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Spatial Cluster Decomposition Properties of the S-Matrix

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Although no nontrivial rigorous model of a local relativistic quantum field theory is known up to now, the general theory of quantized fields is believed to describe short-range interactions of relativistic particles. The physically reasonable properties of the Wightman framework[1]) supporting this are exemplified by the space-like cluster decomposition properties of the vacuum expectation values of products of field operators[2])[3]) and by the existence of asymptotic scattering states in strong convergence[2])[4]).

It is the purpose of this note to prove that the S-matrix in a local relativistic quantum field theory fulfills the spatial cluster decomposition properties, which have recently[5]) been postulated to hold for any reasonable relativistic scattering theory. It will be demonstrated that the outcome of a scattering or production process between massive particles is asymptotically independent of the presence of other particles. This justifies an expansion of the S-matrix in cluster amplitudes[5]) with some analogy to sums of connected Feynman diagrams in perturbation theory.

We consider a Wightman field theory of an at most countable family of local spinor fields $A^r_\mu(x)$. We assume that polynomials B^*_i in the smeared-out fields

$$B^*_i = \sum_{(r)\,(\mu)}^{<\infty} \int {}_i\varphi^{(r)}_{(\mu)} \, (x_1, \ldots x_n) \, A^{r_1}_{\mu_1}(x_1) \ldots A^{r_n}_{\mu_n}(x_n) \, dx_1 \ldots dx_n \tag{1}$$

create from the vacuum Ω states belonging to irreducible representations $[m_i, s_i]$, $m_i > 0$, of the covering group of the inhomogeneous proper orthochronous Lorentz group:

$$B^*_i \, \Omega \in \mathfrak{H}_{[m_i, s_i]} \, , \quad B_i \, \Omega = 0 \, . \tag{2}$$

For a smooth positive frequency solution $f \in \mathfrak{S}^{m_i}_+$ of the Klein Gordon equation to the mass m_i $(f_i(x) = (2\pi)^{-3/2} \int dp \; \theta(p) \, \delta(p^2 - m_i^2) \, \hat{f}_i(\boldsymbol{p}) \, e^{-ipx}, \; \hat{f}_i(\boldsymbol{p}) \in \mathfrak{S}(R^3))$ we define with $B_i(x) = U(x, 1) \, B_i \, U(x, 1)^{-1}$:

$$B_i(f_i, t) = i \int_{x^0 = t} d^3\boldsymbol{x} \, f^*_i(x) \overleftrightarrow{\partial_0} \, B_i(x) \, . \tag{3}$$

Under these assumptions HAAG[4]) and RUELLE[2]) have proved that

$$s - \lim_{t \to \pm\infty} B^*_1(f_1, t) \ldots B^*_n(f_n, t) \, \Omega = |\bar{f}_1, \ldots \bar{f}^{ex}_n), \; \text{ex} = \text{in, out} \, , \tag{4}$$

exists in the strong topology in $\mathfrak{H}$ and defines an asymptotic scattering state belonging to the wave functions $\bar{f}_i = \{\bar{f}_i(\boldsymbol{p})_\alpha, \; \alpha = -s_i, -s_i + 1, \ldots + s_i\}$. The subspaces $\mathfrak{H}^{ex}$ of scattering states have the structure of Fock spaces over $\{\mathfrak{H}_{[m_i, s_i]}\}$. The scattering amplitudes $(\bar{f}_1, \ldots \bar{f}^{out}_m \mid \bar{g}_1 \ldots \bar{g}^{in}_n)$ are tensor-valued tempered distributions

$$S^{(\alpha)\,(\beta)}_{m\,n} \, (p_1, \ldots p_m; -q_1, \ldots -q_n) \in \mathfrak{S}'(R^{3\,(m+n)}) \tag{5}$$

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and define under the additional assumption $\mathfrak{H}^{in} = \mathfrak{H}^{out}$ a unitary S-matrix by $(\bar{f}_1, \ldots \bar{f}_m^{out} \,|\, S \,|\, \bar{g}_1, \ldots \bar{g}_n^{out}) = (\bar{f}_1, \ldots \bar{f}_m^{out} \,|\, \bar{g}_1 \ldots \bar{g}_n^{in})$. For a 3-vector $\boldsymbol{a}$ and $f \in \mathfrak{S}^+$ we set $f^a(x) = f(x^0, \boldsymbol{x} - \boldsymbol{a})$. If $B^*(f, t)\,\Omega = |\,\bar{f})$, then, due to the translation invariance of $B(x)$, one has $B^*(f^a, t)\,\Omega = |\,\bar{f^a})$ with $\bar{f^a}(\boldsymbol{p})_\alpha = e^{i\boldsymbol{p}\boldsymbol{a}} \bar{f}(\boldsymbol{p})_\alpha$. We are now prepared to prove the

Theorem 1: Under the above assumptions

$$|\langle \bar{f}_1 \ldots \bar{f}_k, \bar{f}^a_{k+1}, \ldots \bar{f}^a_{k+l}{}^{out} |\, \bar{g}_1, \ldots \bar{g}_m, \bar{g}^a_{m+1}, \ldots \bar{g}^a_{m+n}{}^{in}\rangle$$
$$- \langle \bar{f}_1 \ldots \bar{f}_k{}^{out} | \bar{g}_1 \ldots \bar{g}_m^{in}\rangle \langle \bar{f}_{k+1} \ldots \bar{f}_{k+l}{}^{out} | \bar{g}_{m+1}, \ldots \bar{g}_{m+n}{}^{in}\rangle| \leqslant c\,(1 + |\boldsymbol{a}|)^{-1/2}\,, \qquad (6)$$

for all $\bar{f}_i(\boldsymbol{p})_\alpha$, $\bar{g}_j(\boldsymbol{q})_\beta \in \mathfrak{D}(R^3)$ and arbitrary integers $k, l, m, n \geqslant 0$, with $c < \infty$ independent of $\boldsymbol{a}$. In particular one has

$$|\langle \bar{f}_1 \ldots \bar{f}_k \bar{f}^a_{k+1} \ldots \bar{f}^a_{k+l}{}^{out} |\, \bar{g}_1, \ldots \bar{g}_m{}^{in}\rangle| \leqslant c\,(1 + |\boldsymbol{a}|)^{-1/2}\,. \qquad (7)$$

Proof: If $B^*\,\Omega \in \mathfrak{H}_{[m,s]}$, $B\,\Omega = 0$, then there exists[2]) for every $\bar{f} = \{\bar{f}(\boldsymbol{p})_\alpha \in \mathfrak{D}(R^3)$, $\alpha = -s, -s+1, \ldots +s\}$ a finite number of quasi-local fields $B_r(x)$, $1 \leqslant r \leqslant N$, and $f_r \in \mathfrak{S}_+^m$ such that

$$B_r^*(x)\,\Omega \in \mathfrak{H}_{[m,s]},\ B_r(x)\,\Omega = 0,\ \sum_{r=1}^{N} B_r^*(f_r, t)\,\Omega = |\,\bar{f})\,. \qquad (8)$$

Without loss of generality we assume for (6) that one has already

$$|\bar{f}_i) = B_i^*(f_i, t)\,\Omega, \qquad |\,\bar{g}_j) = {}'B_j^*(\bar{g}_j, t)\,\Omega\,.$$

The C_∞-function

$$F(s, t, \boldsymbol{a}) = (B_1^*(f_1, s) \ldots B_{k+l}^*(f^a_{k+l}\, s)\,\Omega\,,\ {}'B_1^*(g_1, t) \ldots {}'B^*_{m+n)}g^a_{m+n,}\, t)\,\Omega)$$
$$- (B_1^*(f_1, s) \ldots \Omega,\ {}'B_1^*(g_1, t) \ldots \Omega)\ (\ldots B_{k+l}^*(f_{k+l}, s)\,\Omega,\ \ldots {}'B^*_{m+n}\,(g^*_{m+n}, t)\,\Omega) \qquad (9)$$

converges for $s, -t \to +\infty$ to

$$\langle \bar{f}_1 \ldots \bar{f}_k \bar{f}^a_{k+1} \ldots \bar{f}^a_{k+l}{}^{out}\, |\bar{g}_1, \ldots \bar{g}_m\, \bar{g}^a_{m+1}, \ldots \bar{g}^{a\,in}_{m+n}\rangle$$
$$- \langle \bar{f}_1, \ldots \bar{f}_k{}^{out}\, |\bar{g}_1, \ldots \bar{g}^{in}_m\rangle\, \langle \bar{f}_{k+1} \ldots \bar{f}^{out}_{k+l}\, |\, \bar{g}_{m+1}, \ldots \bar{g}^{in}_{m+n}\rangle\,.$$

In order to show the boundedness of $(1 + |\boldsymbol{a}|)^{1/2}\,|\lim_{s,-t\to\infty} F(s, t, \boldsymbol{a})|$, we write for an arbitrary $A > 0$:

$$\lim_{s,-t\to\infty} F(s, t, \boldsymbol{a}) = -\int_A^\infty ds \int_{-\infty}^{-A} dt\, \frac{d}{ds}\frac{d}{dt}\, F(s, t, \boldsymbol{a}) + \int_A^\infty ds\, \frac{d}{ds}\, F(s, -A, \boldsymbol{a}) +$$
$$+ \int_{-\infty}^{-A} dt\, \frac{d}{dt}\, F(A, t, \boldsymbol{a}) - F(A, -A, \boldsymbol{a}) \equiv \sum_{i=1}^{4} F_i(A, \boldsymbol{a})\,. \qquad (10)$$

Using the Schwarz inequality $F_1(A, \boldsymbol{a})$ can be majorized by

$$\int_A^\infty ds\, \left\| \frac{d}{ds}\, B_1^*(f_1, s) \ldots B_{k+l}^*(f^a_{k+l}, s)\,\Omega \right\| \int_{-\infty}^{-A} dt\, \left\| \frac{d}{dt}\, {}'B_1^*(g_1, t) \ldots {}'B^*_{m+n}(g^a_{m+n}, t)\,\Omega \right\| \qquad (11)$$

and by four terms of the form

$$\int_A^\infty ds\, \left\| \frac{d}{ds}\, B_1^*(f_1, s) \ldots B_k^*(f_k, s)\,\Omega \right\|\, \left\| B_{k+1}^*, (f_{k+1}, s) \ldots B_{k+l}^*(f_{k+l}, s)\,\Omega \right\|$$
$$\times \int_{-\infty}^{-A} dt\, \left\| \frac{d}{dt}\, {}'B_1^*(g_1, t) \ldots {}'B_m^*(g_m, t)\,\Omega \right\|\, \left\| {}'B^*_{m+1}(g_{m+1}, t) \ldots {}'B^*_{m+n}(g_{m+n}, t)\,\Omega \right\|. \qquad (12)$$

We expand $\| d/ds\, B_1^*(f_1, s) \ldots B_{k+l}^*(f_{k+l}^a, s)\, \Omega \|^2$ in a sum of products of truncated vacuum expectation values (TVEV) [2][4]), where the 1-point- and pure 2-point contributions vanish. The TVEV for $r \geqslant 3$ give rise to factors like:

$$\int d\boldsymbol{x}_1, \ldots d\boldsymbol{x}_r \langle B_{i_1}^{(*)}(\boldsymbol{x}, s) \ldots B_{i_r}^{(x)}(\boldsymbol{x}_r, s)\rangle^T h_1^{(a)}(\boldsymbol{x}_1, s) \ldots h_r^{(a)}(\boldsymbol{x}_r, s) \tag{13}$$

with $h_i^{(a)} \in \mathfrak{S}_+$ possibly translated by $\boldsymbol{a}$. (13) can be majorized [2][6]) uniformly in $\boldsymbol{a}$ by

$$\leqslant c \sup_{\boldsymbol{x}_1 (-\boldsymbol{a})} |h_1(\boldsymbol{x}_1, s)| \ldots \sup_{\boldsymbol{x}_{r-1} (-\boldsymbol{a})} |h_{r-1}(\boldsymbol{x}_{r-1}, s)| \ldots$$
$$\times \int d\boldsymbol{x}_r\, |h_r(\boldsymbol{x}_r (-\boldsymbol{a}), s)| \leqslant c'(1 + |s|)^{-3/2\,(r-2)}\,. \tag{14}$$

Since a similar estimate holds for (12), one obtains after s- and t-integration $|F_1(A, \boldsymbol{a})| < c_1(1 + A)^{-1}$, independently of $\boldsymbol{a}$. By expansion in TVEV one sees that terms like $\|B_1^*(f_1, A) \ldots B_{k+l}^*(f_{k+l}^a, A)\, \Omega\|^2$ are uniformly bounded for all A and $\boldsymbol{a}$. Therefore $|F_i(A, \boldsymbol{a})| \leqslant c_i(1 + A)^{-1/2}$, $i = 2, 3$, holds equally, uniformly in $\boldsymbol{a}$.

In order to majorize $F_4(A, \boldsymbol{a})$, we first assume that all the B_i, $'B_j$ in (9) are localized in a compact region $K = \{x_0^2 + \boldsymbol{x}^2 \leqslant R^2\}$, (i.e. supp ${}_i\varphi_{(\mu)}^{(*)} \subset K^{\times n}$ in (1)) and that $\operatorname{supp} f_i|_{x_0=0}$, $\operatorname{supp} g_j|_{x_0=0} < \{\boldsymbol{x}^2 \leqslant R^2\}$. Then $B(f^a, t)$ is localized in $\bigcup_{|\boldsymbol{y}| \leqslant R+t} \{(x_0 - t)^2 + (\boldsymbol{x} - \boldsymbol{a} - \boldsymbol{y})^2 \leqslant R^2\}$. For $A \leqslant |\boldsymbol{a}|/9$ the $B(f_i, A)$, $'B(g_j, -A)$ are localized space-like to the $B(f_{k+i}^a\, A)$, $'B(g_{m+j}^a, -A)$ for sufficiently large $|\boldsymbol{a}|$. Then the majorization of [3]), theorem 1, applies and gives for sufficiently large $|\boldsymbol{a}|$ and $A \leqslant |\boldsymbol{a}|/9$ the estimate $|F_4(A, \boldsymbol{a})| \leqslant c(A)\, |\boldsymbol{a}|^{3/2}\, e^{-M|\boldsymbol{a}|/2}$ ($M > 0$: smallest mass in the theory). The constant $c(A)$ is essentially a sum of products of norms like

$$\|'B_m(g_m, -A) \ldots B_k^*(f_k, A)\, \Omega\|\,,$$

which for $A \leqslant |\boldsymbol{a}|/9$ are uniformly bounded by a polynomial in $|\boldsymbol{a}|$, due to the temperedness of the Wightman distributions.

For arbitrary B_i, $'B_j$, f_i, g_j one uses the method of [3]), section 5, and obtains for any integer $N \geqslant 0$ a constant c_4^N independent of $\boldsymbol{a}$, A such that for $A \leqslant |\boldsymbol{a}|/9$:

$$|F_4(A, \boldsymbol{a})| \leqslant c_4^N(1 + |\boldsymbol{a}|)^{-N}\,. \tag{15}$$

The theorem is proved by setting $A = |\boldsymbol{a}|/9$.

The theorem can be extended to wave-packets $\bar{f}_i$, $\bar{g}_j$ from $\mathfrak{S}$, where the bounds c_i of $F_i(A, \boldsymbol{a})$, $1 \leqslant i \leqslant 3$, and c_4^N can be chosen to stay bounded on bounded sets of the $\{\bar{f}_i, \bar{g}_j\}$ in $\mathfrak{S}$. Using this one shows as in [7]) that (6) entails the following distribution identity:

Corollary: For all integers $k, l, m, n \geqslant 0$ one has in the topology of $\mathfrak{S}'(R^{3(k+l+m+n)})$ for $\varkappa < 1/2$:

$$\lim_{\boldsymbol{a}\to\infty} \Big|1 + |\boldsymbol{a}|\,|^{\varkappa}\Big|\, S_{k+l, m+n}^{(\alpha,\alpha')(\beta,\beta')}(\boldsymbol{p}_1, \ldots \boldsymbol{p}_{k+l}, -\boldsymbol{q}_1, \ldots -\boldsymbol{q}_{m+n})\, e^{i\boldsymbol{a}\left\{\sum_{i=1}^{k} \boldsymbol{p}_i - \sum_{j=1}^{m} \boldsymbol{q}_j\right\}}$$
$$- S_{k,m}^{(\alpha)(\beta)}(\boldsymbol{p}_1, \ldots \boldsymbol{p}_k, -\boldsymbol{q}_1 \ldots -\boldsymbol{q}_m)\, S_{l,n}^{(\alpha')(\beta')}(\boldsymbol{p}_{k+1}, \ldots \boldsymbol{p}_{k+l}, -\boldsymbol{q}_{m+1}, \ldots -\boldsymbol{q}_{m+n})\Big| = 0\,. \tag{16}$$

The theorem can be trivially extended to the decomposition into several clusters along a space-like plane. Furthermore the limit (6) relies only in a rudimentary way on the covariance of the theory under the homogeneous Lorentz group and on the microscopic locality of the fields. Essential for the proof was the existence of quasi-local field operators, which create eigenstates of the mass operator from the vacuum and whose TVEV decrease sufficiently strongly for large space-like separations of the arguments. Therefore the spatial cluster decomposition property of the S-matrix holds as well in the Araki-Haag framework of (macroscopically) localized observables[6]).

The approach of the limit as $0\ (|\boldsymbol{a}|^{-1/2})$ is governed by the decay of the wave-packets f_i, g_j. For non-overlapping wave-packets in velocity-space:

$$\left\{\frac{\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m_i^2}} : \boldsymbol{p} \in \operatorname{supp} \hat{f}_i\right\} \cap \left\{\frac{\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m_j^2}} : \boldsymbol{p} \in \operatorname{supp} \hat{f}_j\right\} = \phi\,, \quad i \neq j\,, \tag{17}$$

the limit (4) is attained faster than any power of $(1+|t|)^{-1}$ (see also H. ARAKI, R. HAAG, and D. W. ROBINSON, to be published). Then the scattering amplitudes show stronger cluster decomposition properties:

Theorem 2: For non-overlapping $\{f_i\}$ and, separately, $\{g_j\}$ (5) can be majorized by $c_N(1+|\boldsymbol{a}|)^{-N}(c_N < \infty$ for all $N)$ in all directions $\boldsymbol{e}$ satisfying for all $0 \leqslant a < \infty$:

$$\boldsymbol{a} = a\,\boldsymbol{e} \neq \frac{\boldsymbol{p}_i}{\sqrt{\boldsymbol{p}_i^2+m_i^2}} - \frac{\boldsymbol{p}_j}{\sqrt{\boldsymbol{p}_j^2+m_j^2}} \tag{18}$$

for $\boldsymbol{p}_i \in \operatorname{supp} \hat{f}_i\ (1 \leqslant i \leqslant k)$, $\boldsymbol{p}_j \in \operatorname{supp} \hat{f}_j\ (k+1 \leqslant j \leqslant k+l)$,
or $\boldsymbol{p}_i \in \operatorname{supp} \hat{g}_i\ (1 \leqslant i \leqslant m)$, $\boldsymbol{p}_j \in \operatorname{supp} \hat{g}_j\ (m+1 \leqslant j \leqslant m+n)$.

Proof: Using the notations of theorem 1 we shall show under (18) for $\boldsymbol{e}$ that $|F_i(a\,\boldsymbol{e}, A)|$, $1 \leqslant i \leqslant 3$, can be majorized by $c_N^i(1+A)^{-N}$, with $c_N^i < \infty$ for all N uniformly in a. Then theorem 2 follows by setting $A = a/9$.

Let $\varepsilon > 0$ be so small that the ε-neighbourhoods of the supports of $\{\hat{f}_i\}$ and, separately, of $\{\hat{g}_j\}$ still fulfill (17) and (18). Let $S(h, s) \equiv \{\boldsymbol{x} = (\boldsymbol{p}/\sqrt{\boldsymbol{p}^2+m^2})\ t,\ \boldsymbol{p} \in U_\varepsilon(\operatorname{supp} \hat{h})\}$. Then the integral (13) over the region $S(h_1, s) \times \ldots \times S(h_r, s)$ is strongly decreasing in s uniformly in a, due to the strong decrease of the supremum of $|\langle B_{i_1}^{(*)}(\boldsymbol{x}_1, s) \ldots B_{i_r}^{(*)}(\boldsymbol{x}_r, s)\rangle^T|$ in this region. For, a purely geometric argument shows that because of (18) the infimum of $|\boldsymbol{x}_i(-\boldsymbol{a}) - \boldsymbol{x}_{i+1}(-\boldsymbol{a})|$ for $\boldsymbol{x}_k \in S(h_k, s)$ increases linearly in s uniformly in a, at least for one $1 \leqslant i \leqslant r-1$. On the other hand the strong decrease of $|h_k(\boldsymbol{x}_k, s)|$ in s uniformly for $\boldsymbol{x}_k \in R^3 - S(h_k, s)$[2]) guarantees a similar majorization of (13) integrated over $R^3 - S(h_k, s)$.

Again, the results can be generalized to more complicated configurations along a space-like plane. No non-trivial temporal cluster decomposition properties of the S-matrix have been obtained.

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