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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **38 (1965)**

Heft I

PDF erstellt am: **30.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-113570

Scattering Theory for Relativistic Particles [a]) [b])

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(29. V. 64)

Relativistic particle quantum mechanics is a possible alternative to quantum field theory on the one hand and analytic S-matrix theory on the other. The scattering theory for one or several two-particle channels can be formulated in the rest frame of the center of mass along the same lines as the nonrelativistic scattering theory. The existence and completeness of the scattering states, their asymptotic properties, and the Lorentz invariance of the S operator follow from known mathematical theorems. To be consistent, a multiparticle theory must satisfy the following requirements. (1) The interaction of two particles should be the same when they are alone or when other particles are present at a large distance. (2) Since a particle system may break up into two or more noninteracting clusters, the dynamical description of the entire system must also contain the correct Lorentz-invariant description of each of the clusters. These requirements have been satisfied for three-particle systems.

I. Introduction

In relativistic field theories, the requirements of relativistic invariance impose potent restrictions on the equations of motion. That is a great virtue since one desires no arbitrary functions and few parameters in a ‚fundamental' theory of ‚elementary' particles. On the other hand, these restrictions may be so strong as to rule out all non-trivial theories, i.e., theories in which the scattering operator differs from the identity.

In particle dynamics without external fields, the restrictions are much less severe. At any rate, one may inquire whether conditions habitually imposed are really necessary or whether they are perhaps the result of unfounded prejudice. At least we must require that Lorentz transformations are represented by unitary operators in the Hilbert space of the states of the system. The irreducible unitary representations of the inhomogeneous Lorentz group are well known [c]) [1]). They are specified by the mass M and the spin j. The elements of the representation space $\mathfrak{H}_{Mj}$ are square-integrable functions $\chi(\boldsymbol{p}, j_3, \eta)$ where $\boldsymbol{p}$ is the momentum, j_3 is a component of the spin, and η stands for all other variables that may be necessary for a complete description. The norm of χ is

$$||\chi||^2 = \int d\boldsymbol{p} \sum_{j_3=-j}^{+j} \sum_{\eta} |\chi(\boldsymbol{p}, j_3, \eta)|^2 \,. \tag{1}$$

[a]) Work performed under the auspices of the U.S. Atomic Energy Commission.

[b]) In 1947 I suggested to Professor Stueckelberg in a casual conversation to formulate relativistic particle dynamics in the rest frame of the center of mass. His immediate reply was: «Obviously you can do this for two particles, but how do you propose to make sense for three particles?» It is a pleasure to dedicate this paper to his 60th birthday.

[c]) Only the positive-mass representations will be of interest to us.

General reducible representations are direct integrals of irreducible representations

$$\mathfrak{H} = \sum_{\oplus j} \int_{\oplus} d\sigma(M)\, \mathfrak{H}_{Mj} \,. \tag{2}$$

The elements of the representation space $\mathfrak{H}$ are square-integrable functions $\chi\,(\boldsymbol{p}, M, j, j_3, \eta)$ with the norm

$$||\chi||^2 = \int d\boldsymbol{p} \int d\sigma(M) \sum_j \sum_{j_3=-j}^{+j} \sum_\eta |\chi(\boldsymbol{p}, M, j, j_3, \eta)|^2 \,. \tag{3}$$

The Hilbert space $\mathfrak{H}$ is therefore a tensor product

$$\mathfrak{H} = \mathfrak{h} \otimes \mathfrak{H}_c \,, \tag{4}$$

where $\mathfrak{H}_c$ is the space of the square-integrable functions $\chi_c(\boldsymbol{p})$ and the 'little Hilbert space' $\mathfrak{h}$ is the representation space of the little group. The elements of $\mathfrak{h}$ are square-integrable functions $\phi\,(M, j, j_3, \eta)$. The dynamical description of the relative motion of the particles may be formulated in the little Hilbert space. Invariance under the little group is only a very weak restriction on possible interactions. Relativistic invariance alone is thus a rather weak requirement. Much stronger restrictions come from the asymptotic conditions to be discussed below.

EDDINGTON[2]) first pointed out that one may specify interactions between particles in terms of the relative coordinates in the rest frame of the center of mass. This frame is physically distinguished and the interactions are not restricted by relativistic requirements. The description of the particle system in any other Lorentz frame is then obtained by applying a Lorentz transformation to the entire system. The resulting theory is Lorentz invariant, but it is not satisfactory. The objections which were pointed out quickly by DIRAC, PEIERLS, and PRYCE[3]) may be paraphrased as follows. The interaction of two particles should be the same when they are alone or when other particles are present at a large distance. A particle system may break up into two or more noninteracting clusters. The dynamical description of the entire system must also contain the correct Lorentz-invariant description of each of the clusters. BAKAMJIAN and THOMAS[4]) have given a mathematical formulation of relativistic particle dynamics without regard to this problem. FOLDY[5]) has clearly formulated the problem and attempted a solution, but without success. The assumption that the individual particle coordinates are observable at all times leads to the requirement that they form invariant world lines. CURRIE, JORDAN, and SUDARSHAN[6] have shown that for two particles this requirement excludes any interaction[d]).

In a quantum theory of strongly interacting particles, it is reasonable to follow HEISENBERG[8]) and assert that the only observables are S-matrix elements and the masses and dynamical variables of free particles. The observable initial and final states of any scattering process are "free-particle states", that is they are vectors in a Hilbert space $\mathfrak{H}_f$ which is constructed by forming tensor products of single-particle spaces and direct sums of such tensor products. The single-particle spaces are representation spaces of irreducible representations of the inhomogeneous Lorentz group. Lorentz transformations on the tensor product are defined by taking the tensor product of the

[d]) This result has been extended to three particles by J. T. CANNON and T. F. JORDAN[7]).

corresponding irreducible representations. The realizations of the ten generators of the inhomogeneous Lorentz group are then additive in the individual particles. The S operator is a Lorentz invariant unitary mapping of this space onto itself. A single-particle state is left unchanged by the operator S. In addition, S must satisfy certain asymptotic conditions [e]) corresponding to the physical requirement that there shall be no scattering for infinite impact parameters. More generally, if a multiparticle system is spatially separated into two subsystems, the S operator must decompose into the tensor product of the separately Lorentz-invariant S operators for the two clusters [f]). This requirement, formulated as a limiting condition in the sense of strong operator convergence [g]), is

$$\underset{|\boldsymbol{a}| \to \infty}{s\text{-lim}} \exp (i\, \boldsymbol{p}'\, \boldsymbol{a})\, S \exp (-\, i\, \boldsymbol{p}'\, \boldsymbol{a}) = S' \otimes S'' , \tag{5}$$

where $\boldsymbol{p}'$ is the total momentum of a subsystem of particles [h]). The strong limit (5) follows from the weak convergence since

$$||S\, e^{i \boldsymbol{p}\, \boldsymbol{a}}\, \chi|| = ||\chi||$$

for all $\boldsymbol{a}$ and

$$||S' \otimes S''\, \chi|| = ||\chi|| .$$

Since we want to discuss a dynamical theory in which S is a derived quantity, we describe briefly the relevant features of ordinary quantum mechanics. States of the system of interacting particles are vectors ψ in a Hilbert space $\mathfrak{H}$. The generator of the time displacements (Hamiltonian) is known as a function of a complete set of operators. The time dependence of any state is then given by

$$\psi(t) = e^{-iHt}\, \psi(0). \tag{6}$$

The knowledge of the one-particle eigenstates of H for both elementary and composite particles provides us with an isometric mapping of the one-particle subspaces of $\mathfrak{H}_f$ onto subspaces of $\mathfrak{H}$. Tensor products and sums of these mappings give a mapping Φ of $\mathfrak{H}_f$ into $\mathfrak{H}$. This mapping is needed to formulate the initial conditions for a scattering process. A scattering state $\psi(t)$ is by definition a state which for $t \to -\infty$ approaches a state of noninteracting particles, i.e.,

$$\lim_{t \to -\infty} ||\psi(t) - \Phi\, e^{-iH_0 t}\, \chi|| = 0 , \tag{7}$$

where

$$\psi(t) = e^{-iHt}\, \psi(0) , \tag{8}$$

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[e]) Within the framework of axiomatic field theory, spatial asymptotic conditions have been formulated by HAAG [9]) and RUELLE [10]).

[f]) The importance of multiparticle scattering in a consistent S-matrix theory was first emphasized by E. C. G. STUECKELBERG in a series of seminar lectures in 1945.

[g]) The cluster decomposition property of the S operator has been formulated by WICHMAN and CHRICHTON [11]) as a weak operator limit.

[h]) For the sake of simplicity we assume here and in the following that the particles are distinguishable. Identical particles can always be treated by projecting out the symmetric or antisymmetric states.

and $\chi \in \mathfrak{H}_f$. The operator H_0 is the energy of the noninteracting particles. The existence of the limit (7) for every χ implies the strong operator limit

$$\Omega_- = \underset{t \to -\infty}{s\text{-lim}}\, e^{iHt}\, \Phi\, e^{-iH_0 t} . \tag{9}$$

The S matrix is defined by

$$S_{fa} = \lim_{t \to +\infty} (\Phi\, e^{-iH_0 t}\, \chi_f, \psi_a(t))$$
$$= (\chi_f, \Omega_+^\dagger\, \Omega_-\, \chi_a) , \tag{10}$$

where

$$\Omega_+ = \underset{t \to +\infty}{s\text{-lim}}\, e^{iHt}\, \Phi\, e^{-iH_0 t} . \tag{11}$$

The operators Ω_+ and Ω_- are generalized Møller operators. The S operator is then

$$S = \Omega_+^\dagger\, \Omega_- . \tag{12}$$

It is unitary if the operator $\Omega_\pm$ are complete, i.e., if Ω_+ and Ω_- have equal ranges.

In the simplest case, that in which all observed particles are elementary, the operator algebras in $\mathfrak{H}$ and $\mathfrak{H}_f$ are isomorphic. The vectors χ and $\Phi\chi$ may then be represented by the same wave functions. It is then convenient to follow the usual practice of identifying $\mathfrak{H}_f$ and $\mathfrak{H}$ with $\Phi = 1$. For rearrangement collosions this possibility does not exist.

In ordinary quantum mechanics the dynamical variables operating in $\mathfrak{H}$ are supposed to be observables. In an S-matrix theory this assumption disappears. Assumptions about the operators operating on $\mathfrak{H}$ are primarily a matter of convenience. Their real content lies only in their consequences for S. In fact, a large equivalence class of Hamiltonians produces the same operator S[12]). Let Ξ be any unitary operator satisfying the condition

$$\underset{t \to \pm\infty}{s\text{-lim}}\, (\Xi - 1)\, \Phi\, e^{-iH_0 t} = 0 . \tag{13}$$

Then the Hamiltonian

$$H' = \Xi^\dagger\, H\, \Xi \tag{14}$$

produces the same S operator as H since

$$\underset{t \to \pm\infty}{s\text{-lim}}\, e^{iH't}\, \Phi\, e^{-iH_0 t} = \Xi^\dagger\, \Omega_\pm . \tag{15}$$

In relativistic field theory, the operators in $\mathfrak{H}$ are the basic fields which may or may not have simple relations to the observed particles. In any relativistic theory we must have a representation of the inhomogeneous Lorentz group $U(a, \Lambda)$ in $\mathfrak{H}$ such that

$$U(a, \Lambda) = \Omega_-\, U_0(a, \Lambda)\, \Omega_-^\dagger \tag{16}$$

and

$$\Omega_+\, U_0(a, \Lambda)\, \Omega_+^\dagger = \Omega_-\, U_0(a, \Lambda)\, \Omega_-^\dagger , \tag{17}$$

where $U_0(a, \Lambda)$ is the group representation in $\mathfrak{H}_f$. A homogeneous Lorentz transformation is labeled by Λ, a space-time translation by a. Corresponding to the equivalence class of Hamiltonians generated by the operators Ξ there is also an equivalence class of representations $U(a, \Lambda)$. The relation (17) guarantees the invariance of the S operator[i]). The S operator has the cluster property (5) is the Møller operators $\Omega_\pm$ have this property, i.e., if

$$\underset{|\boldsymbol{a}|\to\infty}{s\text{-lim}}\,(\Omega_\pm - \Omega'_\pm \otimes \Omega''_\pm)\, e^{i\boldsymbol{p}'\boldsymbol{a}} = 0\,. \tag{18}$$

Equation (5) follows from Equation (18) since

$$S - S' \otimes S'' = \Omega_+^\dagger\,(\Omega_- - \Omega'_- \otimes \Omega''_-) + (\Omega_+ - \Omega'_+ \otimes \Omega''_+)^\dagger\, \Omega'_+ \otimes \Omega''_- \tag{19}$$

and $(S - S' \times S'')\exp i\,\boldsymbol{p}'\,\boldsymbol{a}$ vanishes strongly if it vanishes weakly.

The purpose of the present paper is to explore a scheme in which the dynamical variables operating on $\mathfrak{H}$ are those of a finite number of particles, in particular the case in which they are isomorphic to the observables in $\mathfrak{H}_f$. The operator $\Omega_\pm$ may be constructed from assumed explicit expressions for the generators of the representation U. To do this for just two particles is almost trivial on its face. Explicit expressions for the generators are well known[14])[15]). The scattering problem may be formulated and solved in the rest frame of the center of mass[13]). The resulting S operator is manifestly invariant by construction. The only nontrivial problem is a mathematically rigorous proof of the existence of the Møller operators $\Omega_\pm$ and their asymptotic properties under suitable assumptions about the interaction term in the rest energy. Relevant theorems have been proved by several authors[16–23]). If the interaction energy is an operator of rank one, then the Møller operator can be obtained explicitly in closed form[24]). For spinless particles, JORDAN, MACFARLANE, and SUDARSHAN[25]) exhibit this particular Møller operator and verify by direct calculation that it is unitary and satisfies the required asymptotic conditions.

For more than two particles, the separability requirement of FOLDY[5]) presents a major difficulty. We exhibit a class of three-particle systems with the correct cluster structure. The kinematics of relativistic two-body systems has been thoroughly investigated by JOOS[26]) and MACFARLANE[27]). The kinematics of the two noninteracting clusters is in essence the same as for two particles. In Sec. II, the pertinent information will be reviewed without proof and a convenient notation will be introduced. The scattering theory for two-body systems is discussed in Sec. III. In that section we assemble the mathematical machinery which will allow us to treat three-particle scattering in Sec. IV.

Most of our discussions will be concerned with single-channel scattering, i.e., the same particles appear in the initial and final states. The simplest many-channel systems are those for which the Hilbert space is the direct sum of several n-particle spaces for different numbers and species of particles with possible transitions between channels. The extension of our theory to such systems is straightforward and will be discussed briefly. Multichannel scattering by composite particles involves additional problems which will not be considered in this paper.

As a result we have a scheme for constructing unitary Lorentz-invariant S operators with the correct cluster structure from self-adjoint interaction operators. The latter are as arbitrary as the Hamiltonians in low-energy nuclear theory; for a

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[i]) FONG and SUCHER[13]) show that it is easy to construct Hamiltonians such that Møller operators $\Omega_\pm$ exist but Equation (17) is not satisfied.

physical theory, a concrete proposal for a description of real systems is needed. Such attemps are beyond the scope of the present paper.

II. Kinematics of Particle Systems

The generators of the inhomogeneous Lorentz group are the space translation $\boldsymbol{P}$, the time displacement $\boldsymbol{H}$, the space rotations $\boldsymbol{J}$, and the proper Lorentz transformations $\boldsymbol{K}$. Their commutation relations, with $c = \hbar = 1$, are

$$[P_i, P_j] = [P_i, H] = [J_i, H] = 0 ,$$
$$[J_i, A_j] = i \sum_k \varepsilon_{ijk} A_k , \tag{20}$$

where A_j is the j component of any 3-vector, i.e., J_j, P_j, or K_j; and

$$[K_i, K_j] = - i \sum_k \varepsilon_{ijk} J_k ,$$
$$[H, K_j] = - i P_j , \qquad [P_i, K_j] = - i \delta_{ij} H . \tag{21}$$

In the space of the square-integrable functions $\chi(\boldsymbol{p}, M, j, j_3, \eta)$ [see Equation (3)], the generators are realized by

$$\boldsymbol{P} = \boldsymbol{p} , \quad H = (p^2 + M^2)^{1/2} \equiv p_0$$
$$J = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{j} , \tag{22}$$
$$\boldsymbol{K} = \frac{1}{2} (\boldsymbol{x} H + H \boldsymbol{x}) - (\boldsymbol{j} \times \boldsymbol{p}) (M + H)^{-1} ,$$

where

$$x_k = i \frac{\partial}{\partial p_k} \tag{23}$$

and $p = | \boldsymbol{p} |$. The four vector $(p_0, \boldsymbol{p})$ will be denoted by $\tilde{p}$ and $\tilde{p}^2 = - M^2$. For practical purposes it is more convenient to define dynamical variables and then find the generators as functions of these variables than to follow the more common procedure of defining dynamical variables as functions of the generators. For a single elementary system M, j and η are fixed numbers. In general we may parameterize the little Hilbert space by any set of commuting operators that also commute with $\boldsymbol{x}$ and $\boldsymbol{p}$. In place of M we have then an operator h which commutes with $\boldsymbol{p}$, $\boldsymbol{x}$, and $\boldsymbol{j}$. Any operator A is invariant if and only if it commutes with $\boldsymbol{p}$, $\boldsymbol{x}$, $\boldsymbol{j}$, and h. We note that the spin $\boldsymbol{j}$ is invariant under translations and Lorentz transformations in the direction of $\boldsymbol{p}$. According to WIGNER[1]), any homogeneous Lorentz transformation Λ may be decomposed as

$$\Lambda = \beta(\tilde{p}') \, \mathfrak{R}(\tilde{p}, \Lambda) \, \beta^{-1} (\tilde{p}) , \tag{24}$$

where $\tilde{p}' = \Lambda \tilde{p}$, and $\beta(\tilde{p})$ is a special Lorentz transformation defined by

$$\beta_{ik}(\tilde{p}) = \delta_{ik} + \frac{p_i p_k}{M} (M + p_0), \quad i, k = 1, 2, 3 \tag{25}$$

$$M^2 = - \tilde{p}^2 ,$$
$$\beta^0_\mu = \frac{p_\mu}{M} . \tag{26}$$

It is designed such that

$$\beta(\tilde{p})\ (M, 0, 0, 0) = \tilde{p}\,. \tag{27}$$

$\Re(\tilde{p}, \Lambda)$ is a rotation defined by Equation (24). We have then

$$U^\dagger(\beta(\tilde{p}))\, \boldsymbol{j}\, U(\beta(\tilde{p})) = \boldsymbol{j}\,, \tag{28}$$

$$U^\dagger(\Lambda)\, \boldsymbol{j}\, U(\Lambda) = \Re(\tilde{p}, \Lambda)\, \boldsymbol{j}\,. \tag{29}$$

For two noninteracting systems, we have the operators $\boldsymbol{p}^{(i)}$, $\boldsymbol{x}^{(i)}$, $\boldsymbol{j}^{(i)}$ and $h^{(i)}$ for each system and the Lorentz generators are additive in the two systems. We wish to write the generators of the combined system again in the canonical form. The operators that parameterize the little Hilbert space must commute with $\boldsymbol{PK}$. The individual spins $\boldsymbol{j}^{(1)}$ and $\boldsymbol{j}^{(2)}$ do not have this property.

Let $\tilde{p}$ be the total momentum

$$\tilde{p} = \tilde{p}^{(1)} + \tilde{p}^{(2)}. \tag{30}$$

We define

$$\tilde{k} = \frac{1}{2}\, \beta^{-1}(\tilde{p})\ (\tilde{p}^{(1)} - \tilde{p}^{(2)}) \tag{31}$$

as the relative momentum in the rest frame of the center of mass. The channel spin $\boldsymbol{s}$ is defined by

$$\begin{aligned} \boldsymbol{s} &= U(\beta(\tilde{p}))\ [\boldsymbol{j}^{(1)} + \boldsymbol{j}^{(2)}]\ U^\dagger(\beta(\tilde{p})) \\ &= \Re(\tilde{p}^{(1)}, \beta^{-1}(\tilde{p}))\, \boldsymbol{j}^{(1)} + \Re(\tilde{p}^{(2)}, \beta^{-1}(\tilde{p}))\, \boldsymbol{j}^{(2)}. \end{aligned} \tag{32}$$

The spin of the combined system is then

$$\boldsymbol{j} = \boldsymbol{y} \times \boldsymbol{k} + \boldsymbol{s}\,, \tag{33}$$

where $y_l = i\partial/\partial k_l$. The spin $\boldsymbol{j}$ is the sum of the intrinsic orbital angular momentum and the channel spin.

The transformation coefficients between the $\boldsymbol{p}^{(1)}$, $\boldsymbol{p}^{(2)}$, $j^{(2)}$, $j^{(1)}$, $j_3^{(1)}$, $j_3^{(2)}$ representation and the $\boldsymbol{p}$, $\boldsymbol{k}$, $j^{(1)}$, $j^{(2)}$, s, s_3 representation are closely related to the Clebsch-Gordan coefficients of the inhomogeneous Lorentz group (20, 21). We have

$$\begin{aligned} &\phi(\boldsymbol{p}, \boldsymbol{k}, j^{(1)}, j^{(2)}, s, s_3) \\ &= \int d\boldsymbol{p}^{(1)} \int d\boldsymbol{p}^{(2)} \sum_{j_3^{(1)}} \sum_{j_3^{(2)}} (\boldsymbol{p}, \boldsymbol{k}, s, s_3\, |C(j^{(1)}, j^{(2)})|\, \boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}, j_3^{(1)}, j_3^{(2)}) \\ &\qquad \times \psi(\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}, j^{(1)}, j_3^{(1)}, j^{(2)}, j_3^{(2)})\,, \end{aligned} \tag{34}$$

where

$$\begin{aligned} &(\boldsymbol{p}, \boldsymbol{k}, s, \mu\, |C\ (j', j'')|\, \boldsymbol{p}', \boldsymbol{p}'', m', m'') \\ &= (p_0\, \omega\, /\, p_0'\, p_0'')^{1/2}\, \delta(\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{p}'')\ \delta(\boldsymbol{k} - \boldsymbol{q}\ (\tilde{p}', \tilde{p}'')) \\ &\quad \sum_{\overline{m}'} \sum_{\overline{m}''} (j'\, j''\, \overline{m}'\, \overline{m}''\ |\ s\, \mu)\ D^{j'}_{\overline{m}', m'}\ (\Re_1)\ D^{j'}_{\overline{m}', m'}\ (\Re_2)\,, \end{aligned} \tag{35}$$

$$\tilde{q}(\tilde{p}', \tilde{p}'') = \frac{1}{2}\, \beta^{-1}\ (\tilde{p}^2)\ (\tilde{p}' - \tilde{p}''), \quad \Re_1 = \Re\ (\tilde{p}', \beta^{-1}(\tilde{p}))\,,$$

$$\Re_2 = \Re(\tilde{p}'', \beta^{-1}(\tilde{p})),$$

and

$$\omega^{-1} = (k^2 + M'^2)^{-1/2} + (k^2 + M''^2)^{-1/2}\,.$$

The coefficients $(j'\, j''\, m'\, m'' \mid s\, \mu)$ are the ordinary Clebsch-Gordan coefficients for the addition of angular momenta. The transformation from a $\boldsymbol{k}$, s, s_3 representation to a k, l, s, j, j_3 representation is obviously given by

$$\sum_m (l\, s\, m\, s_3 \mid j\, j_3)\; Y_{l\,m}(\vartheta, \varphi)^* \,, \tag{36}$$

where ϑ and φ are the polar angles of $\boldsymbol{k}$.

If there are more than two independent systems, the little Hilbert space may be parameterized by combining them successively. The resultant choice of variables depends, of course, on the order of the coupling. Representations corresponding to different coupling orders are related by Lorentz-invariant recoupling coefficients.

The commutation rules of the generators are preserved as long as the rest energy h is any operator that commutes with $\boldsymbol{p}$, $\boldsymbol{x}$, and $\boldsymbol{j}$. The conditions that the scattering theory imposes on the operator h will be discussed in the following sections.

III. Two-Body Scattering

The mathematical formulation of the scattering problem in the rest frame of the center of mass is qualitatively the same as for the familiar nonrelativistic theory[28]). Most of the formal developments[29])[30]) can be taken over without change. Existence proofs that rely specifically on the nonrelativistic momentum dependence of the kinetic energy[31])[32]) are not applicable without modification. Directly relevant existence theorems have been proved by several authors[16–23]). The sufficient conditions of Kato[j]) include those of all others.

We identify the Hilbert spaces $\mathfrak{H}_f$ and $\mathfrak{H}$. According to (9) and (11) with $\Phi = 1$ we have,

$$\Omega_\pm(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{iHt}\, e^{-H_0 t} \,, \tag{37}$$

where $H = (p^2 + h^2)^{1/2}$ and $H_0 = (p^2 + h_0^2)^{1/2}$. The operators h and h_0 are the rest-energy operators with and without interaction.

A consistent relativistic scattering theory depends on the following fundamental theorem.

Theorem 1: If the Møller operators

$$\Omega_\pm (h, h_0) = \operatorname*{s-lim}_{\tau \to \pm\infty} e^{ih\tau}\, e^{-ih_0\tau} \tag{38}$$

and

$$\Omega_\pm (h_0, h) = \operatorname*{s-lim}_{\tau \to \pm\infty} e^{ih_0\tau}\, e^{-ih\tau} \tag{39}$$

exist, then the strong limits $\Omega_\pm(H, H_0)$, $\Omega_\pm(H_0, H)$ exist also and

$$\Omega_\pm(H, H_0) = \Omega_\pm(h, h_0) \,, \tag{40}$$

$$\Omega_\pm(H_0, H) = \Omega_\pm(h_0, h) \,. \tag{41}$$

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j) See theorem 2 of reference [23]).

Proof: Consider h and h_0 as operators on the little Hilbert space $\mathfrak{h}$ and define

$$H_0'(p) = (p^2 + h_0^2)^{1/2} , \tag{42}$$

$$H'(p) = (p^2 + h^2)^{1/2} \tag{43}$$

for fixed p. According to KATO (theorems 1 and 2 of ref. [23])), we have

$$\Omega_\pm(H', H_0') = \Omega_\pm(h, h_0) , \tag{44}$$

$$\Omega_\pm(H_0', H') = \Omega_\pm(h_0, h) . \tag{45}$$

Theorem 1 follows from (44), (45) and Lebesgue's bounded convergence theorem.

We assume that $v \equiv h - h_0$ is an integral operator in the space of the square-integrable functions of $\boldsymbol{k}$, and that v may be written in the form

$$v = \bar{\Gamma}^\dagger \Gamma \tag{46}$$

such that $k^2(\boldsymbol{k} \mid \Gamma^\dagger \Gamma \mid \boldsymbol{k})$ and $k^2 (\boldsymbol{k} \mid \bar{\Gamma}^\dagger \bar{\Gamma} \mid \boldsymbol{k})$ are smooth bounded functions of $\boldsymbol{k}$ that vanish as k^2 at the origin. These assumptions are sufficient for the existence of the limits $\Omega_\pm(h, h_0)$ and $\Omega_\pm(h_0, h)$. If Γ and $\bar{\Gamma}$ are Schmidt-class operators [i.e., $\mathrm{Tr}(\Gamma^\dagger \Gamma) < \infty$], then v is by definition of trace class[k]). The existence of the Møller operators is then assured[16])[17]). The operators $\Gamma(h_0 + i)^{-1}$ and $\bar{\Gamma}(h_0 + i)^{-1}$ are always of Schmidt class under our assumptions; KURODA's sufficient conditions[18]) are therefore satisfied.

We proceed to review the consequences of the existence of the limits $\Omega_\pm(h, h_0)$. If a function $f(t)$ has an integrable derivative, then

$$\lim_{t \to \pm\infty} f(t) = f(0) + \int_0^{\pm\infty} dt \, \frac{df}{dt} . \tag{47}$$

The existence of the limit on the right-hand side is necessary and sufficient for the existence of the limit on the left-hand side. From the existence of the limit, it follows that

$$\lim_{t \to \pm\infty} \frac{df}{dt} = 0 . \tag{48}$$

It is easy to generalize these relations to operators and strong limits[l]).

From Equation (47) it follows that

$$\Omega_\pm = 1 + i \int_0^{\pm\infty} dt \, e^{iht} \, v \, e^{-ih_0 t} . \tag{49}$$

From Equation (48) it follows that[m])

$$h \, \Omega_\pm = \Omega_\pm \, h_0 , \tag{50}$$

and therefore

$$\underset{t \to \pm\infty}{s\text{-lim}} \, e^{ih_0 t} \, \Omega_\pm \, e^{-ih_0 t} = 1 . \tag{51}$$

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k) A good summary of the definition and simple properties of trace-class operators can be found in reference [33]). For proofs and more extensive material, see reference [34]).

l) Or see references [29]) and [30]).

m) Strictly speaking, Equation (50) means that $h \, \Omega_\pm f = \Omega_\pm \, h_0 f$ for all f in the domain of h_0.

From Equations (51), (47), and (50), one gets the Lippman-Schwinger equation

$$\Omega_\pm = 1 + i \int_0^{\pm\infty} dt\, e^{i h_0 t}\, v\, \Omega_\pm\, e^{-i h_0 t} . \tag{52}$$

In the same manner, for the S operator we obtain

$$S = 1 - i \int_{-\infty}^{+\infty} dt\, e^{i h_0 t}\, v\, \Omega_-\, e^{-i h_0 t} , \tag{53}$$

and hence

$$S = 1 - i \int_{-\infty}^{+\infty} dt\, e^{i h_0 t}\, v \left(1 - i \int_{-\infty}^{0} dt'\, e^{i h t'}\, v\, e^{-i h_0 t'}\right) e^{-i h_0 t}$$

$$= 1 - i \int_{-\infty}^{+\infty} dt\, e^{i h_0 t} \int T(M)\, dP_0(M)\, e^{-i h_0 t} , \tag{54}$$

where $dP_0(M)$ is the spectral projection of h_0 and

$$T(M) = \lim_{\varepsilon \to 0} v + v\,(M - h + i\,\varepsilon)^{-1}\, v . \tag{55}$$

We assume that the operator

$$K_\pm(M) = \lim_{\varepsilon \to 0} \Gamma\,(M - h_0 \mp i\,\varepsilon)^{-1}\, \bar{\Gamma}^\dagger \tag{56}$$

is completely continuous and find (35)

$$T(M) = \bar{\Gamma}^\dagger (1 - K_-(M))^{-1}\, \Gamma , \tag{57}$$

$$v\, \Omega_\pm = \int \bar{\Gamma}^\dagger (1 - K_\pm(M))^{-1}\, \Gamma\, dP_0(M) . \tag{58}$$

In the nonrelativistic theory in which

$$h_0 \cong M_1 + M_2 + \frac{1}{2}\, k^2 (M_1^{-1} + M_2^{-1}) \tag{59}$$

and v is a local potential, Equation (50) leads to a set of ordinary differential equations for the scattering wave functions. Phase shifts can be easily computed by solving these equations numerically. For relativistic h_0, the simplicity of differential equations is not available and local potentials have no mathematical advantage over nonlocal interactions. For the computations of scattering amplitudes, algebraic techniques are available[35]).

By definition $\Omega_\pm(h, h_0)$ commutes with $\boldsymbol{p}$, $\boldsymbol{x}$, and $\boldsymbol{j}$. From (50), it follows that the operator

$$S = \Omega_+^\dagger\, \Omega_-$$

commutes with h_0. The operator S is therefore Lorentz invariant. It is unitary if the ranges of Ω_+ and Ω_- are the same. This property is assured since the limits $\Omega_\pm(h_0, h)$ as well as $\Omega_\pm(h, h_0)$ exist[19]).

An obvious extension is the case of several coupled two-body channels where all particles are elementary. The Hilbert space is then the direct sum of several two-particle spaces, i.e.,

$$\mathfrak{H} = \sum_{\oplus \varkappa} \mathfrak{H}_\varkappa . \tag{60}$$

The description of the free particles in each channel is the same as before, namely

$$h_0 = \sum_{\varkappa} h_{0\varkappa} . \tag{61}$$

The interaction operator couples different channels, i.e., $v \to v_{\varkappa\varkappa'}$ [n]).

For two coupled channels, we may write

$$h = h_1 + h_2 + v' \tag{62}$$

such that $\mathfrak{H}_\varkappa$ reduces $h_\varkappa$ and v' has no diagonal elements in either channel. Let ϕ_1 and ϕ_2 be states in channels 1 and 2, respectively. We have then

$$(\phi_2, S\,\phi_1) = -i\left(\phi_2, \int\limits_{+\infty}^{-\infty} dt\, e^{i h_{02} t}\, \Omega_{2+}{}^{\dagger}\, v'\, \Omega_-\, e^{i h_{01} t}\, \phi_1\right), \tag{63}$$

where

$$\Omega_{2+} = \lim_{\tau \to \infty} e^{i h_2 \tau}\, e^{-i h_{02} \tau} . \tag{64}$$

In deriving Equation (63) from Equation (12), we have used the transitivity of the Møller operator [19]), i.e.,

$$\Omega_\pm(h, h_0) = \Omega_\pm(h, h_1 + h_2)\, \Omega_\pm(h_1 + h_2, h_0) . \tag{65}$$

A two-body system coupled to other channels of arbitrary complexity may be described by an energy-dependent effective interaction which is the analogue of the optical-model potential in the theory of nuclear reactions [o]).

The states of the system are now vectors in a big Hilbert space

$$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}' , \tag{66}$$

where $\mathfrak{H}^0$ is the space of the two-particle states. All initial and final states of interest are in $\mathfrak{H}^0$. The decomposition (66) reduces the operator h_0 such that

$$h_0 = h_0^0 + h_0' . \tag{67}$$

The interaction v consists of two terms, i.e.,

$$v = v^0 + v' , \tag{68}$$

where

$$v^0 \phi' = 0 \text{ if } \phi' \in \mathfrak{H}' \tag{69}$$

and the range of v^0 is in $\mathfrak{H}^0$. The coupling term v' maps vectors in $\mathfrak{H}^0$ into vectors in $\mathfrak{H}'$ and vice versa. The Møller operators $\Omega_\pm$ are now partial isometries which map $\mathfrak{H}^0$ into part of $\mathfrak{H}$, i.e.,

$$\Omega_\pm\, \phi = \lim_{\tau \to \pm\infty} e^{i h \tau}\, e^{-i h_0 \tau}\, \phi \tag{70}$$

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[n]) As an example, one might investigate the $3/2^+$ meson-baryon resonances by assuming a simple SU_3-invariant interaction. The results of such an investigation should be very similar to those obtained by WALI and WARNOK [36]).

[o]) See, for instance, reference [37]).

for $\phi \in \mathfrak{H}^0$. Let Λ be the projection operator that projects into H^0. Only $\Lambda\, v\Omega_\pm\, \Lambda$ is of interest [see Equation (53)]; and since

$$v\, \Omega_\pm = \Omega_\pm\, h_0 - h_0\, \Omega_\pm\,, \tag{71}$$

it is sufficient to consider $\Lambda\, \Omega_\pm\, \Lambda \equiv \Omega^0_\pm$ and $\Omega'_\pm \equiv (1-\Lambda)\, \Omega_\pm \Lambda$. From Equation (52) it follows that

$$\begin{aligned}\Omega'_\pm &= i \int\limits_0^{\pm\infty} d\tau\, e^{i h_0' \tau}\, v'\, \Omega^0_\pm\, e^{-i h_0 \tau} \\ &= \underset{\epsilon\to 0}{s\text{-lim}} \int (M - h_0' \pm i\,\varepsilon)^{-1}\, v'\, \Omega^0_\pm\, dP_0(M)\end{aligned} \tag{72}$$

and

$$\Omega^0_\pm = 1 + i \int\limits_0^{\pm\infty} d\tau\, e^{i h_0 \tau}\, (v\, \Omega_\pm + v'\, \Omega'_\pm)\, e^{-i h_0 \tau}\,. \tag{73}$$

After substituting Equation (72) into (73), we have

$$\Omega^0_\pm = 1 + i \int\limits_0^{\pm\infty} d\tau\, e^{i h_0 \tau}\, \bar{v}\, \Omega^0_\pm\, e^{-i h_0 \tau}\,, \tag{74}$$

where the effective interaction $\bar{v}$ is defined by

$$\bar{v}\, \Omega^0_\pm = v\, \Omega_\pm + \lim_{\varepsilon\to 0} \int v'(M - h_0' \pm i\,\varepsilon)^{-1}\, v'\, \Omega^0_\pm\, dP_0(M)\,. \tag{75}$$

From (70) and (71) one obtains

$$\Omega^0_\pm = \int [1 + (M - h_0 - \bar{v}(M) \pm i\,\varepsilon)^{-1}\, \bar{v}(M)]\, dP_0(M)\,, \tag{76}$$

with

$$\bar{v}(M) = \lim_{\varepsilon\to 0} \Lambda(v + v'\,(M - h_0 \pm i\,\varepsilon)^{-1}\, v')\, \Lambda; \tag{77}$$

and therefore [see Equation (55)]

$$T(M) = \lim_{\varepsilon\to 0} \bar{v}(M) + \bar{v}(M)\, \big(M - h_0 - \bar{v}(M) + i\,\varepsilon\big)^{-1}\, \bar{v}(M)\,. \tag{78}$$

The coupling to the space $\mathfrak{H}'$ has thus been eliminated. It manifests itself in the effective interaction $\bar{v}$. The operator $\bar{v}$ is Hermitian if M is not in the spectrum of h_0'.

For two-body scattering, the cluster condition (5) is simply

$$\underset{|\boldsymbol{a}|\to\infty}{s\text{-lim}}\, (S - 1)\, \exp\,(i\, \boldsymbol{p}^{(1)}\, \boldsymbol{a}) = 0\,. \tag{79}$$

We note that

$$\boldsymbol{p}^{(1)}\, \boldsymbol{a} = \boldsymbol{k}\, \boldsymbol{a}' - \sqrt{k^2 + M_1^2}\, a_0' \tag{80}$$

where

$$\tilde{a}' = \beta^{-1}(\tilde{p})\, (0, \boldsymbol{a})\,.$$

We have therefore

$$\exp\,(i\, \boldsymbol{p}^{(1)}\, \boldsymbol{a})\, S\, \exp\,(-\, i\, \boldsymbol{p}^{(1)}\, \boldsymbol{a}) = e^{i \boldsymbol{k}\, \boldsymbol{a}'}\, S\, e^{-i \boldsymbol{k}\, \boldsymbol{a}} \tag{81}$$

and the condition (79) is equivalent to

$$\underset{|\boldsymbol{a}|\to\infty}{s\text{-lim}}\, (S - 1)\, e^{i \boldsymbol{k}\, \boldsymbol{a}} = 0\,. \tag{82}$$

Since

$$S - 1 = \Omega_+^\dagger (\Omega_- - \Omega_+) , \tag{83}$$

it will be sufficient to prove

$$\underset{|\boldsymbol{a}| \to \infty}{s\text{-lim}} (\Omega_\pm - 1) e^{i\boldsymbol{k}\,\boldsymbol{a}} = 0 . \tag{84}$$

The relation (84) has been proved by COOK[38]) for nonrelativistic scattering by local potentials. For the proof of Equation (84), we need the following lemma.

Lemma 1: Let

$$v = \sum_{\alpha=1}^{n} \bar{\Gamma}_\alpha^\dagger \Gamma_\alpha , \tag{85}$$

where Γ_α and $\bar{\Gamma}_\alpha$ have the same properties as Γ and $\bar{\Gamma}$ in Equation (46) and

$$\Omega_\pm = \underset{\tau \to \pm\infty}{s\text{-lim}}\, e^{i(h_0+v)\tau} e^{-ih_0\tau} . \tag{86}$$

Then for every $\phi \in \mathfrak{h}$

$$||(\Omega_\pm - 1)\phi||^4 \leqslant 4 \left(\sum_\alpha \int_0^{\pm\infty} d\tau\, ||\bar{\Gamma}_\alpha e^{-ih_0\tau} \phi||^2\right) \left(\sum_\beta \int_0^{\pm\infty} d\tau'\, ||\Gamma_\beta \Omega_\pm e^{-ih_0\tau'} \phi||^2\right) . \tag{87}$$

Proof of Lemma 1: Since $\Omega_\pm^\dagger \Omega_\pm = 1$, we have

$$||(\Omega_\pm - 1)\phi||^2 = (\phi, (1 - \Omega_\pm)\phi) + (\phi, (1 - \Omega_\pm)\phi)^* ; \tag{88}$$

and from Equation (52) it follows that

$$||(\Omega_\pm - 1)\phi||^4 \leqslant 4 \left| \left(\phi, \int_0^{\pm\infty} d\tau\, e^{ih_0\tau} v\, \Omega_\pm e^{-ih_0\tau'} \phi\right)\right|^2 . \tag{89}$$

The lemma follows from (89) by Schwarz's inequality.

To prove Equation (84), we insert $\phi \to e^{i\boldsymbol{k}\,\boldsymbol{a}} \phi$ into Equation (87). There is a dense set of functions $\phi(\boldsymbol{k})$ such that for each ϕ the second factor remains bounded and the first tends to zero as $a \to \infty$. We note that

$$e^{i\boldsymbol{k}\,\boldsymbol{a}} = e^{ikaz} = (i\,k\,a)^{-1} \frac{\partial}{\partial z} e^{ikaz} , \tag{90}$$

$-1 \leqslant z \leqslant +1$, and integrate by parts. The functions $\phi(\boldsymbol{k})$ shall be smooth functions that vanish if the vector $\boldsymbol{k}$ is either parallel or antiparallel to $\boldsymbol{a}$, i.e., $z = \pm 1$. Thus

$$\int_0^{\pm\infty} d\tau\, ||\bar{\Gamma}_\alpha e^{-ih_0\tau} e^{i\boldsymbol{k}\,\boldsymbol{a}} \phi||^2$$

$$\leqslant \frac{2\pi}{a^2} \int d\boldsymbol{k} \int d\boldsymbol{k}' (k\,k')^{-1} \delta(\omega - \omega') \frac{\partial}{\partial z} \frac{\partial}{\partial z'} \phi^*(\boldsymbol{k}') (\boldsymbol{k}' \mid \bar{\Gamma}_\alpha^\dagger \Gamma_\alpha \mid \boldsymbol{k}) \phi(\boldsymbol{k})$$

$$< \text{const}, a^{-2} , \tag{91}$$

where

$$\omega \equiv (k^2 + M_1^2)^{1/2} + (k^2 + M_2^2)^{1/2} .$$

IV. Three-Particle Scattering

In practice, the multiparticle scattering amplitudes of interest are those for two particles in the initial state and n particles in the final state. The cluster properties that the S matrix must have as a matter of principle are best studied by considering scattering amplitudes for n particles in both the initial and final state. The Møller operators for n-body scattering are needed in any case to correctly describe the final-state interactions in a channel in which they are produced. Equation (63) is still valid if channel 2 is an n-particle channel.

The kinematical complexities increase rapidly with particle number. That must be expected since even in nonrelativistic mechanics there is no very simple way to eliminate the center of mass from the dynamical variables. We shall therefore discuss only threebody systems. Throughout this section it is assumed that the interactions do not produce bound states.

The variables of a [(12)3] coupling scheme of a three-particle system are defined as follows. The total momentum is

$$\tilde{p} = \tilde{p}^{(1)} + \tilde{p}^{(2)} + p^{(3)} . \tag{92}$$

The total momentum of the (12) cluster is

$$\tilde{p}^{(1\,2)} = \tilde{p}^{(1)} + \tilde{p}^{(2)} . \tag{93}$$

The relative momentum of the (12) cluster is

$$\tilde{k}^{(1\,2)} = \frac{1}{2}\, \beta^{-1}\,(\tilde{p}^{(1\,2)})\,(\tilde{p}^{(1)} - \tilde{p}^{(2)}) . \tag{94}$$

By Equation (32), the channel spin of the (12) cluster is

$$\boldsymbol{s}^{(1\,2)} = \Re(\tilde{p}^{(1)}, \beta^{-1}\,(\tilde{p}^{(1\,2)}))\, \boldsymbol{j}^{(1)} + \Re(\tilde{p}^{(2)}, \beta^{-1}\,(\tilde{p}^{(1\,2)}))\, \boldsymbol{j}^{(2)} . \tag{95}$$

The spin of the (12) cluster is

$$\boldsymbol{j}^{(1\,2)} = \boldsymbol{y}^{(1\,2)} \times \boldsymbol{k}^{(1\,2)} + \boldsymbol{s}^{(1\,2)} . \tag{96}$$

The relative momentum of the particle 3 and the (12) cluster is

$$\tilde{k}_3 = \beta^{-1}(\tilde{p})\,(\tilde{p}^{(3)} - \tilde{p}^{(1\,2)}) . \tag{97}$$

The rest energy of the noninteracting system is

$$h_0 = \left(k^2 + h_0^2\,(1\;2)\right)^{1/2} + (k_3^2 + M_3^2)^{1/2} , \tag{98}$$

where

$$h_0\,(1\;2) = (k^{(1\,2)2} + M_1^2)^{1/2} + (k^{(1\,2)2} + M_2^2)^{1/2} . \tag{99}$$

The next task is to modify $h_0 \to h$ such that we have a theory of interacting particles. The existence of Møller operators and a unitary Lorentz-invariant S operator is no longer sufficient for a satisfactory theory. We must recover the Lorentz-invariant two-body theory in the limit in which the third particle is removed.

An interaction operator $v = h - h_0$ that satisfies Kuroda's conditions for the existence of $\Omega_\pm(h, h_0)$ vanishes if any of the three particles is removed to infinity, i.e.,

$$\operatorname*{s-lim}_{a\to\infty} v \exp\,(i\, \boldsymbol{p}^{(j)}\, \boldsymbol{a}) = 0 , \tag{100}$$

$j = 1, 2, 3$. From this it follows that

$$\underset{a \to \infty}{s\text{-lim}} \, (\Omega_{\pm} - 1) \exp (i \, \boldsymbol{p}^{(j)} \boldsymbol{a}) = 0 \, . \tag{101}$$

Equation (101) can be proved in the same manner as Equation (84). For such three-body interactions there is no scattering if any one of the three particles is far away.

More instructive is the special case in which particles 1 and 2 interact and there is no interaction with particle 3. In that case we have

$$h = \left(k_3^2 + h^2 \, (1\,2)\right)^{1/2} + (k_3^2 + M_3^2)^{1/2} \tag{102}$$

and

$$h(1\,2) = h_0(1\,2) + v(1\,2) \, . \tag{103}$$

A satisfactory theory must fulfill the requirement that

$$\Omega_{\pm}(h, h_0) = \Omega_{\pm}\left(h(1\,2), h_0(1\,2)\right) \equiv \Omega_{\pm}^{(1\,2)}. \tag{104}$$

Equation (104) does indeed hold as a consequence of theorem 1. The cluster conditions for S are therefore satisfied. Our results would be the same for any number of additional noninteracting particles.

Define now a two-body interaction $V(12)$ in the rest frame of the three-particle system by the relation

$$V(1\,2) = \{k_3^2 + [h_0(1\,2) + v(1\,2)]^2\}^{1/2} - \left(k_3^2 + h_0^2(1\,2)\right)^{1/2} . \tag{105}$$

Similar definitions hold for any pair of particles so that the full rest-energy operator of the three-particle system is

$$h = h_0 + v \, , \tag{106}$$

with

$$v = V(1\,2) + V(1\,3) + V(2\,3) \, . \tag{107}$$

Kato's proof[23]) of the existence and completeness of the Møller operators $\Omega_{\pm}(h, h_0)$ is based on the assumption that $\phi(h) - \phi(h_0)$ is of trace class for some suitable monotonic function ϕ. That assumption is not valid for multiparticle Hamiltonians of the form (106) even in the non-relativistic case [p]). Inspection of the proofs of Kato[17])[23]) and Kuroda[18]) indicates that it should be possible to extend them in such a manner that they cover multiparticle scattering if the two-body interactions guarantee the existence of two-body Møller operators. Along these lines, the existence of $\Omega_{\pm}$ is obviously easier to prove than the completeness. Since according to Equation (49)

$$||(\Omega_{\pm} - 1) \, \phi|| = ||\int\limits_0^{\pm\infty} d\tau \, e^{i h \tau} \, v \, e^{-i h_0 \tau} \, \phi||$$

$$\leqslant \frac{1}{2} \sum_{ik} \int\limits_0^{\pm\infty} d\tau \, ||V(i\,k) \, e^{-i h_0 \tau} \, \phi|| \, , \tag{108}$$

it is sufficient to prove the convergence of the integral in Equation (108) for a dense set of vectors ϕ. For two-body interactions of the form considered in Sec. III, there

p) For nonrelativistic three-body scattering, Fadeev[39]) has given a proof of the existence and completeness of the Møller operators.

is a suitable dense set of smooth functions of the momenta for which the convergence of the integral can be proved by partial integration.

The following theorem due to PROSSER[40]) establishes the existence and unitarity of $\Omega_\pm$ for certain multiparticle systems.

Theorem 2: Let h_0 and v be self-adjoint operators. Assume that the operator v may be written in the form

$$v = \sum_{\alpha=1}^{n} \bar{\Gamma}_\alpha^\dagger \Gamma_\alpha , \tag{109}$$

where Γ_α and $\bar{\Gamma}_\alpha$ are bounded operators such that for each ϕ in the domain of h_0

$$||\Gamma_\alpha e^{-i h_0 \tau} \bar{\Gamma}_\beta^\dagger \phi|| \leqslant K_{\alpha\beta}(\tau) \, ||\phi|| , \tag{110}$$

where $K_{\alpha\beta}(\tau)$ is independent of ϕ and

$$\sum_{\alpha\beta} \int_{-\infty}^{+\infty} K_{\alpha\beta}(\tau) = \eta < 1 . \tag{111}$$

Then the strong limits $\Omega_\pm(h, h_0)$ exist and are unitary.

Prosser proves this theorem for $n = 1$. Our generalization $n > 1$ does not require any changes in the course of his proof. It is this generalization, however, that makes the theorem useful for multiparticle systems.

The cluster property of Møller operators may be demonstrated as follows. Since [19])

$$\Omega_\pm(h, h_0) = \Omega_\pm\big(h, h_0 + V(1\,2)\big) \, \Omega_\pm\big(h_0 + V(1\,2), h_0\big) , \tag{112}$$

we use Equation (104) to derive the inequality

$$\begin{aligned} ||(\Omega_\pm - \Omega_\pm^{(1\,2)}) \phi||^4 &= ||(\Omega'_+ - 1) \, \Omega_\pm^{(1\,2)} \phi||^4 \\ &\leqslant 4 \, |(\Omega_\pm^{(1\,2)} \phi, (\Omega'_\pm - 1) \, \Omega_\pm^{(1\,2)} \phi|^2 , \end{aligned} \tag{113}$$

where

$$\Omega'_\pm = \Omega_\pm\big(h, h_0 + V(1\,2)\big) . \tag{114}$$

From

$$\Omega'_\pm - 1 = - i \int_0^{\pm\infty} d\tau \, e^{i(h_0 + V(1\,2))\tau} \big(V(1\,3) + V(2\,3)\big) \, \Omega'_\pm \, e^{-i(h_0 + V(1\,2))\tau} , \tag{115}$$

$$\big(h_0 + V(1\,2)\big) \, \Omega_\pm^{(1\,2)} = \Omega_\pm^{(12)} \, h_0 , \tag{116}$$

and Schwarz's inequality, it follows that

$$\begin{aligned} ||(\Omega_\pm - \Omega_\pm^{(1\,2)}) \phi||^4 \leqslant 4 \int_0^{\pm\infty} d\tau (||\bar{\Gamma}_{1\,3} \, \Omega_\pm^{(1\,2)} e^{-i h_0 \tau} \phi||^2 + ||\bar{\Gamma}_{2\,3} \, \Omega_\pm^{(1\,2)} e^{-i h_0 \tau} \phi||^2) \\ \times \int_0^{\pm\infty} d\tau' \, (||\Gamma_{1\,3} \, \Omega_\pm e^{-i h_0 \tau'} \phi||^2 + ||\Gamma_{2\,3} \, \Omega_\pm \, e^{-i h_0 \tau'} \phi||^2) . \end{aligned} \tag{117}$$

From the inequality (117) we can prove the asymptotic condition

$$\underset{a \to \infty}{s\text{-lim}} \, (\Omega_\pm - \Omega_\pm^{(1\,2)}) \, e^{i \boldsymbol{k} \, \boldsymbol{a}} = 0$$

by the same arguments that were used to establish the asymptotic condition (84).

Part of this paper has benefited greatly from unpublished lecture notes by R. HAAG. I wish to thank R. HAAG and H. EKSTEIN for many long discussions and critical comments on the subject of this paper.

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