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On the Clebsch-Gordan Series of Semisimple Lie Algebras

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Abstract. Starting from a formula of Steinberg, we derive a simple representation theorem for the highest weights in the decomposition of a tensor product of irreducible modules into irreducible constituents which is valid for arbitrary split semisimple Lie algebras (over a field of characteristic 0). Furthermore we use the formula of Steinberg to evaluate the multiplicities of the irreducible modules corresponding to these highest weights for special Lie algebras.

Introduction

In the physical literature, there exist quite a few papers about the Clebsch-Gordan series of SU_3[1]). But it seems that it has been overlooked that Steinberg[2]) has given a formula for the decomposition of a tensor product of irreducible modules into irreducible constituents which is valid for arbitrary split semisimple Lie algebras over a field of characteristic 0. The formula of Steinberg expresses the multiplicities of the irreducible constituents by a double sum over the Weyl group W. Hence to determine the multiplicities, one only has to know the root system.

In § 1 we discuss briefly the formula of Steinberg. Starting from this formula, we prove a general representation theorem for the highest weights in the decomposition of the tensor product in § 2. With the help of this theorem, we can easily determine the multiplicities of the irreducible modules corresponding to these highest weights for special Lie algebras. This is carried out in § 3 for the algebras A_2, G_2, and A_3.

§ 1. The Formula of Steinberg

Let $\mathfrak{M}_{\Lambda'}$ and $\mathfrak{M}_{\Lambda''}$ be two finite dimensional irreducible modules with the highest weights Λ' and Λ'' of a finite dimensional semisimple Lie algebra $\mathfrak{L}$ over a field of characteristic 0. Further we assume that $\mathfrak{L}$ has a splitting Cartan subalgebra $\mathfrak{H}$ (the characteristic roots of every $ad(h)$, $h \in \mathfrak{H}$, are in the base field). If the base field is algebraically closed, any finite dimensional Lie algebra is of course split.

The tensor product $\mathfrak{M}_{\Lambda'} \otimes \mathfrak{M}_{\Lambda''}$ is, according to a general theorem, completely reducible (this is the case for arbitrary finite dimensional Lie algebras over a field of characteristic 0). Let

$$\mathfrak{M}_{\Lambda'} \otimes \mathfrak{M}_{\Lambda''} = \bigoplus_{\Lambda} m_{\Lambda}\, \mathfrak{M}_{\Lambda} \qquad (1)$$

be its decomposition into irreducible modules with the multiplicities m_Λ, then the formula of Steinberg reads

$$m_\Lambda = \sum_{S, T \in W} \det (S\, T)\, P\, [S\,(\Lambda' + \delta) + T\,(\Lambda'' + \delta) - (\Lambda + 2\,\delta)]\,. \tag{2}$$

The sum on the right hand side of (2) extends over the Weyl group W. This group is finite and is generated by the reflections at the simple roots (hence $\det (ST) = \pm 1$). δ is one half of the sum of all positive roots: $\delta = {}^1/_2 \sum_{\alpha > 0} \alpha$. $P\,[M]$ is the number of solutions of $\sum_{\alpha > 0} k_\alpha\, \alpha = M$, where the k_α are non-negative integers. From this definition follows that $P\,[M]$ is different from zero only if M is an integral linear function over the Cartan algebra $\mathfrak{H}$*).

It is perhaps useful to see how one can immediately obtain from (2) the usual Clebsch-Gordan series for the Lie algebra A_1. Let α be the only positive root; $\lambda = \alpha/2$ is the fundamental dominant weight; $\delta = \alpha/2$. The Weyl group consists simply of I and S_α (S_α: reflection at the root α), i.e. W is the cyclic group Z_2. We put $\Lambda' = m'\,\lambda$, $\Lambda'' = m''\,\lambda$, $\Lambda = m\,\lambda$; m, m', m'' non-negative integers. If we assume that $m' \geqslant m''$, then the only terms which contribute to the sum of the right hand side of (2) are $(S, T) = (1, 1)$ and $(S, T) = (1, S)$. We obtain

$$m_\Lambda = P\left[\frac{m' + m'' - m}{2}\,\alpha\right] - P\left[\frac{m' - m'' - m - 2}{2}\,\alpha\right]$$

which means: $m_\Lambda = 1$ for $m = m' + m''$, $m' + m'' - 2, \dots m' - m''$ and $m_\Lambda = 0$ in all other cases.

§ 2. A Representation Theorem for Λ in (1)

In this paragraph we prove the following

Theorem. The highest weights in (1) necessarily have the form

$$\Lambda = \Lambda' + \Lambda'' - \sum_{j=1}^{l} n_j\, \alpha_j$$

with non-negative integers n_j and the simple system of roots $\pi = (\alpha_1, \alpha_2, \dots \alpha_l)$.

Proof: To prove this theorem, we need the following

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*) M is an integral linear function over $\mathfrak{H}$ if $M \in \mathfrak{H}^*$ ($\mathfrak{H}^*$ dual space of $\mathfrak{H}$) has the property $M(h_i)$ integer for $i = 1, 2, \dots l$ (l = rank of $\mathfrak{L}$). Here the h_i are those elements of the Cartan algebra which belong to the set of canonical generators. They are defined in the following way: Let $\pi = (\alpha_1, \dots \alpha_l)$ be a simple system of roots with the characteristic property that every root $\alpha = \sum_{i=1}^{l} k_i\, \alpha_i$, $\alpha_i \in \pi$, where the k_i are all either non-negative or non-positive integers. To every linear function $\alpha_i \in \mathfrak{H}^*$ we attribute the vector $h_{\alpha_i} \in \mathfrak{H}$ such that $\alpha_i(h) = (h_{\alpha_i}, h)$ for all $h \in \mathfrak{H}$ (scalar product = Killing form); then $h_i = 2\, h_{\alpha_i}/(\alpha_i, \alpha_i)$. The integral linear functions form a lattice with the fundamental dominant weights (defined by the property $\lambda_i(h_j) = \delta_{ij}$) as a basis. There is a 1 : 1 correspondence between the isomorphism classes of finite dimensional irreducible modules for $\mathfrak{L}$ and the set of dominant integral linear functions of $\mathfrak{H}$ (Λ dominant integral function if $\Lambda(h_i) \geqslant 0$).

Lemma. For $S \in W$ and $S \neq I$, $\delta - S\,\delta$ is a non zero sum of distinct positive roots*).

This lemma can be found in [3]). For reasons of completeness repeat the short proof: Since the Weyl group simply permutes the roots, $S\,\delta = \delta - \Sigma\,\beta$, where the summation is taken over the $\beta = -S\,\alpha > 0$. If there would be no such β, then $S\,\alpha > 0$ for all α. Then simple roots would be carried over into simple ones (compare footnote pag. 57), i.e. $S\,\pi = \pi$. According to a well known theorem[4]), we could conclude $S = I$, contrary to hypothesis.

Since also the weights are simply permuted under the Weyl group, especially $S\,\Lambda$ is a weight if Λ is the highest weight (dominant integral linear function on $\mathfrak{H}$) of an irreducible module. According to a well known theorem it can be represented as $S\,\Lambda = \Lambda - \sum_{j=1}^{l} k_j\,\alpha_j$ with non-negative integers k_j.

Hence, using the lemma, we get for $S \neq I$

$$S\,(\Lambda + \delta) = \Lambda + \delta - \sum k_j\,\alpha_j - (\delta - S\,\delta) = \Lambda + \delta - \sum \varkappa_j\,\alpha_j$$

where the $\varkappa_j$ are non-negative integers which do not vanish simultaneously.

The general argument of P in (3), which we simply denote with $X_{S,\,T}$, is for $(S, T) \neq (1, 1)$ therefore of the form

$$X_{S,\,T} = \Lambda' + \Lambda'' - \Lambda - \sum w_j\,\alpha_j\,;$$
$$w_j \text{ non-negative integers, not all} = 0\,.$$

From this one easily concludes, that a necessary condition for $m_\Lambda \neq 0$ is

$$P\,[\Lambda' + \Lambda'' - \Lambda] \neq 0\,. \tag{4}$$

In order to translate this condition into an explicit form, we put

$$\Lambda' = \sum m_s'\,\lambda_s,\ \Lambda'' = \sum m_s''\,\lambda_s,\ \Lambda = \sum m_s\,\lambda_s\,,$$

λ_s; $s = 1, \dots l$ are the fundamental dominant weight (compare footnote pag. 57). If we expand the λ_s in terms of the simple roots, the condition $\lambda_j(h_i) = \delta_{ij}$ immediately shows that the expansion matrix is the inverse Cartan matrix, i.e.

$$\lambda_i = \sum (A^{-1})_{j\,i}\,\alpha_i\,, \tag{5}$$

where

$$A_{ij} \overset{\text{Def.}}{=} \frac{2\,(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \alpha_j(h_i)\,, \tag{6}$$

hence

$$\Lambda' + \Lambda'' - \Lambda = \sum \alpha_j \left(\sum_s (A^{-1})_{j\,s}\,\Delta\,m_s\right),$$

with

$$\Delta\,m_s = m_s' + m_s'' - m_s$$

*) In the subspace $\mathfrak{H}_0^* \subset \mathfrak{H}^*$ over the rationals with basis $\alpha_1 \dots \alpha_l$, we introduce the usual ordering: $\alpha = \Sigma\,\lambda_i\,\alpha_i > 0$ if $\lambda_1 = \dots = \lambda_h = 0$, $\lambda_{h+1} > 0$, $h < l$. $\alpha > \beta$ if $\alpha - \beta > 0$. The simple roots then can not be written as a sum of positive roots.

(4) requires that

$$\sum_s (A^{-1})_{js} \Delta m_s = n_j ,$$

with non-negative n_j. From this we get

$$m_s = m_s' + m_s'' - \sum A_{sj} n_j , \tag{7}$$

or

$$\Lambda = \Lambda' + \Lambda'' - \sum A_{sj} n_j \lambda_s$$

and with (5)

$$\Lambda = \Lambda' + \Lambda'' - \sum n_j \alpha_j \tag{8}$$

what we intended to prove.

We remark that not for every Λ of the form (8) (with Λ dominant) m_Λ has to be different from zero. Indeed, one easily finds counter examples. On the other hand, the weight $\Lambda = \Lambda' + \Lambda''$ always appears with multiplicity one. For practical purposes the formula (7) is more useful. Of course, the n_j are restricted by the condition

$$\sum A_{sj} n_j \leq m_s' + m_s'' .$$

§ 3. Evaluation of Steinberg's Formula for Special Lie algebras

To decompose the tensor product (1), we can now, according to the theorem of § 2, restrict ourself to dominant weights Λ of the form (8). For the calculation of the multiplicities m_Λ, we have to know explicitly the $X_{S,T}$, i.e. we have to determine expressions of the type $S(\Lambda + \delta)$ (Λ = highest weight, $S \in W$).

We first derive a generally valid recursion formula which is useful for this purpose. A reflection S_i at a simple root α_i is given by

$$S_i \alpha_j = \alpha_j - A_{ij} \alpha_i . \tag{9}$$

Now, the following equation holds: $S_i \delta = \delta - \alpha_i$. This is due to the fact that $S_i \alpha > 0$ if $\alpha > 0$, except for $\alpha = \alpha_i$, where of course $S_i \alpha_i = - \alpha_i$ (compare [5]). Hence

$$S_i \delta = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ \alpha \neq \alpha_i}} \alpha - \frac{1}{2} \alpha_i = \delta - \alpha_i .$$

From this we get

$$S_i (\Lambda + \delta) = \sum_{s,j} m_s (A^{-1})_{js} S_i \alpha_j + \delta - \alpha_i$$

or with (9)

$$S_i (\Lambda + \delta) = \Lambda + \delta - (m_i + 1) \alpha_i . \tag{10}$$

Now we put for $S \in W$

$$S(\Lambda + \delta) - (\Lambda + \delta) = - \sum_j \sigma_j (S) \alpha_j$$

then we get from (10) the following recursion formula

$$\sum_j \sigma_j (S_i S) \alpha_j = (m_i + 1) \alpha_i + \sum_j \sigma_j (S) \alpha_j - \sum_j \sigma_j (S) A_{ij} \alpha_i . \tag{11}$$

We turn now to special Lie algebras.

1. Example: A_2

Let α_1 and α_2 be the two simple roots for A_2. The Cartan matrix is

$$(A_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(7) reads in this case

$$m_1 = m_1' + m_1'' - (2 n_1 - n_2)$$
$$m_1 = m_2' + m_2'' - (2 n_2 - n_1)$$

n_1, n_2 non-negative integers.

The Weyl group consists of the following six elements: $W = \{1, S_1, S_2, S_1 S_2, S_1 S_2 S_1, (S_1 S_2)^2\}$. The defining relation (beside $S_1^2 = S_2^2 = 1$) is $S_2 = (S_1 S_2)^2 S_1$. We also remark here that the Weyl group for A_l is isomorphic to the symmetric group S_{l+1}[6]). With the help of the recursion formula (11), we obtain now for the multiplicities the following explicit expression

$$m_\Lambda = \sum_{S, T \in W} \det (S, T) \; P \left[\sum_i (n_i - \sigma_i'(S) - \sigma_i''(T)) \alpha_i\right] \tag{13}$$

$\sigma_i'(S)$ and $\sigma_i''(S)$ can be read off in table 1 (substitute for m_j in $\sigma_i(S)$ respectively m_i' and m_i'').

Table 1

S	$\sigma_1(S)$	$\sigma_2(S)$
1	0	0
S_1	$1+m_1$	0
S_2	0	$1+m_2$
$S_1 S_2$	$1+m_1$	$2+m_1+m_2$
$S_1 S_2 S_1$	$2+m_1+m_2$	$2+m_1+m_2$
$(S_1 S_2)^2$	$2+m_1+m_2$	$1+m_2$

For concrete examples the sum in (13) is carried out immediately. We illustrate this for the tensor product $(1,1) \otimes (3,0)$. (For a more general example compare the appendix). The possible n-values in (12) are: $n \equiv (n_1, n_2) = (3,2), (2,1), (2,0), (1,1), (1,0), (0,0)$. For $n = (3,2)$ the following terms contribute in (13): $(S, T) = (1,1), (1, S_2), (S_1, 1), (S_2, 1), (S_1, S_2)$ and one gets

$$m_{\Lambda(0,0)} = P[3\alpha_1 + 2\alpha_2] - P[3\alpha_1 + \alpha_2] - P[\alpha_1 + 2\alpha_2] -$$
$$- P[3\alpha_1] + P[\alpha_1 + \alpha_2] = 3 - 2 - 2 - 1 + 2 = 0 .$$

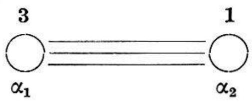
Still easier one sees that $m_{\Lambda(0,3)} = 0$ (corresponding to $n = (2,0)$), while in all other cases $m_\Lambda = 1$. Thus we get the well known decomposition

$$(1,1) \otimes (3,0) = (1,1) \oplus (3,0) \oplus (2,2) \oplus (4,1)$$

or

$$8 \otimes 10 = 8 \oplus 10 \oplus 27 \oplus 35$$

2. *Example:* G_2

From the Dynkin diagram: 

one can read off the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

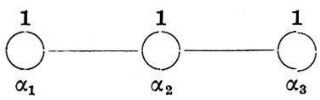
The Weyl group and $\sigma_i(S)$ $i = 1, 2$ are given in table 2. For low dimensional representations only few terms contribute in (13).

Table 2

S	$\sigma_1(S)$	$\sigma_2(S)$
1	0	0
S_1	(m_1+1)	0
S_2	0	m_2+1
S_2S_1	m_1+1	$3\,m_1+m_2+4$
$S_1S_2S_1$	$3\,m_1+m_2+4$	$3\,m_1+m_2+4$
$(S_2S_1)^2$	$3\,m_1+m_2+4$	$3\,(2\,m_1+m_2+3)$
$S_1(S_2S_1)^2$	$4\,m_1+2\,m_2+6$	$3\,(2\,m_1+m_2+3)$
$(S_2S_1)^3$	$4\,m_1+2\,m_2+6$	$6\,m_1+4\,m_2+10$
$S_1(S_2S_1)^3$	$3\,m_1+2\,m_2+5$	$6\,m_1+4\,m_2+10$
$(S_2S_1)^4$	$3\,m_1+2\,m_2+5$	$3\,m_1+3\,m_2+6$
$S_1(S_2S_1)^4$	m_1+m_2+2	$3\,m_1+3\,m_2+6$
$(S_2S_1)^5 = S_1S_2$	m_1+m_2+2	m_2+1

3. *Example:* A_3

Because the Lie algebra A_3 is possibly of physical interest, we give here the explicit expressions for this example. From the Dynkin diagram



one obtains for the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(8) reads here

$$m_1 = m_1' + m_1'' - (2\, n_1 - n_2)$$
$$m_2 = m_2' + m_2'' - (2\, n_2 - n_1 - n_3)$$
$$m_3 = m_3' + m_3'' - (2\, n_3 - n_2)\, .$$

The construction of the Weyl group from the reflections at the simple roots is here somewhat tedious. Beside $S_j^2 = 1$, $i = 1, 2, 3$, the defining relations of this group are:

$$S(13) = S(31),\ S(121) = S(212),\ S(232) = S(323), \text{ where for example}$$
$$S(231) \equiv S_2\, S_3\, S_1\, .$$

The different elements of the Weyl group and $\sigma_i(S)$ $i = 1, 2, 3$ are given in table 3. For given $n = (n_1, n_2, n_3)$ only those terms contribute of course to m_Λ for which the inequalities $\sigma_j'(S) + \sigma_j''(T) \leqslant n_j$, $j = 1, 2, 3$, are fulfilled. This condition restricts the summation over the Weyl group in many cases to a few terms only.

Table 3

S	$\sigma_1(S)$	$\sigma_2(S)$	$\sigma_3(S)$
1	0	0	0
S_1	m_1+1	0	0
S_2	0	m_2+1	0
S_3	0	0	m_3+1
$S(12)$	m_1+m_2+2	m_2+1	0
$S(21)$	m_1+1	m_1+m_2+2	0
$S(13)$	m_1+1	0	m_3+1
$S(23)$	0	m_2+m_3+2	m_3+1
$S(32)$	0	m_2+1	m_2+m_3+2
$S(121)$	m_1+m_2+2	m_1+m_2+2	0
$S(123)$	$m_1+m_2+m_3+3$	m_2+m_3+2	m_3+1
$S(231)$	m_1+1	$m_1+m_2+m_3+3$	m_3+1
$S(132)$	m_1+m_2+2	m_2+1	m_2+m_3+2
$S(321)$	m_1+1	m_1+m_2+2	$m_1+m_2+m_3+3$
$S(232)$	0	m_2+m_3+2	m_2+m_3+2
$S(1231)$	$m_1+m_2+m_3+3$	$m_1+m_2+m_3+3$	m_3+1
$S(3121)$	m_1+m_2+2	m_1+m_2+2	$m_1+m_2+m_3+3$
$S(1232)$	$m_1+m_2+m_3+3$	m_2+m_3+2	m_2+m_3+2
$S(2321)$	m_1+1	$m_1+m_2+m_3+3$	$m_1+m_2+m_3+3$
$S(2312)$	m_1+m_2+2	$m_1+2\,m_2+m_3+4$	m_2+m_3+2
$S(12321)$	$m_1+m_2+m_3+3$	$m_1+m_2+m_3+3$	$m_1+m_2+m_3+3$
$S(12312)$	$m_1+m_2+m_3+3$	$m_1+2\,m_2+m_3+4$	m_2+m_3+2
$S(21321)$	m_1+m_2+2	$m_1+2\,m_2+m_3+4$	$m_1+m_2+m_3+3$
$S(123121)$	$m_1+m_2+m_3+3$	$m_1+2\,m_2+m_3+4$	$m_1+m_2+m_3+3$

Final Remarks

In the derivation of STEINBERG's formula, an explicit formula of KONSTANT[7]) for the multiplicities n_M of the weights M in the irreducible module with highest weight

Λ is essential (n_M = dimensionality of the weight space if M is a weight, and $n_M = 0$ if M is not a weight). This formula too is very useful also for practical purposes. Because KONSTANT's formula has not yet been used in the physical literature, we give it here

$$n_M = \sum_{S \in W} \det(S)\, P\,[S\,(\Lambda + \delta) - (M + \delta)]\ .$$

With the earlier formulas, we can immediately evaluate the right hand side for special Lie algebras. For a weight $M = \Lambda - \sum_{j=1} n_j\, \alpha_j$ we obtain

$$n_M = \sum_{S \in W} \det(S)\, P\,[\sum_i (n_i - \sigma_i(S)\, \alpha_i)]$$

with the same tables for $\sigma_i(S)$.

Finally, we would like to remark that the algebraic theory of characters for Lie algebras[8]) certainly gives simple formulae (which only contain the root system) for the following problem: Let $\mathfrak{L}'$ be a sub-algebra of $\mathfrak{L}$ and let be given an irreducible module for $\mathfrak{L}$. This module is then completely reducible for $\mathfrak{L}'$ (for semisimple $\mathfrak{L}'$). One can now ask for the irreducible constituents with respect to $\mathfrak{L}'$. This question will be discussed in a future paper.

Acknowledgements

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Appendix

To demonstrate the power of the method which we have presented in this paper, we show in detail how one can immediately decompose the tensor product $(m_1, m_2) \otimes (1,1)$ of a general irreducible representation (m_1, m_2) of A_2 with the eightdimensional representation (1,1). The possible n-values in (12) are $n \equiv (n_1, n_2) = (0,0), (0,1), (1,0), (1,1), (1,2), (2,1), (2,2), (2,3), (2,4) \ldots$. If both $m_1, m_2 > 1$ it is easy to see from table 1, that for the above first seven n's only the following term contribute in (13): $(S, T) = (1,1), (1, S_1), (1, S_2)$. Furthermore, the corresponding multiplicities are respectively:

$$\begin{aligned} m = P[0],\ P[\alpha_2],\ P[\alpha_1],\ P[\alpha_1 + \alpha_2],\ P[\alpha_1 + 2\,\alpha_2] - \\ - P[\alpha_1],\ P[2\,\alpha_1 + \alpha_2] - P[\alpha_2],\ P[2\,\alpha_1 + 2\,\alpha_2] - P[\alpha_1] - P[\alpha_2]\ , \end{aligned}$$

i.e. $m_\Lambda = 1$ except for $n = (1,1)$, where $m_\Lambda = 2$. But these seven irreducible constituents give the complete decomposition as one can see for instance by comparing the dimensions. We remember that the dimension of an irreducible module $\mathfrak{M}_{\Lambda(\mu_2,\, \mu_1)}$ with the highest weight $\Lambda = (\mu_1, \mu_2)$ is given by

$$\dim \mathfrak{M}_{\Lambda\,(\mu_1,\, \mu_2)} = (\mu_1 + 1)\,(\mu_2 + 1)\left[1 + \frac{\mu_1 + \mu_2}{2}\right].$$

The special cases where m_1 and m_2 are not both larger than one are easily discussed. For example, in the case $n = (1,1)$, we get for $(m_1, m_2) \neq (0,0)$ $m_\Lambda = P[\alpha_1 + \alpha_2] = 2$. If $m_1 = 0$, $m_2 \neq 0$ one obtains $m_\Lambda = P[\alpha_1 + \alpha_2] - P[\alpha_2] = 1$; the same holds for $m_1 \neq 0$, $m_2 = 0$, while for $m_1 = m_2 = 0$ we get $m_\Lambda = 0$.

Thus we have the following result:

$$(m_1, m_2) \otimes (1,1) = (1) \oplus (2) \oplus \cdots \oplus (7) ,$$

where

$(1) = (m_1 + 2, m_2 - 1)$ with $m_\Lambda = 0$, except for $m_2 = 0$.

$(2) = (m_1 - 1, m_2 - 1)$ with $m_\Lambda = 0$, except for m_1 or $m_2 = 0$.

$(3) = (m_1 - 2, m_2 + 1)$ with $m_\Lambda = 0$, except for $m_1 = 0,1$.

$(4) = (m_1 + 1, m_2 + 1)$ with $m_\Lambda = 0$.

$(5) = (m_1 - 1, m_2 + 2)$ with $m_\Lambda = 0$, except for $m_1 = 0$.

$(6) = (m_1 + 1, m_2 - 2)$ with $m_\Lambda = 0$, except for $m_2 = 0,1$.

$(7) = (m_1, m_2)$ with $m_\Lambda = 2$ for $(m_1, m_2) \neq (0,0)$.

$m_\Lambda = 1$ for $m_1 = 0$, $m_2 \neq 0$ or $m_1 \neq 0$, $m_2 = 0$.

$m_\Lambda = 0$ for $m_1 = m_2 = 0$.

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