18) of $A(x)$.

§ 2. Integral Representations

The reduction formulae in section 1 gave a large class of 'off-shell' extrapolations for the 2-particle scattering amplitude T_{22} and for the 1-particle matrix element $\langle p_2 \mid \tilde{j}(p_1)\, \tilde{j}(-q_1) \mid q_2\rangle$ of the currents $\tilde{j}(p) = (p^2 - m^2)\, \tilde{A}(p)$. All these retarded expressions derived from the 4-point function $\langle A(x_1) \ldots A(x_4)\rangle_0$ turned out to be equivalent on the mass shell.

In this section we shall construct a class of 'sharp' admissible extrapolations, which are tempered $i\, L_+^\uparrow$-invariant distributions with the necessary support properties in x- and p-space, for the proof of the analyticity properties of T_{22}. The construction is based on two lemmas on invariant distributions.

Let $\mu > 0$ and $\overline{V}_+^\mu \equiv \{p : p_0 \geqslant \sqrt{\boldsymbol{p}^2 + \mu^2}\}$. Let $\mathring{\mathfrak{S}}'(\overline{V}_+^\mu \times R^{4n}; L_+^\uparrow)$ be the subspace of $L_+^\uparrow$-invariant tempered distributions $T \in \mathfrak{S}'(R^{4(n+1)})$ with supp $T \subset \overline{V}_+^\mu \times R^{4n}$. Let $[\mu^2, \infty\rangle$ be the interval $\mu^2 \leqslant t < \infty$, let O_+ be the proper real rotation group in 3 dimensions and $\mathring{\mathfrak{S}}'([\mu^2, \infty\rangle \times R^{4n}; O_+)$ be the subspace of tempered distributions $\tilde{T}(t, q_1^0, \boldsymbol{q}_1, \ldots q_n^0, \boldsymbol{q}_n)$ with supp $\tilde{T} \subset [\mu^2, \infty\rangle \times R^{4n}$, which are O_+-invariant in $\boldsymbol{q}_1, \ldots \boldsymbol{q}_n$. Then one has:

Lemma 2.1: $\mathring{\mathfrak{S}}'(\overline{V}_+^\mu \times R^{4n}; L_+^\uparrow)$ and $\mathring{\mathfrak{S}}'([\mu^2, \infty\rangle \times R^{4n}; O_+)$ are (topologically) isomorphic.

Proof: For any $p \in V_+$ let $L(p)$ be the pure Lorentz transformation into the standard rest-frame of p corresponding to the $A(p) \in SL\,(2, C)$ (σ_i: Pauli matrices):

$$A(p) = \left[2\sqrt{(p,p)}\, \left(\sqrt{(p,p)} + p_0\right)\right]^{-1} \left\{ \left(\sqrt{(p,p)} + p_0\right) \sigma_0 - \boldsymbol{p}\, \boldsymbol{\sigma} \right\}. \tag{2.1}$$

Then for any $\mu > \overline{\mu} > 0$ and $\varphi \in \mathfrak{S}(V_+^{\overline{\mu}} \times R^{4n})$ the function

$$(M\,\varphi)\,(t, p_1, \ldots p_n) \equiv \int dp\; \delta((p,p) - t)\; \varphi(p, L^{-1}(p)\, p_1, \ldots L^{-1}(p)\, p_n)) \tag{2.2}$$

lies in $\mathfrak{S}(\langle\overline{\mu}^2, \infty\rangle \times R^{4n})$ (see e.g. [7]). One can further prove that the linear mapping

$$M\colon \mathfrak{S}(V_+^{\overline{\mu}} \times R^{4n}) \to \mathfrak{S}(\langle\overline{\mu}^2, \infty\rangle \times R^{4n}) \tag{2.3}$$

is onto and continuous. The homomorphism M defines then an isomorphism M' between $\mathring{\mathfrak{S}}'(\overline{V}_+^\mu \times R^{4n}; L_+^\uparrow)$ and $\mathring{\mathfrak{S}}'([\mu^2, \infty\rangle \times R^{4n}; O_+)$ by

$$\langle T, \varphi\rangle = \langle M'\, T, M\,\varphi\rangle\,. \tag{2.4}$$

In this sense $T \in \mathring{\mathfrak{S}}'(\overline{V}_+^\mu \times R^{4n}; L_+^\uparrow)$ is given by its 'value' $\tau \in \mathring{\mathfrak{S}}'([\mu^2, t\rangle \times R^{4n}; O_+)$ in the standard rest-frame of p:

$$T(p, p_1, \ldots p_n) = \tau\big((p,p),\, L(p)\, p_1, \ldots L(p)\, p_n\big)\, \theta_0(p)\,, \tag{2.5}$$

a formal notation, which will be convenient in the sequel.

Let $\mathfrak{S}'(R^6; O_+)$ be the subspace of tempered distributions $F(\boldsymbol{q}_1, \boldsymbol{q}_2)$, which are O_+-invariant in $\boldsymbol{q}_1$, $\boldsymbol{q}_2$. Let D be the closed convex set

$$\{(a, b, c) \in R^3 \colon a, c \geqslant 0, a\, c \geqslant b^2\}\,.$$

Then the following lemma holds:

Lemma 2.2: The spaces $\mathfrak{S}'(R^6; O_+)$ and $\mathring{\mathfrak{S}}'(D)$ are (topologically) isomorphic.

Proof: Let dR be the invariant Haar measure of O_+ normalized to $\int_{O_+} dR = 1$. Then the mapping

$$N_1\colon\ \varphi(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \int_{O_+} dR\, \varphi(R^{-1}\, \boldsymbol{x},\, R^{-1}\, \boldsymbol{y}) \tag{2.6}$$

is a continuous projection from $\mathfrak{S}(R^6)$ onto the subspace $\mathfrak{S}(R^6; O_+)$ of the O_+-invariant testing functions ($\varphi(\boldsymbol{x}, \boldsymbol{y}) = \varphi(S^{-1}\, \boldsymbol{x}, S^{-1}\, \boldsymbol{y})$ for all $S \in O_+$). As in [18]) one can see that the mapping

$$N_2\colon\ \psi(a, b, c) \rightarrow \psi(\boldsymbol{x}^2, \boldsymbol{x}\, \boldsymbol{y}, \boldsymbol{y}^2) \tag{2.7}$$

gives a topological isomorphism between $\mathfrak{S}(R^6; O_+)$ and $\mathfrak{S}(D)$. For N_2 is evidently one-to-one, linear and continuous. Furthermore to every $\varphi \in \mathfrak{S}(R^6; O_+)$ there exists an in D continuous function ψ such that $\psi(\boldsymbol{x}^2, \boldsymbol{x}\, \boldsymbol{y}, \boldsymbol{y}^2) = \varphi(\boldsymbol{x}, \boldsymbol{y})$. By evaluation of the derivatives of ψ in special O_+-frames for φ one can show by complete induction that $\psi \in \mathfrak{S}(D)$ and that N_2^{-1} is continuous. Setting $N \equiv N_2^{-1} \circ N_1$ the isomorphism N' between $\mathfrak{S}'(R^6; O_+)$ and $\mathring{\mathfrak{S}}'(D)$ is given by duality:

$$\langle T, \varphi \rangle = \langle N'\, T, N\, \varphi \rangle\,. \tag{2.8}$$

If $T(\boldsymbol{x}, \boldsymbol{y})$ is a O_+-invariant tempered continuous function, then there exists in D a tempered continuous function $\tilde{T}$ with $\tilde{T}(\boldsymbol{x}^2, \boldsymbol{x}\, \boldsymbol{y}, \boldsymbol{y}^2) = T(\boldsymbol{x}, \boldsymbol{y})$. For $\varphi \in \mathfrak{S}(R^6)$ and $\tilde{\varphi} = N\, \varphi \in \mathfrak{S}(D)$ one has

$$\langle T, \varphi \rangle = 2\, \pi^2 \int \tilde{T}(\boldsymbol{x}^2, \boldsymbol{x}\, \boldsymbol{y}, \boldsymbol{y}^2)\, \tilde{\varphi}(\boldsymbol{x}^2, \boldsymbol{x}\, \boldsymbol{y}, \boldsymbol{y}^2)\, d\boldsymbol{x}^2\, d\boldsymbol{x}\, \boldsymbol{y}\, d\boldsymbol{y}^2 \tag{2.9}$$

and therefore $N'\, T = 2\, \pi^2\, \tilde{T} \in \mathring{\mathfrak{S}}'(D)$. In the sense of (2.8) we shall sometimes use the functional notation (2.9) in section 3.

Let $I\colon R^{12} \rightarrow R^6$ be the mapping into the $L_+^\uparrow$-invariants:

$$I\colon\ (p_1, p_2, p_3) \in R^{12} \rightarrow \big((p_1, p_1), \ldots (p_3, p_3)\big) \in R^6\,. \tag{2.10}$$

Let $G \subset R^6$ be the closed convex I-image of $\overline{V}_+^\mu \times R^8$. Then by combining lemma 2.1 and 2.2 and using the nuclear theorem of L. SCHWARTZ[7]) one obtains:

Lemma 2.3: The spaces $\mathring{\mathfrak{S}}'(\overline{V}_+^\mu \times R^8; L_+^\uparrow)$ and $\mathring{\mathfrak{S}}'(G)$ are (topologically) isomorphic.

In trying to generalize lemma 2.3 to more than 3 four-vectors one encounters the difficulty that the image of $\overline{V}_+^\mu \times R^{4n}$, $n \geqslant 3$, in the space of the $L_+^\uparrow$-invariants is an algebraic variety $\hat{V} \subset R^m$ $\left(m = (n+1)\,(n+2)/2 + \binom{n+1}{4}\right)$ with singularities, on which no reasonable spaces of testing functions have yet been defined. For $L_+^\uparrow$-invariant continuous functions $T(p_0, \dots p_n)$ with supp $T \subset \overline{V}^{\mu}_{+} \times R^{4n}$, however, there exists a continuous function $\hat{T}$ on $\hat{V}$ with $T(p_0, \dots p_1) = \hat{T}((p_0, p_0), \dots (p_n, p_n))$, a frequently-used result in relativistic scattering theory.

Lemma 2.1 justifies the study of Lorentz invariant distributions in special Lorentz frames, if certain support properties are satisfied. Furthermore the construction of a sharp extrapolation for T_{22} or for $\langle p_2 \mid \tilde{j}(p_1)\, \tilde{j}(-q_1) \mid q_2 \rangle$ in the Breit- or centre-of-mass system can be $i\, L^{\uparrow}_{+}$-invariantly extended into an arbitrary Lorentz frame. This we shall illustrate by constructing and discussing the sharp 2-fold retarded commutator $\langle \tilde{R}_0(p_1, p_2)\, \tilde{R}_0(-q_1, -q_2) \rangle_0$.

Let E_0 be the projection on the vacuum and E_1 the projector on the 1-particle space $\mathfrak{H}_{[m,0]}$. Then in order to derive analyticity properties of the 2-particle scattering amplitude we have to make some assumptions on the mass spectrum of the theory. For convenience we postulate for the spectral measure $E(\Delta) = \int_{\Delta} dE(p)$ corresponding to $U(a, 1) = \int e^{i(p,a)}\, dE(p)$ [11]:

$$\text{(D)} \qquad (\mathbf{1} - E_0 - E_1)\, E(\Delta) = 0 \ \text{ for } \Delta \cap \{p_0 \geqslant \sqrt{4\, m^2 + \boldsymbol{p}^2}\} = \phi\,,$$

(D) follows from (C) for an asymptotic complete theory of one kind of $[m, 0]$-particles. More general mass-spectra are treated in the literature [1]).

The 2-fold commutator $\langle [\tilde{A}(p_1)\, \tilde{A}(p_2)]\, E_0^{\perp}\, [\tilde{A}(-q_1)\, \tilde{A}(-q_2)] \rangle_0$ has support in $\{p_1 + p_2 \in \overline{V}^m_+\}$ and is of the form:

$$\delta(p_1 + p_2 - q_1 - q_2)\; \theta_0(p_1 + p_2)$$
$$\tau_{00}\Big((p_1 + p_2)^2,\;\; L(p_1 + p_2)\, \frac{p_1 - p_2}{2}\,,\;\; L(p_1 + p_2)\, \frac{q_1 - q_2}{2}\Big)\,, \tag{2.11}$$

as a consequence of translation invariance and of Lemma 2.1. The tempered distribution $\tau_{00}\,(s, k_2, k_3)$ is O_+-invariant in $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and has its support in $\{s \geqslant m^2,\ |\, k_i^0 \,| \geqslant \sqrt{m^2 + \boldsymbol{k}_i^2} - {}^1/_2 \sqrt{s},\ i = 1, 2\}$. Since the partial Fourier transform $\tilde{\tau}_{00}\,(s, \xi_2, \xi_3)$ vanishes for $(\xi_2, \xi_2) < 0$ or $(\xi_3, \xi_3) < 0$, $\tau_{00}\,(s, k_2, k_3)$ can be extended [5])[22]) to a rotation-symmetric tempered solution $\hat{\tau}_{00}\,(s, K_2, K_3)$ of the 6-dimensional wave equation in K_2, K_3 $(K_i = (k_i^0, \boldsymbol{k}_i, k_i^4, k_i^5))$.

With the same methods as for a simple 6-dimensional wave-equation (see e.g. [20])) it can be shown that the Cauchy problem has a unique solution in terms of the Cauchy data on the surface $\{k_2^0 = k_3^0 = 0\}$. This gives a 2-fold Jost-Lehmann-Dyson (JLD) representation [12])[5])[22]) for $\tau_{00}(s, k_2, k_3)$, which is (because of the symmetry of supp τ_{00}) of the form:

$$\tau_{00}(s, k_2, k_3) = \int d\boldsymbol{u}_2\, d\boldsymbol{u}_3\, d\varkappa_2^2\, d\varkappa_3^2\, \varepsilon(k_2^0)\, \varepsilon(k_3^0)\, \delta\big((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 - \varkappa_2^2\big)$$
$$\times\, \delta\big((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 - \varkappa_3^2\big)\, [\Phi_{11} + k_2^0 \Phi_{21} + k_3^0\, \Phi_{12} + k_2^0\, k_3^0\, \Phi_{22}] \tag{2.12}$$

The integral (2.12) is to be understood in the sense that for $\varphi\,(k_2, k_3) \in \mathfrak{S}(R^8)$

$$\int dk_2\, dk_3\, \varphi(k_2, k_3)\, \varepsilon(k_2^0)\, \varepsilon(k_3^0)\, \delta\big((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 - \varkappa_2^2\big)\, \delta\big((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 - \varkappa_3^2\big)$$

is a testing function in the variables $\boldsymbol{u}_2, \boldsymbol{u}_3, \varkappa_2^2, \varkappa_3^2$. The $\Phi_{ij} = \Phi_{ij}\,(\boldsymbol{u}_2, \boldsymbol{u}_3, \varkappa_2^2, \varkappa_3^2, s)$ (being derivatives of the initial values of $\hat{\tau}_{00}$, $\hat{\tau}_{00,\,k_2^0}$, $\hat{\tau}_{00,\,k_3^0}$, $\hat{\tau}_{00,\,k_2^0,\,k_3^0}$ [22])) are tempered distributions, in $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ O_+-invariant, which have for fixed $s \geqslant m^2$ their support in

$$G(s) = \left\{\boldsymbol{u}_i^2 \leqslant \frac{s}{4}, \ \varkappa_j \geqslant \max\left\{0, m - \left(\frac{s}{4} - \boldsymbol{u}_j^2\right)^{1/2}\right\}, \ i, j = 1, 2\right\}. \qquad (2.13)$$

In the standard construction[1]) of a sharp retarded commutator $\langle \tilde{R}(p_1, p_2)\, E_0^{\perp}\, \tilde{R}(-q_1, -q_2)\rangle_0$ one studies the integral

$$\tau(s, k_2, k_3) = \int d\boldsymbol{u}_2\, d\boldsymbol{u}_3\, d\varkappa_2^2\, d\varkappa_3^2\, \chi\, [\Phi_{11} + k_2^0 \Phi_{21} + k_3^0 \Phi_{12} + k_2^0\, k_3^0 \Phi_{22})]\,, \qquad (2.14)$$

with the kernel χ defined by:

$$\chi = \left(\frac{i}{2\pi}\right)^2 \frac{((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 + \varkappa_{20}^2)^{N_2}\, ((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 + \varkappa_{30}^2)^{N_3}}{((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 - \varkappa_2^2)\, ((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 - \varkappa_3^2)\, (\varkappa_2^2 + \varkappa_{20}^2)^{N_2}\, (\varkappa_3^2 + \varkappa_{30}^2)^{N_3}} \qquad (2.15)$$

(2.14) converges in the weak sense, for $\varkappa_{20}^2, \varkappa_{30}^2 > 0$ and sufficiently large integers $N_2, N_3 \geqslant 0$, uniformly for all $(k_2, k_3) \in D(s)$:

$$D(s) = \{(k_i^0)^2 \neq (\boldsymbol{k}_i - \boldsymbol{u}_i)^2 + \varkappa_i^2, \ i = 2,3 \ \ \forall\, (\boldsymbol{u}_2, \boldsymbol{u}_3, \varkappa_2^2, \varkappa_3^2) \in G(s)\}, \qquad (2.16)$$

$\tau(s, k_2, k_3)$ is O_+-invariant and holomorphic in $(k_2, k_3) \in D(s)$ and a tempered measure in $s \geqslant m^2$ [11]). $D(s)$ contains for all $s \geqslant m^2$ the direct product of the foreward and backward tubes $(\mathfrak{T}_+ \cup \mathfrak{T}_-) \times (\mathfrak{T}_+ \cup \mathfrak{T}_-)$. $\tau(s, k_2, k_3)$ fulfills due to the temperedness of the Φ_{ij} the growth condition in $(\mathfrak{T}_+ \cup \mathfrak{T}_-) \times (\mathfrak{T}_+ \cup \mathfrak{T}_-)$ (see [7])[20])), which guarantees the existence of the boundary values:

$$\underset{aa}{\tau_{rr}}(s, k_2, k_3) = \lim_{Im\, k_2,\, Im\, k_3 \in V_{\pm} \to 0} \tau(s, k_2, k_3)\,, \qquad (2.17)$$

as O_+-invariant tempered distributions. From (2.14) one proves the following support properties of the partial Fourier transform $\underset{aa}{\tilde{\tau}_{rr}}(s, \xi_2, \xi_3)$:

$$\tilde{\tau}_{rr} - \tilde{\tau}_{ra} - \tilde{\tau}_{ar} - \tilde{\tau}_{aa} = \tilde{\tau}_{00}\,,$$

$$\underset{aa}{\tilde{\tau}_{rr}}(s, \xi_2, \xi_3) = 0 \ \text{ for } \ (\xi_2, \xi_2) < 0 \ \text{ or } \ (\xi_3, \xi_3) < 0 \qquad (2.18)$$

and $\underset{aa}{\tilde{\tau}_{rr}}(s, \xi_2, \xi_3) = 0$, if ξ_2^0 or $\xi_3^0 < 0$ for (r, r); $-\ \xi_2^0$ or $\xi_3^0 < 0$ for (a, r); ξ_2^0 or $-\ \xi_3^0 < 0$ for (r, a); ξ_2^0 or $\xi_3^0 > 0$ for (a, a).

Therefore the $i\, L_+^{\uparrow}$-invariant tempered distribution

$$\langle \tilde{R}_0(p_1, p_2)\, E_0^{\perp}\, \tilde{R}_0(-q_1, -q_2)\rangle_0 \equiv i^2\, \delta(p_1 + p_2 - q_1 - q_2)\, \theta_0(p_1 + p_2)$$
$$\times\, \tau_{rr}\left((p_1 + p_2)^2,\ L(p_1 + p_2)\, \frac{p_1 - p_2}{2},\ L(p_1 + p_2)\, \frac{q_1 - q_2}{2}\right) \qquad (2.19)$$

is a sharp 2-fold retarded commutator, which can be used for the off-shell extrapolation of $\langle p_2 \,|\, \tilde{j}(p_1)\, \tilde{j}(-q_1) \,|\, q_2\rangle = \langle p_2 \,|\, \tilde{j}(p_1)\, E_0^{\perp}\, \tilde{j}(-q_1) \,|\, q_2\rangle$ in (1.31).

$\tau_{rr}(s, k_2, k_3)$ is unique up to O_+-invariant tempered distributions $\theta_i(s, k_2, k_3)$, $i = 2, 3$, which are polynomials in k_i, $i = 2, 3$, and which result e.g. from a different choice of the subtractions in (2.15). In the reduction formula (1.31) this ambiguity does not contribute to

$$(p_1^2 - m^2)\,(p_2^2 - m^2)\,(q_1^2 - m^2)\,(q_2^2 - m^2)\;\theta_0(p_1 + p_2)$$
$$\times\; \tau_{rr}\Big((p_1 + p_2)^2,\; L(p_1 + p_2)\,\frac{p_1 - p_2}{2},\; L(p_1 + p_2)\,\frac{q_1 - q_2}{2}\Big).$$

By (2.18) the latter distribution differs from any sharp retarded commutator

$$\tau'_{rr}\Big((p_1 + p_2)^2,\; L(p_1 + p_2)\,\frac{p_1 - p_2}{2},\; L(p_1 + p_2)\,\frac{q_1 - q_2}{2}\Big)\;\theta_0\,(p_1 + p_2)$$

constructed as above from

$$\langle[\tilde{j}(p_1)\,\tilde{j}(p_2)]\;E_0^{\perp}[\tilde{j}(-q_1)\,\tilde{j}(-q_2)]\rangle_0$$

only by trivial terms of similar structure. For the discussion of the analyticity properties of the sharp off-shell extrapolations one has therefore only to study τ'_{rr}, where the support of the spectral distributions in (2.14) is due to (D) determined by the mass $2\,m$ of the lowest 2-particle state.

Sometimes it is convenient to treat the 'pole term' in $\langle\tilde{R}_0\,(p_1, p_2)\,E_0^{\perp}\,\tilde{R}_0\,(-q_1, -q_2)\rangle_0$ separately. Since E_1 is of the form:

$$E_1 = \int \frac{d^2\boldsymbol{p}}{2\,\omega_p}\;|\;a_{ex}^*\,(\boldsymbol{p})\;\Omega)\;(a_{ex}^*\,(\boldsymbol{p})\;\Omega\;| \qquad (2.20$$

$\langle\tilde{R}_0(p_1, p_2)\;E_1\;\tilde{R}_0(-q_1, -q_2)\rangle_0$ leads to a product of vertex functions

$$\int dp\;\theta_0(p)\;\delta(p^2 - m^2)\;\langle 0\;|\;\tilde{R}_0(p_1, p_2)\;|\;p\rangle\;\langle p\;|\;\tilde{R}_0(-q_1, -q_2)\;|\;0\rangle\,.$$

Then the continuum contribution

$$\langle\tilde{R}_0(p_1, p_2)\;E_2\;\tilde{R}_0(-q_1, -q_2)\rangle_0\;\;(E_2 \equiv \mathbf{1} - E_0 - E_1)$$

starts from $s \geqslant 4\,m^2$.

Executing the same standard construction for the other cases in the Breit- or centre-of-mass system, one obtains

Theorem 2.1: Sharp retarded commutators (RC) can be defined in the form *):

$$\langle p_2\;|\;\tilde{R}_0(p_1, -q_1)\;|\;q_2\rangle = i\,\delta(p_1 + p_2 - q_1 - q_2)\;\theta_0(p_2)\;\delta(p_2^2 - m^2)\;\theta_0(q_2)\;\delta(q_2^2 - m^2)$$
$$\times\;\tau_{1r}\Big(\underline{L(p_2 + q_2)\,\frac{p_2 - q_2}{2}}\,,\;L(p_2 + q_2)\,\frac{p_1 + q_1}{2}\Big)\,, \qquad (2.21a)$$

$$\langle\tilde{R}_0(p_1, p_2)\;|\;q_1\,q_2^{in}\rangle = i\,\delta(p_1 + p_2 - q_1 - q_2)\;\theta_0(q_1)\;\delta(q_1^2 - m^2)\;\theta_0(q_2)\;\delta(q_2^2 - m^2)$$
$$\times\;\tau_{2r}\Big(L(q_1 + q_2)\,\frac{p_1 - p_2}{2},\;\underline{L(q_1 + q_2)\,\frac{q_1 - q_2}{2}}\Big)\,, \qquad (2.21b)$$

*) The space-part of $q \in R^4$ is denoted here by $\underline{q}$.

$$\langle \tilde{R}_0(p_1, p_2)\, E_1\, \tilde{R}_0(-q_1, -q_2)\rangle_0 = i^2\, \delta(p_1 + p_2 - q_1 - q_2)\, \theta_0(p_1 + p_2)$$
$$\times\, \delta\Big((p_1 + p_2)^2 - m^2\Big)\, \tau_{3r}\Big(L(p_1 + p_2)\, \frac{p_1 - p_2}{2}\Big)\, \tau_{4r}\Big(L(p_1 + p_2)\, \frac{q_1 - q_2}{2}\Big), \qquad (2.21c)$$

$$\langle \tilde{R}_0(p_1, p_2)\, E_2\, \tilde{R}(-q_1, -q_2)\rangle_0 = i^2\, \delta(p_1 + p_2 - q_1 - q_2)\, \theta_0(p_1 + p_2)$$
$$\times\, \tau^2_{rr}\Big((p_1 + p_2)^2,\ L(p_1 + p_2)\, \frac{p_1 - p_2}{2},\ L(p_1 + p_2)\, \frac{q_1 - q_2}{2}\Big). \qquad (2.21d)$$

Here $\tau_{1r}(\boldsymbol{\Delta}, \omega)$, $\tau_{2r}(k_2, \boldsymbol{k}_3)$, $\tau_{3r}(k_2)$, $\tau_{4r}(k_3)$ and $\tau^2_{rr}(s, k_2, k_3)$ are O_+-invariant tempered distributions, boundary values from $\mathfrak{T}_+$ or $\mathfrak{T}_+ \times \mathfrak{T}_+$ of weakly convergent integral representations of the type (2.14), which are derived from the JLD-representations of the corresponding commutator matrix-elements.

Although the 'sharp' RC (2.21) are not of the class considered in section 1, it is easy to see that theorem 1.2 and 1.3 also hold for them, the difference between a 'sharp' and a 'smooth' RC being a tempered distribution, which does not contribute in (1.20), (1.30), (1.31), by an argument like (1.23).

We conclude this section with a remark on a doubt expressed recently as to the validity of the proof of dispersion relations [6]). It follows directly from the temperedness of the Φ'_{ij} in (2.14) that for a sufficiently large $M \geqslant 0$ the integral

$$\int\limits_{4m^2}^{\infty} \frac{ds'\, \tau'_{22}\,(s', k_2, k_3)}{(s' - s)^{M+1}} \qquad (2.22)$$

converges uniformly for all complex $s \notin [4\, m^2, \infty\rangle$ and all (k_2, k_3) mapped by (3.9) into the compact set (3.10), with M independent of s and ε. For, it is seen by a tedious, but straightforward majorization that for any choice of N_2, $N_3 \geqslant 0$ and $\varkappa^2_{20}$, $\varkappa^2_{30} > 0$ in (2.15) the function:

$$\chi_{N_2 N_3} = \left(\frac{i}{2\pi}\right)^2 \frac{((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 + \varkappa^2_{20})^{N_2}\, ((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 + \varkappa^2_{30})^{N_3}}{((k_2^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{u}_2)^2 - \varkappa^2_2)\, ((k_3^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{u}_3)^2 - \varkappa^2_3)} \qquad (2.23)$$

is for all $(k_2, k_3) \in (3.10)$ a multiplier $\in O_M$ in a small neighbourhood of the support of the tempered distributions Φ'_{ij}. Since in the (formal) integral:

$$\int\limits_{4m^2}^{\infty} \frac{ds'\, \tau'_{22}\,(s', k_2, k_3)}{(s' - s)^{M+1}} = \int \frac{ds'\, d\varkappa^2_2\, d\varkappa^2_3}{(s' - s)^{M+1}\, (\varkappa^2_2 + \varkappa^2_{20})^{N_2}\, (\varkappa^2_3 + \varkappa^2_{30})^{N_3}} \times$$
$$\times \int d\boldsymbol{u}_2\, d\boldsymbol{u}_3\, \chi_{N_2 N_3}\, \{\Phi'_{11} + k_2^0 \Phi'_{21} + k_3^0\, \Phi'_{12} + k_2^0\, k_3^0 \Phi'_{22}\}, \qquad (2.24)$$

one has $\boldsymbol{u}_2^2$, $\boldsymbol{u}_3^2 \leqslant s/4$ in supp Φ'_{ij}, only the polynomial growth in s of the inner integral will be affected by changing the powers N_2, $N_3 \geqslant 0$. Since the $\Phi'_{ij}\, \chi_{N_2 N_3}$ increase only polynomially in $\varkappa^2_2$, $\varkappa^2_3$, the left-hand side of (2.24) will converge uniformly for sufficiently large N_2, $N_3 \geqslant 0$ and $M = M(N_2, N_3) \geqslant 0$.

§ 3. Analyticity on the Mass Shell

In this section the off-shell extrapolations of the 2-particle scattering amplitude T_{22} by sharp retarded commutators will be used to derive analyticity properties of T_{22}

on the mass shell. The investigation is scarcely original and will be centered around the question as to how the classical results of dispersion theory[1]) can be rigorously proved from the general postulates (A), (B), and (D) without making additional 'technical assumptions'.

The behaviour of T_{22} in the momentum transfer $t = (p_1 - q_1)^2$ can be easily obtained by the well-known argument of LEHMANN[13]). Combining the results of theorems 1.3 and 2.1 one gets for T_{22} the distribution identity (for $p_1^2, p_2^2 < 4\,m^2$):

$$T_{22}(p_1, p_2, -q_1, -q_2) = 2\pi i\, \delta(p_1 + p_2 - q_1 - q_2)\, \theta_0(q_1)\, \delta(q_1^2 - m^2)\, \theta_0(q_2)\, \delta(q_2^2 - m^2)$$
$$\times \lim_{a,b\to\infty} \lim_{c,d\to\infty} \delta^m_{ab}(p_1)^*\, \delta^m_{cd}(p_2)^*$$
$$\times \left[(p_1^2 - m^2)(p_2^2 - m^2)\, \tau_{2r}\left(L(q_1+q_2)\,\frac{p_1-p_2}{2},\ \underline{L(q_1+q_2)\,\frac{q_1-q_2}{2}}\right)\right]. \tag{3.1}$$

The distribution

$$(p_1^2 - m^2)(p_2^2 - m^2)\, \tau_{2r}\left(L(q_1+q_2)\,\frac{p_1-p_2}{2},\ \underline{L(q_1+q_2)\,\frac{q_1-q_2}{2}}\right)$$

can be replaced (neglecting an O_+-invariant distribution, which is a polynomial in $L(q_1 + q_2)\,((p_1 - p_2)/2)$ by a retarded commutator

$$\tau'_{2r}\left(L(q_1+q_2)\,\frac{p_1-p_2}{2},\ \underline{L(q_1+q_2)\,\frac{q_1-q_2}{2}}\right)$$

of the currents, defined by an integral representation $\tau'_{2r}(k_2, \boldsymbol{k}_3) = \lim_{Im\,k_2 \in V_+ \to 0} \tau_2(k_2, \boldsymbol{k}_3)$:

$$\tau'_2(k_2, \boldsymbol{k}_3) = \left(\frac{i}{2\pi}\right) \int \frac{d^4u\, d\varkappa^2\, \Phi'(u, \varkappa^2, \boldsymbol{k}_3)\, ((u-k_2)^2 + \varkappa_0^2)^N}{((u-k_2)^2 - \varkappa^2)\, (\varkappa^2 + \varkappa_0^2)^N}. \tag{3.2}$$

Now, the support of the JLD-spectral distribution Φ' of the commutator $(\Omega, [\tilde{j}(p_1)\, \tilde{j}(p_2)] \mid q_1\, q_2^{in})$ is such that $\tau'_2(k_2, \boldsymbol{k}_3)$ (and therefore $\tau'_{2r}(k_2, \boldsymbol{k}_3)$) is for fixed $\boldsymbol{k}_3$ analytic in k_2 in a neighbourhood of the mass shell $\{k_2^0 = 0,\ \boldsymbol{k}_2^2 = \boldsymbol{k}_3^2\}$[13]). Thus in the limit $a, b \to \infty$, $c, d \to \infty$ (3.1) is just the product of $\theta_0(p_1)\, \delta(p_1^2 - m^2)\, \theta_0(p_2)\, \delta(p_2^2 - m^2)$ with $(p_1^2 - m^2)(p_2^2 - m^2)\, \tau_{2r}$, which is a real-analytic function in the critical variables essentially given by (3.2).

$\tau'_2(k_2, \boldsymbol{k}_3)$ is for real $k_2, \boldsymbol{k}_3$ a tempered O_+-invariant distribution and thus by lemma 2.2 a distribution in the invariants k_2^0, $\boldsymbol{k}_2^2$, $\boldsymbol{k}_2\, \boldsymbol{k}_3$, $\boldsymbol{k}_3^2$. The well-defined restriction to the mass shell is then a tempered distribution in $s = (p_1 + p_2)^2 = 4\,m^2 + 4\,\boldsymbol{k}_3^2$ and $t = (p_1 - q_1)^2 = 2\,(\boldsymbol{k}_2\, \boldsymbol{k}_3 - \boldsymbol{k}_3^2)$.

For $\boldsymbol{k}_3^2 \neq 0$ (to be understood in the weak sense, i.e. integrated over a testing function $\chi(\boldsymbol{k}_3)$ with sufficiently small support) the only dependence of the O_+-invariant distribution $\tau'_2(k_2, \boldsymbol{k}_3)$ on $\boldsymbol{k}_2\, \boldsymbol{k}_3$ can be brought into the kernel of (3.2) by taking the mean over the rotation group around $\boldsymbol{k}_3$. In this way it can be shown[13]) that on the mass shell $\tau'_2(k_2, \boldsymbol{k}_3)$ is, for fixed $s > 4\,m^2$, holomorphic in t in the small Lehmann ellipse:

$$E_1(s) = \left\{ |t| + |t + s - 4\,m^2| < \left[(s - 4\,m^2)^2 + \frac{36\,m^4}{s}\,(s - 4\,m^2)\right]^{1/2} \right\}. \tag{3.3}$$

The same holds for the reduced 2-particle scattering amplitude, which is defined by $\lim\,[((k_2^0)^2 - \boldsymbol{k}_2^2 + \boldsymbol{k}_3^2)^2 - 4\,(k_2^0)^2\,(m^2 + \boldsymbol{k}_3^2)]\ \tau_{2r}(k_2, \boldsymbol{k}_3)$:

Theorem 3.1: Under the assumptions (A), (B), (D) the 2-particle scattering amplitude is of the form:

$$T_{22}(p_1, p_2, -q_1, -q_2) = \delta(p_1 + p_2 - q_1 - q_2)\ \theta_0(p_1)\ \delta(p_1^2 - m^2)\ \theta_0(p_2)\ \delta(p_2^2 - m^2) \times \theta_0(q_1)\ \delta(q_1^2 - m^2)\ \theta_0(q_2)\ \delta(q_2^2 - m^2)\ T_{22}((p_1 + p_2)^2, (p_1 - q_1)^2)\,. \tag{3.4}$$

The reduced scattering amplitude $T_{22}(s, t)$ is a tempered distribution in the physical domain $\{(s, t) : s \geqslant 4\,m^2,\ 2\,m^2 - s/2 \leqslant t \leqslant 0\}$, which has for fixed $s > 4\,m^2$ an analytic continuation in $t \in E_1(s)$.

Next we discuss the behaviour of $T_{22}(s, t)$ for fixed momentum transfer t in the centre-of-mass energy s, following closely the classical idea of BOGOLIUBOV[4]). By theorem 1.2 and 2.1 one can use the following sharp off-shell extrapolation for T_{22}:

$$T_{22}(p_1, p_2, -q_1, -q_2) = 2\,\pi\,i\ \delta(p_1 + p_2 - q_1 - q_2)\ \theta_0(p_2)\ \delta(p_2^2 - m^2)\ \theta_0(q_2)\ \delta(q_2^2 - m^2) \times \lim_{a,b\to\infty}\ \lim_{c,d\to\infty}\ \delta^m_{ab}(p_1)^*\ \delta^m_{cd}(q_1) \times \left[(p_1^2 - m^2)\ (q_1^2 - m^2)\ \tau_{1r}\Big(\underline{L(q_2 + p_2)\ \tfrac{p_2 - q_2}{2}}\,,\ L(q_2 + p_2)\ \tfrac{p_1 + q_1}{2}\Big)\right], \tag{3.5}$$

$\tau_{1r}(\boldsymbol{\Delta}, \omega)$ is defined as the limit, $\operatorname{Im}\,\omega \in V_+ \to 0$, of

$$\tau_1(\boldsymbol{\Delta}, \omega) = \left(\frac{i}{2\,\pi}\right) \int \frac{d\boldsymbol{u}\ d\varkappa^2\ \{\varphi_1\,(\boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta}) + \omega_0\,\varphi_2\}\ (\omega_0^2 - (\boldsymbol{\omega} - \boldsymbol{u})^2 + \varkappa_0^2)^N}{(\omega_0^2 - (\boldsymbol{\omega} - \boldsymbol{u})^2 - \varkappa^2)\ (\varkappa^2 + \varkappa_0^2)^N}\,. \tag{3.6}$$

The distribution $[(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4\,(\boldsymbol{\omega}\ \boldsymbol{\Delta})^2]\ \tau_{1r}\,(\boldsymbol{\Delta}, \omega)$ is, up to a polynomial in ω, identical with the boundary value of a $\tau_1'(\boldsymbol{\Delta}, \omega)$, defined as in (3.6) with spectral distributions $\varphi_i'(\boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta})$ corresponding to the commutator of the currents:

$$\langle p_2\,|\,[\tilde{j}(p_1)\ \tilde{j}(-q_1)]\,|\,q_2\rangle = \delta(p_1 + p_2 - q_1 - q_2)\ \theta_0(p_2)\ \delta(p_2^2 - m^2)\ \theta_0(q_2)\ \delta(q_2^2 - m^2) \times \tau_{10}'\Big(\underline{L(p_2 + q_2)\ \tfrac{p_2 - q_2}{2}}\,,\ L(p_2 + q_2)\ \tfrac{p_1 + q_1}{2}\Big)\,, \tag{3.7}$$

where

$$\tau_{10}'(\boldsymbol{\Delta}, \omega) = \int d\boldsymbol{u}\ d\varkappa^2\ \varepsilon(\omega_0)\ \delta(\omega_0^2 - (\boldsymbol{\omega} - \boldsymbol{u})^2 - \varkappa^2)\ \{\varphi_1' + \omega_0\,\varphi_2'\}. \tag{3.8}$$

$\tau_1(\boldsymbol{\Delta}, \omega)$ is, for fixed $\boldsymbol{\Delta}$, holomorphic in the domain $D_1(\boldsymbol{\Delta}) = \{\omega : \omega_0^2 \neq (\boldsymbol{\omega} - \boldsymbol{u})^2 + \varkappa^2\ \forall\ (\boldsymbol{u}, \varkappa^2) \in G_1(\boldsymbol{\Delta})\}$ with $G_1(\boldsymbol{\Delta}) = \{(\boldsymbol{u}, \varkappa^2) : \boldsymbol{u}^2 \leqslant m^2 + \boldsymbol{\Delta}^2, \varkappa \geqslant \max\{0, 2m - \sqrt{m^2 + \boldsymbol{\Delta}^2 - \boldsymbol{u}^2}\}\}$.

$D_1(\boldsymbol{\Delta})$ contains $\mathfrak{T}_+ \cup \mathfrak{T}_-$ and real ω with $|\omega_0| < \sqrt{4\,m^2 + \boldsymbol{\omega}^2} - \sqrt{m^2 + \boldsymbol{\Delta}^2}$, except for 'pole terms' singular at $|\,\omega_0\,| = \sqrt{m^2 + \boldsymbol{\omega}^2} - \sqrt{m^2 + \boldsymbol{\Delta}^2}$. The domain of holomorphy $D_1(\boldsymbol{\Delta})$ is well known[6]), and one can see that no conclusive analyticity properties of $T_{22}(s, t)$ can be proved using only the information contained in (3.5).

Following the argument of BOGOLIUBOV et al.[4]) we shall therefore construct an analytic continuation of $((\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4\,(\boldsymbol{\omega}\ \boldsymbol{\Delta})^2)\ \tau_1(\boldsymbol{\Delta}, \omega)$ by a Cauchy integral representation for 'virtual masses'. The commutator (3.7) will appear in the integrand,

and we shall use for (3.7) the far-reaching information contained in the reduction formula (1.31) with a sharp retarded commutator (2.19).

$$(p_1^2 - m^2)\,(p_2^2 - m^2)\,(q_1^2 - m^2)\,(q_2^2 - m^2)\,\langle \tilde{R}_0(p_1, p_2)\, E_0^{\perp}\, \tilde{R}_0(-q_1, -q_2)\rangle_0$$

is the boundary value of an analytic function $\tau'(s, k_2, k_3)$, which is, due to its O_+-invariance, a holomorphic function $\bar{\tau}'$ [10]) in the invariants k_2^0, k_3^0, $\boldsymbol{k}_0^2$, $\boldsymbol{k}_2\, \boldsymbol{k}_3$, $\boldsymbol{k}_3^2$. These variables are for $s > 0$ biholomorphically related to the Lorentz invariants:

$$\begin{aligned} z_1 &= p_1^2 = \left(\frac{\sqrt{s}}{2} + k_2^0\right)^2 - \boldsymbol{k}_2^2 & z_2 &= p_2^2 = \left(\frac{\sqrt{s}}{2} - k_2^0\right)^2 - \boldsymbol{k}_2^2\,, \\ z_3 &= q_1^2 = \left(\frac{\sqrt{s}}{2} + k_3^0\right)^2 - \boldsymbol{k}_3^2 & z_4 &= q_2^2 = \left(\frac{\sqrt{s}}{2} - k_3^0\right)^2 - \boldsymbol{k}_3^2\,, \\ z_5 &= t = (k_2^0 - k_3^0)^2 - (\boldsymbol{k}_2 - \boldsymbol{k}_3)^2\,. \end{aligned} \tag{3.9}$$

The analysis of the domain of analyticity $\bigcap_{s=4m^2}^{\infty} D(s) \cap D(m^2)$ of $\bar{\tau}' = \bar{\tau}'(s, z_1, \ldots z_5)$ shows [4]), [13]), [1]), that $\bar{\tau}'$ is for $s = m^2$, $s \geqslant 4\,m^2$ holomorphic in $z_1, \ldots z_5$ from:

$$\begin{aligned} &|\, z_1 - \zeta^2 \,| \leqslant \delta,\ |\, z_2 - m^2 \,| \leqslant \delta,\ |\, z_3 - \zeta^2 \,| \leqslant \delta\,, \\ &|\, z_4 - m^2 \,| \leqslant \delta,\ -8\,m^2 + \varepsilon \leqslant \operatorname{Re} z_5 \leqslant 0,\ |\operatorname{Im} z_5\,| \leqslant \frac{\delta}{s}\,, \end{aligned} \tag{3.10}$$

with $\delta = \delta(\varepsilon) > 0$ for all $\varepsilon > 0$ and $3\,m^2 \leqslant \zeta^2 \leqslant m^2$.

The physical values of p_1^2, p_2^2, q_1^2, q_2^2, t are for $-8\,m^2 < t \leqslant 0$ contained in (3.10). The same applies to

$$(p_1^2 - m^2)\,(p_2^2 - m^2)\,(q_1^2 - m^2)\,(q_2^2 - m^2)\,\langle \tilde{R}_0(-q_1, p_2)\, E_0^{\perp}\, \tilde{R}_0(p_1, -q_2)\rangle_0\,,$$

which is given by $\bar{\tau}'(u, z_1, \ldots z_5)$, where $u \equiv s + t - (p_1^2 + p_2^2 + q_1^2 + q_2^2)$. Using this one sees that in the reduction formula (1.31) the limit $\lim\limits_{a,b\to\infty} \delta_{ab}^m(p_2)^* \lim\limits_{c,d\to\infty} \delta_{cd}^m(q_2)$ leads to the product of the δ-distributions $\theta_0(p_2)\,\delta(p_2^2 - m^2)\,\theta_0(q_2)\,\delta(q_2^2 - m^2)$ with the functions $\bar{\tau}'$, which are again regular in the critical variables.

In order to insert the reduction formula (1.31) in (3.7), we compute the invariants (3.9) in the variables $\boldsymbol{\Delta} = L(q_2 + p_2)\,(p_2 - q_2)/2$ and $\omega = L\,(p_2 + q_2)\,(p_1 + q_1)/2$ of the Breit-system:

$$p_1^2,\, q_1^2 = \omega^2 \pm 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2;\ t = -4\,\boldsymbol{\Delta}^2;\ s, u = m^2 + \boldsymbol{\Delta}^2 + \omega^2 \pm 2\,\omega_0 \sqrt{m^2 + \boldsymbol{\Delta}^2}. \tag{3.11}$$

Combining (1.31), (2.19) and (3.11) one obtains the important identity:

$$\tau_{10}'(\boldsymbol{\Delta}, \omega) = 2\,\pi\,\{\bar{\tau}'(s, p_1^2, m^2, q_1^2, m^2, t) - \bar{\tau}'(u, p_1^2, m^2, q_1^2\, m^2, t)\} \tag{3.12}$$

for real $\boldsymbol{\Delta}$, ω satisfying $\boldsymbol{\Delta}^2 < 2\,m^2$, $\omega^2 \pm \boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2 \leqslant m^2 + \delta$. It follows from the regularity properties of $\bar{\tau}'$ in (3.10) that $\tau_{10}'(\boldsymbol{\Delta}, \omega)$ is a measure in ω_0 and real-analytic in ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$, $\boldsymbol{\Delta}^2$ in the above domain.

For sufficiently large M and for real ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$, $\boldsymbol{\Delta}^2$ with $\boldsymbol{\Delta}^2 \leqslant 2\,m - \varepsilon$, $\omega^2 \pm 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2 \leqslant m^2 + \delta(\varepsilon)$, the ω_0'-integral of the measure $\bar{\tau}_{10}'(\omega_0', \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$ over the testing function $(\omega_0' - \omega_0)^{-(M+1)}$ exists for all $\mathrm{Im}\,\omega_0 \neq 0$ and $\varepsilon > 0$. Using (3.12) and changing variables one obtains:

$$\tau^M(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2) \equiv \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\omega_0'\,\bar{\tau}_{10}'(\omega_0', \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)}{(\omega_0' - \omega_0)^{M+1}}$$

$$= - i\,M! \int_{m^2}^{\infty} ds\left(2\sqrt{m^2+\boldsymbol{\Delta}^2}\right)^M \bar{\tau}'(s, \omega^2 + 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2, m^2 \omega^2 - 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2, m^2, -4\,\boldsymbol{\Delta}^2)$$

$$\times \left\{\left(s - \omega^2 - m^2 - \boldsymbol{\Delta}^2 - 2\,\omega_0 \sqrt{m^2 + \boldsymbol{\Delta}^2}\right)^{-(M+1)} + (-1)^M \left(s - \omega^2 - m^2 - \boldsymbol{\Delta}^2 + 2\,\omega_0 \sqrt{m^2 + \boldsymbol{\Delta}^2}\right)^{-(M+1)}\right\}. \tag{3.13}$$

According to section 2 the right-hand side of (3.13) converges for sufficiently large M also for complex values of ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$. Therefore $\tau^M(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$ is for $0 \leqslant \boldsymbol{\Delta}^2 < 2\,m^2$ a C^∞-function in $\boldsymbol{\Delta}^2$, which is for fixed $\boldsymbol{\Delta}^2$ holomorphic in ω_0, ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$ in a domain, which contains the points:

$$|\,\mathrm{Im}\,\omega^2\,| < 2\sqrt{m^2 + \boldsymbol{\Delta}^2}\;|\,\mathrm{Im}\,\omega_0\,|$$

$$|\,\omega^2 \pm 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2 - \zeta^2\,| < \delta(\boldsymbol{\Delta}) \quad \text{for} \quad -3\,m^2 \leqslant \zeta^2 \leqslant m^2\,. \tag{3.14}$$

In order to relate $\tau^M(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$ to an analytic continuation $[(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4\,(\boldsymbol{\omega}\,\boldsymbol{\Delta})^2]\,\tau_1(\boldsymbol{\Delta}, \omega)$ of the 2-particle scattering amplitude, we prove:

Lemma 3.1: Let $f_\chi^N(\omega; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2)$ be defined as

$$f_\chi^N(\omega; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2) \equiv \frac{i}{2\pi} \int_{O_+} dR\, \frac{\chi(R^{-1}\,\boldsymbol{\Delta})\,(\omega_0^2 - (\boldsymbol{\omega} - R^{-1}\,\boldsymbol{u})^2 + \varkappa_0^2)^N}{(\varkappa^2 + \varkappa_0^2)^N\,(\omega_0^2 - (\boldsymbol{\omega} - R^{-1}\,\boldsymbol{u})^2 - \varkappa^2)}\,, \tag{3.15}$$

for

$$\chi \in \mathfrak{D}(R^3),\; X = \mathrm{supp}\,\chi \subset \{\boldsymbol{\Delta}^2 \leqslant 2\,m^2 - \varepsilon, \varepsilon > 0\},$$

$$(\boldsymbol{u}, \varkappa^2) \in \bigcup_X G_1(\boldsymbol{\Delta}),\; \omega \in \bigcap_X D_1(\boldsymbol{\Delta}),\; \varkappa_0^2 > 0\,.$$

Then one has for integer $M \geqslant M(N) \geqslant 0$:

$$\left(\frac{\partial}{\partial\omega_0}\right)^M f_\chi^N(\omega; \boldsymbol{\Delta}, u, \varkappa^2)\,\Big|_{\omega^2 = \omega_0^2 - \cdots^2 \text{ fixed}}$$

$$= \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} d\omega_0' \int_{O_+} dR\, \frac{\varepsilon(\omega_0')\,\delta((\omega_0')^2 - (\boldsymbol{\omega} - R^{-1}\,\boldsymbol{u})^2 - \varkappa^2)\,\chi(R^{-1}\,\boldsymbol{\Delta})}{(\omega_0' - \omega_0)^{M+1}}\,, \tag{3.16}$$

for all $\mathrm{Im}\,\omega_0 \neq 0$, $\omega^2 < 0$ and sufficiently small real $\boldsymbol{\omega}\,\boldsymbol{\Delta}$.

For the proof one remarks that, for $(\boldsymbol{u}, \varkappa^2) \in \bigcup_X G_1(\boldsymbol{\Delta})$ and $\omega \in \bigcap_X D_1(\boldsymbol{\Delta})$, $f_\chi^N(\omega; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2)$ is O_+-invariant in $\boldsymbol{\Delta}$, $\boldsymbol{u}$ and holomorphic in ω with $f_\chi^N(\omega_1; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2) = f_\chi^N(\omega_2; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2)$ for $\omega_1^2 = \omega_2^2$, $\omega_{1,0} = \omega_{2,0}$, $\boldsymbol{\omega}_1\,\boldsymbol{\Delta} = \boldsymbol{\omega}_2\,\boldsymbol{\Delta}$. Therefore f_χ^N is a holomorphic function $\bar{f}_\chi^N(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}; \boldsymbol{\Delta}, \boldsymbol{u}, \varkappa^2)$ in the variables ω_0, ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$ in the image $\overline{\bigcap_X D_1(\boldsymbol{\Delta})}$ of the saturated domain $\bigcap_X D_1(\boldsymbol{\Delta})$ [10]. For fixed $\omega^2 < 0$ and real $\boldsymbol{\omega}\,\boldsymbol{\Delta}$ sufficiently small, $\bar{f}_\chi^N$ can be seen to be holomorphic in the complex ω_0-plane, except possibly for singularities on the real axis. After having replaced $(\omega_0^2 - (\boldsymbol{\omega} - \boldsymbol{u})^2 - \varkappa^2)^{-1}$ in the definition (3.15) of f_χ^0 by the following expression (under the R-integration equivalent):

$$\frac{\omega^2 - \boldsymbol{u}^2 - \varkappa^2 + 2(\boldsymbol{\omega}\,\boldsymbol{\Delta})\,(\boldsymbol{u}\,\boldsymbol{\Delta})\,(\boldsymbol{\Delta}\,\boldsymbol{\Delta})^{-1}}{(\omega^2 - \boldsymbol{u}^2 - \varkappa^2 + 2(\boldsymbol{\omega}\,\boldsymbol{\Delta})\,(\boldsymbol{u}\,\boldsymbol{\Delta})\,(\boldsymbol{\Delta}\,\boldsymbol{\Delta})^{-1})^2 - 4(\omega_0^2 - \omega^2 - (\boldsymbol{\omega}\,\boldsymbol{\Delta})\,\boldsymbol{\Delta}^{-2})\,(\boldsymbol{u}^2 - (\boldsymbol{u}\,\boldsymbol{\Delta})^2\,\boldsymbol{\Delta}^{-2}} \quad \text{for } \boldsymbol{\Delta}^2 \neq 0$$

$$\frac{\omega^2 - \boldsymbol{u}^2 - \varkappa^2}{\omega^2 - \boldsymbol{u}^2 - \varkappa^2)^2 - 4(\omega_0^2 - \omega^2)\,\boldsymbol{u}^2} \quad \text{for } \boldsymbol{\Delta}^2 = 0 \tag{3.17}$$

one obtains (3.16) for $N = 0$ by CAUCHY's theorem. The general case follows from the fact that for sufficiently large $M \geqslant M(N)$ one has:

$$\left(\frac{\partial}{\partial\omega_0}\right)^M \bar{f}_\chi^N(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}; \boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta}) = \left(\frac{\partial}{\partial\omega_0}\right)^M \bar{f}_\chi^0(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}; \boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta}). \tag{3.18}$$

We apply Lemma 3.1 to

$$\int d\boldsymbol{\Delta}\, \tau_1'(\boldsymbol{\Delta}, \omega)\, \chi(\boldsymbol{\Delta}) = \int d\boldsymbol{\Delta}\, \bar{\tau}_1'(\boldsymbol{\Delta}, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \omega_0)\, \chi(\boldsymbol{\Delta})$$
$$= \int d\boldsymbol{u}\, d\varkappa^2\, d\boldsymbol{\Delta}\, \{\varphi_1'(\boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta}) + \omega_0\, \varphi_2'(\boldsymbol{u}, \varkappa^2\, \boldsymbol{\Delta})\}\, \bar{f}_\chi^N(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}; \boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta}) \tag{3.19}$$

and obtain for a sufficiently large M:

$$\left(\frac{\partial}{\partial\omega_0}\right)^M \int \bar{\tau}_1' (\boldsymbol{\Delta}, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \omega_0)\, \chi(\boldsymbol{\Delta})\, d\boldsymbol{\Delta} =$$

$$= \frac{M!}{2\pi i} \int d\boldsymbol{u}\, d\varkappa^2\, d\boldsymbol{\Delta} \int_{-\infty}^{+\infty} d\omega_0' \int_{O_+} dR\, \frac{\varepsilon(\omega_0')\, \delta((\omega_0')^2 - (\boldsymbol{\omega} - R^{-1}\boldsymbol{u})^2 - \varkappa^2)\, \chi(R^{-1}\boldsymbol{\Delta})\, \{\varphi_1' + \omega_0\, \varphi_2'\}}{(\omega_0' - \omega_0)^{M+1}}$$

$$= \int d\boldsymbol{\Delta}\, \chi(\boldsymbol{\Delta})\, \frac{M!}{2\pi i} \int_{-\infty}^{+\infty} d\omega_0'\, \frac{\bar{\tau}_{10}'(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)}{(\omega_0' - \omega_0)^{M+1}}\,. \tag{3.20}$$

In (3.20) the R-integration can be omitted, as the $\varphi_1(\boldsymbol{u}, \varkappa^2, \boldsymbol{\Delta})$ are O_+-invariant, and the interchange of the ω_0'- and $(\boldsymbol{u}, \varkappa^2)$-integration is justified by FUBINI's theorem[17]). Since $\tau_1'(\boldsymbol{\Delta}, \omega)$ and $[(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4(\boldsymbol{\omega}\,\boldsymbol{\Delta})^2]\, \tau_1(\boldsymbol{\Delta}, \omega)$ only differ by a polynomial in ω, one finally obtains for sufficiently large M the distribution identity:

$$\left(\frac{\partial}{\partial\omega_0}\right)^M [(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4(\boldsymbol{\omega}\,\boldsymbol{\Delta})^2]\, \tau_1(\boldsymbol{\Delta}, \omega) \,\Big|_{\,\omega^2 = \omega_0^2 - \boldsymbol{\omega}^2 \text{ fixed}}$$
$$= \frac{M!}{2\pi i} \int d\omega_0'\, \frac{\bar{\tau}_{10}'(\omega_0', \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)}{(\omega_0' - \omega_0)^{M+1}} = \tau^M(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2), \tag{3.21}$$

for $\boldsymbol{\Delta}^2 \leqslant 2\,m^2 - \varepsilon$, $\omega^2 < 0$, sufficiently small real $\boldsymbol{\omega}\,\boldsymbol{\Delta}$, $\operatorname{Im}\omega_0 \neq 0$ and $\varepsilon > 0$. Since the left-hand side is analytic in $\overline{D_1(\boldsymbol{\Delta})}$ and the right-hand side in the set $\overline{D_2(\boldsymbol{\Delta})}$ defined by (3.14), (3.21) holds identically in $\overline{D_1(\boldsymbol{\Delta})} \cap \overline{D_2(\boldsymbol{\Delta})}$. This shows that the M-fold ω_0-integral of τ^M is an analytic continuation of $[(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4(\boldsymbol{\omega}\,\boldsymbol{\Delta})^2]\, \tau_1(\boldsymbol{\Delta}, \omega)$ except for an O_+-invariant distribution $P(\boldsymbol{\Delta}, \omega)$, which as a distribution in the invariants ω_0, ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}$, $\boldsymbol{\Delta}^2$ is a polynomial of M^{th} degree in ω_0.

For all fixed real $\boldsymbol{\Delta}$ the physical values of $\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}$ (corresponding to real 4-vectors ω) lie on the boundary of $\overline{\mathfrak{T}_\pm(\boldsymbol{\Delta})}$ and also on the boundary of the domains $\overline{D_\pm(\boldsymbol{\Delta})} \equiv \overline{D_2(\boldsymbol{\Delta})} \cap \{\mathrm{Im}\,\omega_0 \lesseqgtr 0\}$ for $0 \leqslant \boldsymbol{\Delta}^2 < 2\,m^2$, as well as on the boundary of $\overline{D_\pm(\boldsymbol{\Delta})} \cap \overline{\mathfrak{T}_\pm(\boldsymbol{\Delta})}$[1]). The boundary value of $\tau^M(\omega_0 - i\,\varepsilon, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$ for real ω_0, ω^2, $\boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2$ with $\boldsymbol{\Delta}^2 < 2\,m^2, \omega^2 \pm 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2 < m^2 + \delta(\boldsymbol{\Delta})$ is for $\varepsilon \downarrow 0$ the tempered distribution in ω_0:

$$\lim_{\varepsilon \downarrow 0} \tau^M(\omega_0 + i\,\varepsilon, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2) = \frac{1}{2}\left(\frac{\partial}{\partial \omega_0}\right)^M \bar{\tau}'_{10}(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$$

$$+ \frac{M!}{2\,\pi\,i}\, P \int_{-\infty}^{+\infty} d\omega_0'\, \frac{\bar{\tau}'_{10}(\omega_0', \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)}{(\omega_0' - \omega_0)^{M+1}}\,, \qquad (3.22)$$

and the same limit distribution is obtained taking a sequence from $\overline{\mathfrak{T}_+(\boldsymbol{\Delta})} \cap \overline{D_+(\boldsymbol{\Delta})}$ converging to $\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2$. This shows that τ^M is for these points an analytic continuation of the M^{th} ω_0-derivative (for $\omega^2 = \omega_0^2 - \boldsymbol{\omega}^2$ fixed) of the off-shell extrapolation $[(\omega^2 - m^2 - \boldsymbol{\Delta}^2)^2 - 4(\boldsymbol{\omega}\,\boldsymbol{\Delta})^2]\ \tau_{1r}(\boldsymbol{\Delta}, \omega)$ of the 2-particle scattering amplitude.

The mass-shell is characterized by $\omega^2 = m^2 + \boldsymbol{\Delta}^2$, $\boldsymbol{\omega}\,\boldsymbol{\Delta} = 0$. It follows from (3.22) that $\tau^M(\omega_0, \omega^2, \boldsymbol{\omega}\,\boldsymbol{\Delta}, \boldsymbol{\Delta}^2)$ is, for physical points with $\boldsymbol{\Delta}^2 \leqslant 2\,m^2 - \varepsilon$, $\varepsilon > 0$, $\omega^2 \pm 2\,\boldsymbol{\omega}\,\boldsymbol{\Delta} - \boldsymbol{\Delta}^2 \leqslant m^2 + \delta(\varepsilon)$, regular in ω^2 and $\boldsymbol{\omega}\,\boldsymbol{\Delta}$ in a neighbourhood of the mass shell. Therefore the limit of $\delta^m_{ab}(p_1)^*\ \delta^m_{cd}(q_1)\ \tau^M$ again exists as the product of τ^M with $\theta_0(p_1)\ \delta(p_1^2 - m^2)\ \theta_0(q_1)\ \delta(q_1^2 - m^2)$.

From (3.21) and (3.4) $\tau^M(\omega_0, m^2 + \boldsymbol{\Delta}^2, 0, \boldsymbol{\Delta}^2)$ is up to a factor $(2\sqrt{m^2 + \boldsymbol{\Delta}^2})^M$ identical with $(\partial/\partial s)^M\ T_{22}(s, t)$. Therefore $(\partial/\partial s)^M\ T_{22}(s, t)$ is for $-8\,m^2 < t \leqslant 0$ infinitely often differentiable in t and for fixed t a boundary distribution of an analytic function for $\mathrm{Im}\,s \neq 0$, which fulfills the integral representation (3.13). Integrating M times with respect to s one obtains the same result for $T_{22}(s, t)$, except for a polynomial $\sum_{\mu=0}^{M-1} c_\mu(t)\, s^\mu$ in s with coefficient distributions in t. But $T_{22}(s, t)$ is for fixed $s > 4\,m^2$ holomorphic in $t \in E_1(s)$ and the M-fold s-integral of τ^M is again C^∞ in t for $0 \geqslant t > -8\,m^2$. Therefore the $c_\mu(t)$ must also be C^∞-functions in t for $0 \geqslant t > -8\,m^2$, and we obtain the

Theorem 3.2: Under (A), (B), (D) the reduced 2-particle scattering amplitude $T_{22}(s, t)$ is the boundary value from $\mathrm{Im}\,s > 0$ of a holomorphic function in s for $\mathrm{Im}\,s \neq 0$, which is C^∞ in t and fulfills in s an M-fold subtracted dispersion relation. Furthermore $\mathrm{Im}\,T_{22}(s, t)$ is a measure in s and for fixed s holomorphic in t in the 'large Lehmann ellipse'

$$E_2(s) = \left\{t\colon |\,t\,| + |\,t + s - 4\,m^2\,| < (s - 4\,m^2)\left(2\left[1 + \frac{36\,m^4}{s(s - 4\,m^2)}\right]^{1/2} - 1\right)\right\}. \qquad (3.23)$$

To prove the last statement we remark that, for physical points, $\mathrm{Im}\,T_{22}(s, t)$ is essentially given by the measure

$$\bar{\tau}'(s, m^2, m^2, m^2, m^2, t) - \bar{\tau}'(u(s, t), m^2, m^2, m^2, m^2, t)\,.$$

Then the analyticity properties of Im $T_{22}(s, t)$ follow by studying the domain of analyticity $D'(s)$ of $\bar{\tau}'(s, z_1, \ldots z_5)$ (see [13]) [1]) [6])).

The theorems 3.1 and 3.2 contain the necessary analyticity properties for the proof of MANDELSTAM and LEHMANN[14]) that $T_{22}(s, t)$ is the boundary value of a function simultaneously holomorphic in s and t.

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