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Fundamental Representations of Lie Groups*)

by **D. Speiser****)

Department of Physics and Stanford Linear Accelerator Center, Stanford University
Stanford, California, USA

To E. C. G. STUECKELBERG for his 60th birthday!

(21. IX. 64)

Summary. WIGNER's calculus of weight diagrams is extended to orthogonal and symplectic groups. A straightforward method for decomposing direct products is presented. The results are proved within the theory of characters. Finally, it is shown that some important invariants can be expressed through a complex monomial.

Introduction

Lie groups have played a role in different fields of physics. In quantum mechanics, conservation of angular momentum leads to the theory of the group of rotations in ordinary space. The group of rotations in 4-space provided the key to PAULI's derivation of the Balmer formula in matrix mechanics before wave mechanics were discovered[8]). WIGNER proved the importance of the group of unitary transformations in a 4-dimensional complex space (SU_4) for nuclear physics[4]). Also the so-called symplectic groups have found application in nuclear physics to *j-j* coupling[10]). The same holds true for the group of complex unitary unimodular transformations in a 3-dimensional complex space (SU_3) since that group governs the harmonic oscillator[9]). The same group is now recognized as being fundamental for elementary particle physics[11]); and yet even this may not be the end of this puzzling story[15])!

The roles which these groups play in the examples quoted, and the various reasons for this role, are different enough. Some of these symmetries are exact, some are broken. Some are fundamental, some are viewed as an accident. In some cases we believe that we may understand them; others present us with what are probably the most intriguing problems of present-day physics. This great variety leads one to expect that also in the future in one way or another groups may be important for applications of quantum mechanics. An easy access to some of the results and techniques most needed for computations may therefore be useful.

In this note we want to show a method for computing representations of compact simple groups and for decomposing direct products, which avoids the necessity for knowledge of higher mathematics. Such knowledge is needed only for proofs, which

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**) Permanent address: Centre de Physique Nucléaire, Université de Louvain, Belgium.

may be omitted by everyone interested only in the experimental side of mathematics *). With the help of a little geometry and still less combinatorics everyone can orient himself easily in this field[12]).

The geometric method adopted in this note seems the one best suited for making the regularities governing all aspects of Lie groups transparent[3])[6]). For these regularities shed an organizing light on what seems at first sight a mere pile of theorems. The basic tools used here are the so-called *weight diagrams* introduced into physics by WIGNER **), for presenting his theory of supermultiplets of nuclear levels[4]). Only a little knowledge of theory of regular bodies is needed to be able to guess at most theorems[12]).

Throughout the paper the accent is put on the simplicity of the results so that the reader can proceed to compute further without difficulty, if he needs more details. First the simplest cases will be considered, then the results found will be generalized, and finally they will be proved. Proofs (which are of secondary importance) are partly transferred to the appendices.

The paper is directed by CARTAN's fundamental theorem[1]): if the l so-called fundamental representations of a simple group of rank l are known, then from these all other ones can be constructed. To put it precisely: one can form direct products of two known representations, so that every product contains at most one representation which so far has not been known. Proceeding in this way, step by step, each representation may be computed. According to FROBENIUS, however, it suffices to know the character (the trace) of a representation, and the *weight diagram is nothing but a geometric picture of the character.*

The way to proceed then is clear:

In Sec. I the weight diagrams of the fundamental representations of the orthogonal and the symplectic groups (B_l, C_l, D_l) are built up. Unitary groups are mentioned only as examples, since they have received ample consideration in the literature[5])[6]). It will be obvious how all results can be formulated in coordinates, in order to be applicable to groups of any rank.

In Sec. II these results are proved in the framework of the theory of HOPF, SAMELSON and STIEFEL. Furthermore, a method for reducing direct products is explained.

In Sec. III the fundamental invariants of representations of rank 2 are related to monomials of a complex variable.

Only parts of the paper are new in substance. The emphasis is on the simplicity of the method. I began to develop it while at the Institute for Advanced Study at Princeton when I learned about WIGNER's treatment of SU_4 from C. N. YANG. Parts of it have found their way into publications of various authors through seminars.

*) This expression is the very opposite of derogatory and alludes to a particularly exciting aspect of mathematics. It seems, besides, an appropriate description of the method by which STUECKELBERG often found his way through the jungle of formalisms to the far-reaching results for which the physicists are indebted to him.

**) WIGNER, when asked about weight diagrams, replies that they are as old as Noah. I have, however, been unable to find them in earlier works, and they are unknown to many mathematicians. It is not unlikely that most of what will be said here was known long ago by WIGNER.

Parts of the paper, written two years ago, are still in press[3]). Parts were taken from another publication[6]) in press, in order to make the paper reasonably self-contained.

This work was completed while I was a guest of the Department of Physics at Stanford University and of the Stanford Linear Accelerator Center, Stanford, California. It is a pleasure to thank Professors F. Bloch, S. Drell, H. P. Noyes, W. K. H. Panofsky, L. Schiff, and D. Walecka for the spendid and cordial hospitality extended to me by both institutions. I am indebted to C. N. Yang for having brought to my attention the work of Wigner. For stimulating suggestions and partly for reading the manuscript, I wish to thank J. S. Bell, R. J. Oakes, M. Bander, F. Cerulus, H. Ruegg, E. Shrauner, and last but not least, J.-P. Antoine.

It gives me great pleasure to dedicate this paper to E. C. G. Stueckelberg whose originality as a scientist inspired many, and whose gentleness and equanimity preserved through much hardship won the admiration of all who had the good fortune to come into contact with him!

Notation

G	group
T	toroid
n	dimension of the group; in the Appendices, dimension of representations
l	rank of the group
$m = n - l/2$	number of families of parallel hyperplanes of Γ
D	representation
WD	weight diagram
χ	character $= X/\Delta$
X	characteristic (Stiefel's notation)
Δ	"square root" of Jacobian = characteristic of the identity representation
Γ	Cartan-Stiefel diagram
g^c	points of maximal intersection of Γ, cf. Figure 8
g^e	sublattice of g^c which represents the unit element
γ	lattice spanned by the roots
S	representation under which the scalar transforms (identity representation)
Sp	representation under which the spinor transforms
V	representation under which the vector transforms
A_2	representation under which the skew tensor transforms
A_ν	fundamental representation of B_l or D_l of dim (n/ν)
F_ν	fundamental representation of C_l of dim $(n/\nu) - (n/\nu - 1)$
$A_l = SU_{l+1}$	group of all unitary unimodular matrices of dim $l+1$
$B_l = O_{2l+1}$	group of all rotations in an Euclidean space of dim $2l+1 = E_{2l+1}$
$C_l = Sp(2l)$	symplectic groups – groups of all unitary matrices which leave invariant a skew form
$D_l = O_{2l}$	group of all rotations in an Euclidean space of dim $2l = E_{2l}$
C_2, C_3, C_4, C_6	Casimir-Racah operators (only in Sec III)
C_ν	points, edges, ... hypercubes. (only in the appendices)

I. A Collection of Simple Rules for Obtaining Representations of Lie Groups

1. *The Groups* $A_l = SU_{l+1}$

A. SU_2

The representations of SU_2 are well known: there is exactly one of every dimension. We represent each by a one-dimensional figure, composed of small circles. The number of circles is equal to the dimension of the representation:

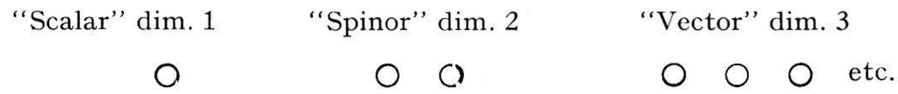


Fig. 1

These figures are the "weight diagrams" (WD's). The significance of the name will become apparent later. These figures can be used for instance for decomposing the direct product of two representations into its irreducible parts in the following way.

Mark for D_2 a similar figure consisting of points rather than circles. Take the WD representing D_1 like a "rubber stamp" and stamp five times without, however, rotating the stamp, such that the center of the WD falls once on each point:

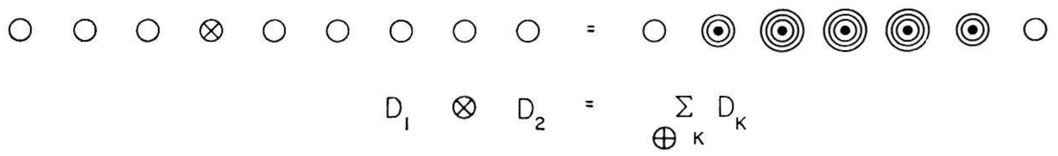


Fig. 2

Looking now at the catalogue of all WD's of SU_2 we can easily resolve this figure into its constituents:

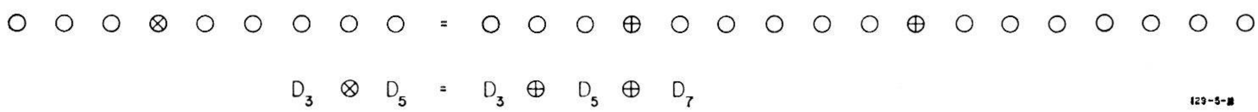


Fig. 3

In this last equation the index denotes the dimension. This procedure is in a way nothing but the old vector-diagram rule.

We note especially that one can build up in the described way all representations, starting with only the "spinor":

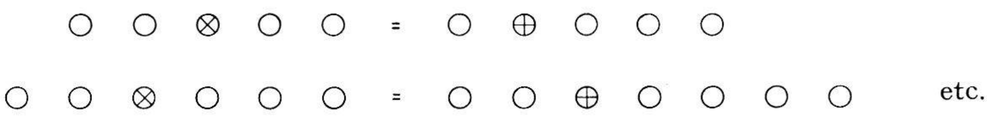


Fig. 4

B. SU_3

Can these WD's and the procedure outlined here for SU_2 be generalized also for other (semi-simple) groups? This can be done indeed, and it shall be shown here. SU_3 is usually defined by its representation of dim 3 (the group of all unimodular matrices of dimension 3) or its complex conjugate. To each of these representations must therefore correspond a WD with 3 weights.

We choose them in the following way:

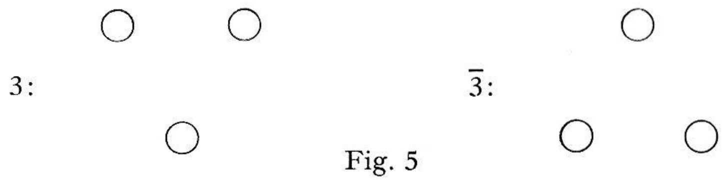


Fig. 5

They will be called 3 and $\bar{3}$ respectively. $\bar{3}$ is the inversion of 3: a relation which, as we shall see, holds for every pair of complex conjugate representations. Can we now find all other weight diagrams in this case too, and can we build up all of them starting by 3 and $\bar{3}$? Using our rubber stamp rule we get for instance:

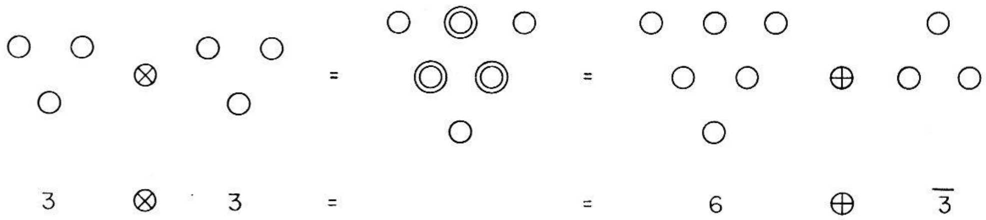


Fig. 6

Since we know the WD $\bar{3}$ we can separate it from the sum. Because of its dimension we call the first representation "the 6". Clearly if we replace the triangle 3 by its inversion, $\bar{3}$, we obtain the inversion of 6, which we call $\bar{6}$.

$$\bar{3} \otimes \bar{3} = \bar{6} \oplus 3 .$$

We try to tabulate these representations in a geometrical pattern. But since now corresponding to the rank of SU_3 we deal with 2-dimensional figures, we list them in a 2-dimensional pattern, and since moreover the WD's display a triangular symmetry we choose a triangular lattice. This choice will turn out to be of deeper significance and quite useful, too. For convenience we write on the border of the diagram at all points the number 0.

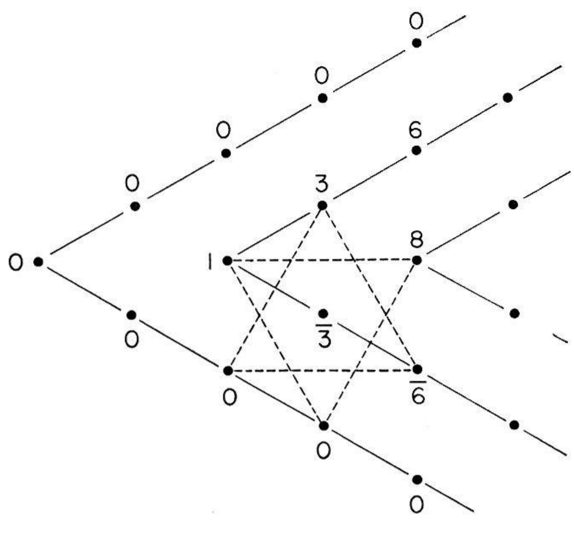


Fig. 7

Now we observe: the points $\bar{6}$ and 3 lie (together with a third one) on a regular triangle with the point $\bar{3}$ at its center. Moreover the triangle is oriented the same way as the triangle $\bar{3}$. The same is true for $\bar{3}$ and 6. Our procedure then can be described as follows: take one of the two triangles; put its center at one lattice point, with known number a, say. If the numbers at two of the lattice points on which the summits of the triangle fall are known, say b and c, then the third one x can be computed

$$x = 3 \cdot a - b - c .$$

This rule is the special case of a geometric procedure for the decomposition of direct products: for from

$$D_1 \otimes D_2 = \sum_{\oplus\, k} D_k$$

it follows:

$$\dim D_1 \cdot \dim D_2 = \sum_k \dim D_k .$$

The general rule and its proof will be given later.

What numbers then belong to the other lattice points, i.e., what are the dimensions of all other representations? For this is indeed what we shall obtain. Putting the triangle 3 around the point $\bar{3}$ one finds: $3 \otimes \bar{3} = 1 \oplus 8$ (cf. Fig. 7). Furthermore:

$$\bar{3} \otimes 6 = 3 \oplus 15$$

$$3 \otimes 8 = 3 \oplus \bar{6} \oplus 15 \text{ etc.}$$

One sees ("geometric completed induction") that with the two triangles 3 and $\bar{3}$ we can indeed find the dimensions of all representations. Moreover, we suspect that, by using the other WD's in the same way, we might be able to decompose every direct product.

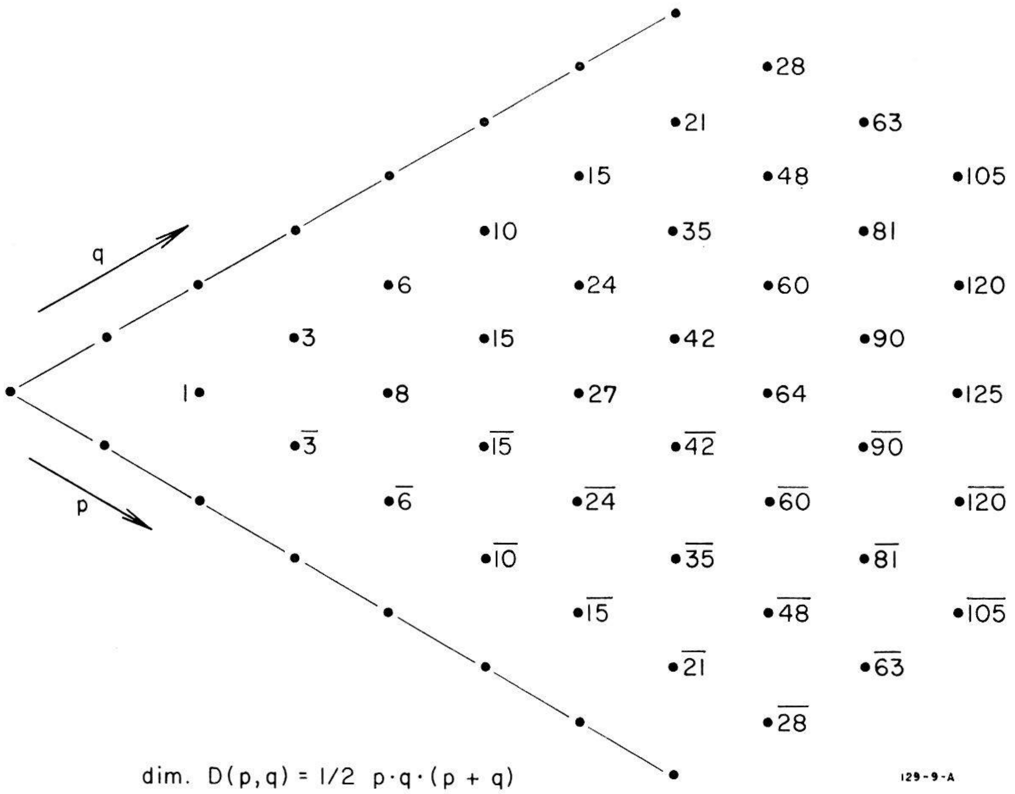


Fig. 8

So far we have determined only the dimensions of the representations. But we can build up all WD's, too. For instance multiplying 3 and $\overline{3}$ we find the WD 8:

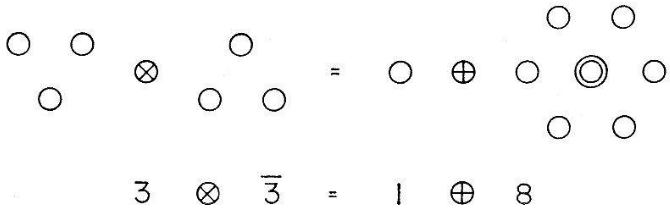


Fig. 9

Multiplying 3 and 8 we find:

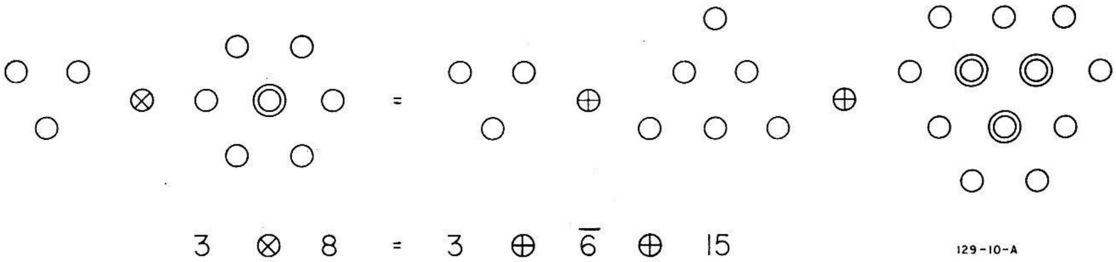


Fig. 10

That this can be accomplished starting from 3 and $\overline{3}$ only, carrying out the multiplication in such order [i. e., that every product contains only one unknown representation ($\approx$ WD)] is proved by a similar "geometric completed induction" as can be seen from inspection of the lattice Figure 8.

C. SU_4

From WIGNER's work[4]) it is well known that the WD's of the lowest representations are two tetrahedra, one the space-inversion of the other:

Fig. 11

The analogy between this fact and what was done so far suggests two things. First: for SU_2 (rank 1) one needs one, for SU_3 (rank 2) two figures in order to obtain all the other ones. Therefore in the case of SU_4 we will probably need 3 figures and presumably we must plot the dimensions of our representations on a 3-dimensional lattice. Second: segment (SU_2), triangle (SU_3), tetrahedron (SU_4) are all regular simplices. We guess then, that SU_n will be represented by an n-dimensional simplex, and that we need in this case $n - 1 = l$ diagrams to build up all the other ones. What then is the third diagram we need for SU_4? We write:

$$\begin{array}{lccccccc}
SU_2 & & & & 2 & & & \\
SU_3 & & & 3 & & \bar{3} & & \\
SU_4 & & 4 & & ? & & \bar{4} & \\
SU_5 & 5 & & ? & & ? & & \bar{5} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Remembering PASCAL's triangle, which is composed of the numbers $\binom{n}{k}$, we guess that we need for SU_4 one more diagram of 6, and for SU_5 two more diagrams of 10 weights each respectively, etc. (Note: this corresponds to the antisymmetric multilinear forms in tensor calculus.) What does the third fundamental diagram of SU_4 look like? In order to answer this question, ask: what is the most symmetric figure built up of 6 weights? Answer: the octahedron.

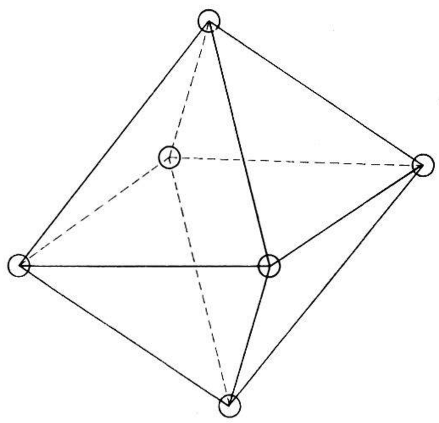
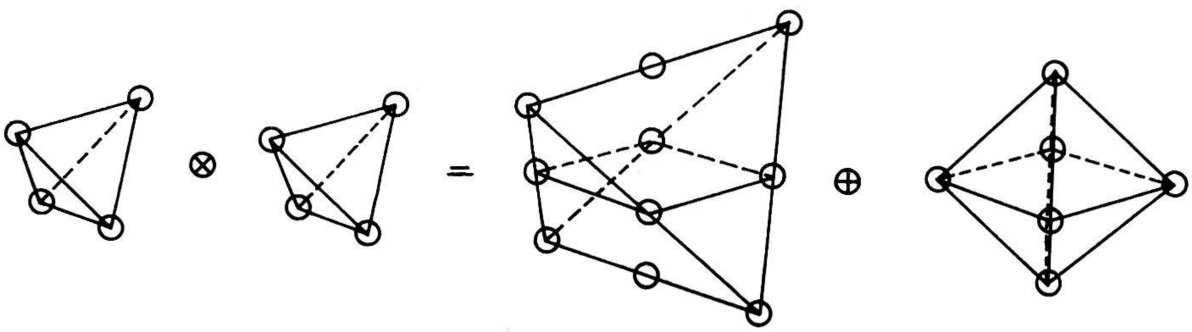


Fig. 12

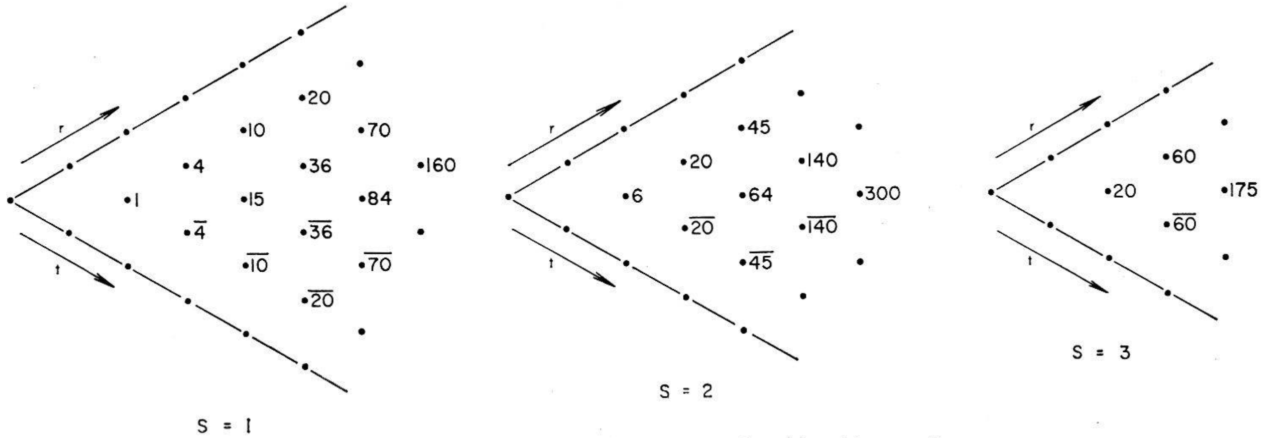
and indeed this seems to fit if we "compute" in exactly the same way as we did for SU_3:



$4 \otimes 4 = 10 \oplus 6$

Fig. 13

The dimensions are best plotted in a 3-dimensional lattice of which we show a corner of the first 3 sheets:



dim. D(r,s,t) = 1/12 · r · s · t · (r + s) · (s + t) · (r + s + t)

Fig. 14

The n-dimensional simplex, which characterizes SU_{l+1}, displays the close relationship between these groups and the groups of all $(l+1)!$ permutations of $l+1$ elements: every summit of the regular simplex has the same location with respect to all others.

We shall not enter further into the consideration of the groups SU_{l+1}, since much attention has been paid to them, cf. [5]). For a detailed geometric treatment similar to this presentation, cf. [6]), where the decomposition of g^c into $l+1$ sublattices is explained.

The Groups $B_l = O_{2l+1}$

All WD's of the group A_l can be obtained from l "fundamental" ones. The same is true for all simple groups, and it is therefore important to know how to construct them in the simplest possible way.

Since the regular simplices turned out to be fundamental for the groups A_l we ask: where will the regular cubes play a role? An l-dimensional cube has 2^l summits. $2^{\frac{2l+1-1}{2}} = 2^l$, however, is the number of components of a spinor in an n (odd) dimensional space. We guess then that spinor representations (if the dimension of the space is odd) are characterized by a cube of l dimensions.

For $l = 1$ this is indeed true since $O_3 = SU_2$! We can easily check our guess:

$$\text{for } O_3 = B_1 \text{ we have } \underline{Sp \otimes Sp = S \oplus V}$$
$$2 \otimes 2 = 1 \oplus 3$$

Thus we expect for $B_2 = O_5$:

$$Sp \otimes Sp = S \oplus V \oplus A$$
$$4 \otimes 4 = 1 \oplus 5 \oplus 10$$

S, V, A denote the representations under which a scalar, a vector and a skew-tensor respectively transform. A vector has 5 and a skew-tensor (angular momentum) 10 components. Indeed a diagram computation yields:

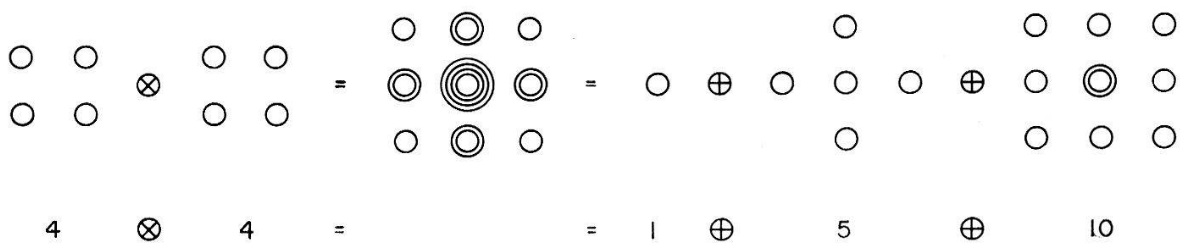


Fig. 15

Again we plot the dimensions in a two-dimensional lattice ($l = 2$!) but this time we use an ordinary square lattice. Also here we assign "dim 0" to all lattice points on the boundaries, but we add one *further row* below the diagram and write to the first two points -1 and -5:

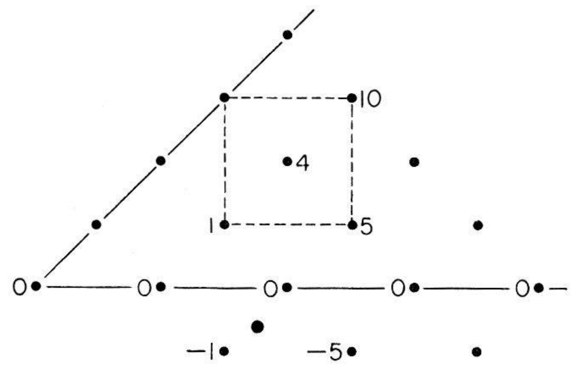


Fig. 16

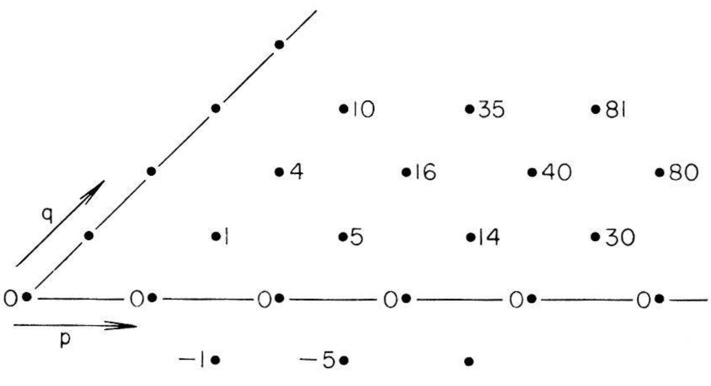
Using now $\begin{matrix}\circ & \circ \\ \circ & \circ\end{matrix}$ (spinor) and $\begin{matrix} & \circ & \\ \circ & \circ & \circ \\ & \circ & \end{matrix}$ (vector) as our basic tools we can easily enough fill the whole diagram. Putting the "spinor" on point 4 we find:

$$4 \times 4 = \begin{matrix} 0 + x \\ +1 + 5 \end{matrix} \quad \text{therefore } x = 10 \, .$$

If we put the "vector" on points of the first line, we must take also the negative numbers into account:

$$5 \times 5 = \begin{matrix} 10 \\ +1 + 5 + y \\ -5 \end{matrix} \quad \text{therefore } y = 14 \, .$$

We then write -14 below 14, etc.



dim. $D(p,q) = \frac{1}{3!}\, p \cdot q \cdot (p+q)(2p+q)$

Fig. 17

Note: the lattice points (= basic vectors) 5 and 10 span a sublattice to which one-half of the lattice points belong. The representations on this lattice are the often (somewhat unprecisely) so-called "integer representations" ("single-valued") which represent faithfully only the adjoint group. The numbers 1, 3, and 1, 5, 10, respectively, again belong to Pascal's triangle: they are the first ones in the $2\,l + 2^{nd}$ row. It is an obvious guess that the general formula*) will be:

$$Sp \otimes Sp = S \oplus V \oplus A \oplus A_3 \oplus A_4 \oplus \cdots$$
$$2^l \times 2^l = 1 + n + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{\nu}\,.$$

Here A_ν denotes the antisymmetric tensor with $\binom{n}{\nu}$ components.

$$\text{indeed: } \sum_{\nu=0}^{l} \binom{n}{\nu} = 2^{n-1}\,.$$

For B_3 we find the following fundamental representations:

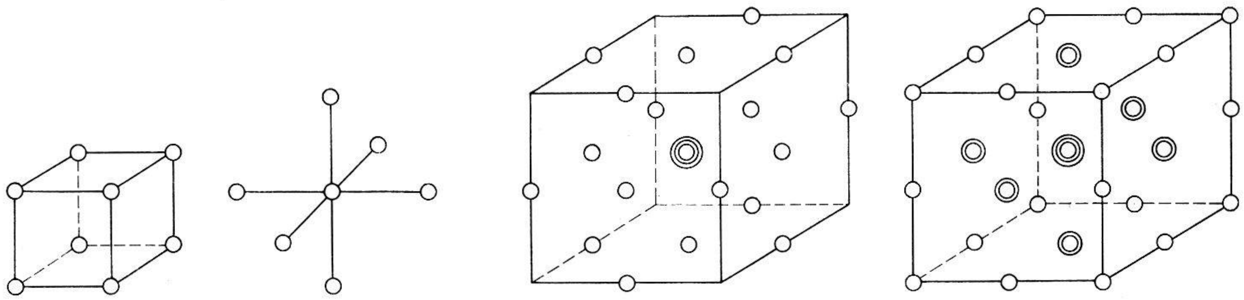
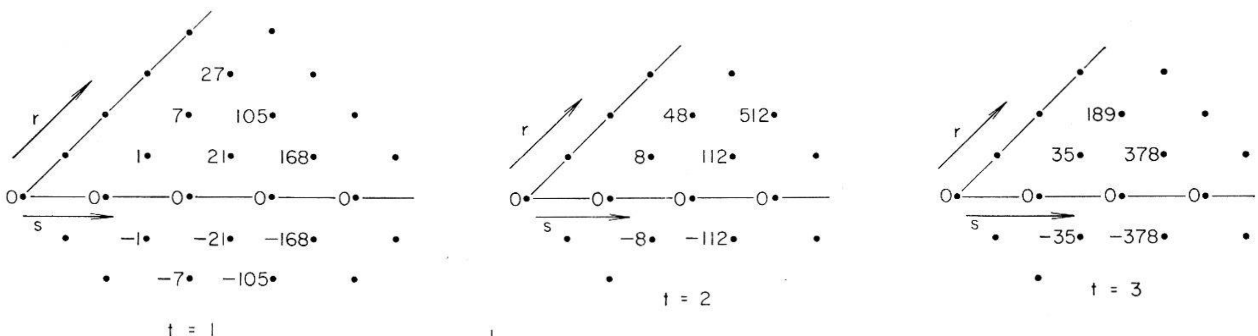


Fig. 18

The same remark as to the sublattice spanned by these 3 points applies here. Thus in the case of the rotation groups (the same will be shown to hold for the even-dimensional case) we need not go through all the trouble with the antisymmetrization. All fundamental representations can be obtained at once from the spinor. As we see, we obtain even one more representation! The spin representation and the $l-1$ first ones (after the scalar) in the product $S\,p \otimes S\,p$ span the whole lattice. The same $l-1$ representations and the l^{th} in this product span the lattice of integer representations.



$$\text{dim. } D(r,s,t) = \frac{1}{6!}\, r\cdot s\cdot t\cdot(r+s)(s+t)(2s+t)(r+s+t)\cdot(r+2s+t)\cdot(2r+2s+t)$$

Fig. 19

Appendix I contains a table for the fundamental WD's of the groups B_l.

The Groups $C_l = Sp\ (2\,l)$

Having used the simplex and the cube we ask: where will the third regular body, the octahedron play a role? The l-dim octahedron has $2\,l$ summits. Therefore the symplectic groups, i.e., the groups of unitary transformations which leave invariant a skew form, are obvious condidates. For the representations by which we just defined

*) In these formulae $n = 2\,l+1$.

these groups are cearly of dimension $2\,l$. In one dimension simplex, cube, octahedron are all the same: and indeed $A_1 = B_1 = C_1$. In two dimensions cube and octahedron differ only by their orientation and again: $B_2 = C_2$. Only for $l = 3$ do we get something new.

$$6 \otimes 6 = 1 \oplus 14 \oplus 21 .$$

The product $6 \otimes 14$ contains another representation of dimension 14. These numbers 6, 14, 14′, are the differences between the $l + 1$ first numbers in the $2\,l + 2$ row of PASCAL's triangle.

For C_4 we find the following dimensions of the 4 fundamental representations: 8, 28, 48, 42. 8, of course, is the number of summits of a 4 dimensional octahedron.

Contrary to what we saw when studying groups B_l and D_l these are already all $(= l)$ fundamental representations. The 1st, 3rd, 5th, etc., representations represent the universal covering group faithfully, the 2nd, 4th, etc., only the adjoint group.

C_2

Fig. 20

C_3

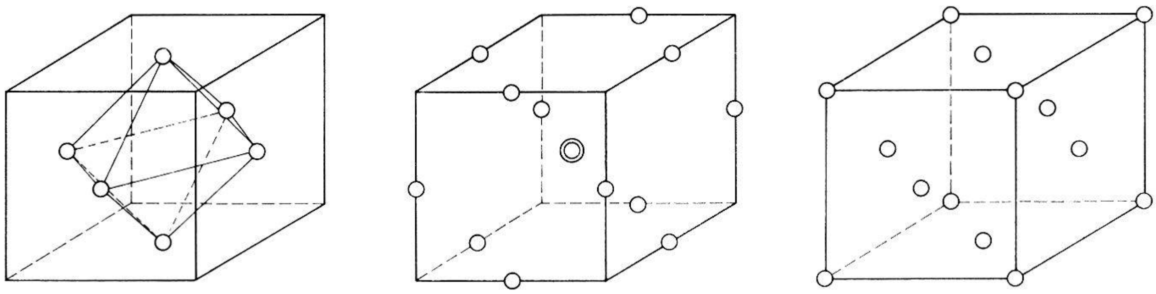


Fig. 21

It will be explained in the appendix how the complicated antisymmetrization procedure can be avoided. The fundamental WD's of C_l can be obtained from the l WD's which span the sublattice of single-valued representations of the groups B_l.

The Groups $D_l = O_{2\,l}$

The list of regular bodies is by now essentially exhausted. Icosahedron and dodecahedron have no analogy beyond $l = 4$. Therefore we must look for another hint.

The group $O_{2\,l} \cong D_l$ is of the same rank as the group $O_{2\,l+1} \equiv B_l$. The spinor, however, if $O_{2\,l+1}$ is reduced with respect to $O_{2\,l}$ decays into two half-spinors each of half this dimensionality. The representations under which the two half-spinors transform are inequivalent. In order to find the WD's of the two spin representations, one expects therefore the following rule: start from one summit and go along the edges to all others, retain those which you reach after an even number of edges; the other ones will form the second half-spinor.

The dimensions of the other fundamental representations here again can be found in PASCAL's triangle. They are the $l - 1^{st}$ first numbers following the 1 in the $2l + 1$ row. The next number, i.e., the one in the middle column, corresponds to a representation which is irreducible for the group of rotations and reflections. Under the proper group it decays into two inequivalent representations of equal, i.e., $\frac{1}{2}\binom{2l}{l}$ dimensions.

The connection between the two half-spinors and the other $l - 2$ fundamental representations is here slightly more complicated than it was for the B_l. We must distinguish between l even and odd.

A. $l =$ odd Opposite points of the cube belong to different half-spinors. This means, as will be shown, that these two representations are complex conjugates. If $l = 3$ we find two tetrahedra (since $D_3 \approx A_3$) cf. Figure 11.

Denoting representations by these dimensions one obtains in general*) ($n \equiv 2l$):

$$S\,p \otimes S\,p' = 1 \oplus \binom{n}{2} \oplus \binom{n}{4} + \cdots + \binom{n}{l-1}$$

$$S\,p \otimes S\,p \;=\; \binom{n}{1} \oplus \binom{n}{3} \oplus \cdots \frac{1}{2}\binom{n}{l}$$

The symbol $\frac{1}{2}\binom{n}{l}$ indicates that this representation ($\sim$ the numbers on the middle column of PASCAL's triangle) irreducible under the rotation-reflection group, decays under the group of proper rotations into two inequivalent parts. One part belongs to the product $S\,p \otimes S\,p$, the other to the product $S\,p' \otimes S\,p'$.

B. $l =$ even Opposite points of the cube belong to the same WD. This means, as will be shown that the two representations are either real or symplectic. The products $S\,p \otimes S\,p$ and $S\,p' \otimes S\,p'$ therefore contain the identity.

One obtains ($n = 2l$):

$$S\,p \otimes S\,p' = \binom{n}{1} \oplus \binom{n}{3} \oplus \cdots \binom{n}{l-1}$$

$$S\,p \otimes S\,p = 1 \oplus \binom{n}{2} \oplus \cdots \frac{1}{2}\binom{n}{l}\,.$$

The same remark as before applies to the symbol $\frac{1}{2}\binom{n}{l}$.

Also note that in either case the two representations

$$\frac{1}{2}\binom{n}{l} \text{ and } \frac{1}{2}\binom{n}{l}'$$

are not fundamental ones.

For details of this slightly more complicated case, again cf. [6]): the decomposition for g^c into sublattices is explained there.

II. Proof of the Assertions Made so Far

1. *Survey of the Theory of Compact Lie Groups*

Weight diagrams are usually derived from the Lie algebra as the solutions of eigenvalue equations. But this method is unnecessarily complicated.

---

*) In these formulae $n = 2l$.

A much easier and more straightforward way is accessible starting from the work of HOPF, SAMELSON, and STIEFEL[3]).

We recall from it the following results[6]):

(a) An abelian, connected, compact Lie group is called *toroid* (denoted by T). One proves that every toroid is the direct product of several groups O_2 (rotations in a 2-dimensional Euclidean plane E_2). A toroid therefore is completely characterized by its dimension. It is a l-dimensional cube, corresponding points on opposite surfaces of which are identified.

A toroid which is a subgroup of a Lie-group G but not a subgroup of a toroid of a higher dimension, is called *maximal toroid* of G. The fundamental theorem of HOPF says that every element of a compact group is contained in (at least) one maximal toroid. The dimension of $T_{\max}$ is an invariant (denoted by l) and called the *rank* of the group.

(b) If an element of a maximal toroid does not belong to any other maximal toroid, it is called *regular*, otherwise *singular*.

(c) The universal covering group of T is isomorphic to an l-dimensional Euclidean space E_l. Every element of T then will be represented by an infinite point lattice in E_l rather than by one point only. In particular, the image of the unit element is a lattice g^e.

(d) The image of the singular elements of T consists of m families of l-dimensional hyperplanes. Here $m = n - l/2$, n is the order of the group, l its rank. One proves that no two singular hyperplanes may coincide. The set of all singular hyperplanes, i.e., the union of all singular points, is called *the diagram Γ of* CARTAN *and* STIEFEL. The points of maximal intersection, i.e., the points which belong to one hyperplane of every set, represent the center of G.

(e) The essential property of Γ is the following: Γ remains invariant under a reflection in any of the hyperplanes of which it is composed. Thus Γ possesses very special symmetry properties.

(f) If G is *semi-simple*, its center is a discrete group. Therefore, its image in E_l is a point lattice g^c, generated by l basis vectors. This lattice g^c contains g^e as a sublattice.

(g) The finite discrete group, generated by the reflections in the hyperplanes, passing through the origin of g^c, was called by WEYL the *group* S. S is the crystal-class of g^c. The fundamental domains D_i of S are infinite pyramids whose corners are at the origin of g^c. D_i has l edges.

(h) The connection to the infinitesimal theory (the "Lie algebra") is the following. The $2m$ vectors, orthogonal to the m hyperplanes passing through the origin of g^c, twice as long as the distance to the next parallel hyperplane, are the *roots* of CARTAN. Thus also the set of roots, the *root diagram*, is invariant under the transformations of S. Whence follows immediately:

$$\frac{2(\lambda\,\mu)}{(\lambda\,\lambda)} = n = \text{integer}\,,$$

where λ, μ are two roots and (λ, μ) their scalar product. This relation is the source of VAN DER WAERDEN's classification of all simple groups which was greatly simplified by COXETER and DYNKIN[5]). In the framework of the infinitesimal theory, the roots are the non-zero eigenvectors of the characteristic equation of the group. This equation is constructed in terms of the infinitesimal generators of the group, cf. 1, 2, 5. The toroid is the manifold generated by a "complete set of commuting variables" of the Lie algebra and of its linear combinations.

$$e^{i \Sigma c^{\alpha} H_{\alpha}}.$$

(i) The diagram Γ of CARTAN and STIEFEL does not characterize a group completely, but rather the family of all groups which are locally isomorphic. Consider the origin of g^c and g^e and all points into which it can be transformed through a series of successive reflections in hyperplanes of Γ. The set of these points is a sublattice γ of g^c. Thus one has:

$$\gamma \subset g^e \subset g^c ,$$

Global properties are determined by the lattice g^e, or better by the way g^e contains γ and is itself contained in g^c. For instance $g^e = g^c$ means that e is the only element of the center of the group. This characterizes the adjoint group G_A.

On the other hand, $g^e = \gamma$ characterizes the universal covering group. (For a proof of this statement, cf.[3]).)

e.g.,

$$SU_3 \qquad g^e = \gamma \qquad C = Z_3$$

$$SU_3/Z_3: \quad g^e = g^c \qquad C = e .$$

Here C denotes the center of G.

In between these two extreme cases there may be room for intermediate possibilities, cf. Part II, groups of type A and type D[6]).

(j) WEYL showed that the character of an irreducible representation of a semisimple Lie group may be written in the following form[2]):

$$\chi = \frac{X}{\Delta} \tag{1}$$

where

$$\chi = \chi(\boldsymbol{K}_0) = \sum_s \delta_s \, e^{i(s\boldsymbol{K}_0 \cdot \varphi)} \qquad \boldsymbol{K}_0 \in D_0 \,, \ \boldsymbol{K}_0 \in g^c$$

$$\Delta = \chi(\boldsymbol{R}_0) \,. \tag{2}$$

$\boldsymbol{K}_0$ is any lattice vector of that part of g^c which lies inside the fundamental D_0. $\boldsymbol{R}_0$ is the smallest lattice vector inside D_0. The summation runs in this formula over all elements of S and $\delta_s = \pm 1$, according to whether s is a proper $(+1)$ or improper (-1) rotation. The φ^k are the group parameters, introduced as coordinates into the toroid ($0 \leqslant \varphi^k < 2\pi$, $\varphi^i = 0$ or 2π represents the unit element of T). X is called the *characteristic* of the representation *). Δ is the characteristic of the unit representation.

With respect to the group S, X, and Δ are alternating functions, therefore an automorphic function:

$$s\,\chi = \delta_s\,\chi \qquad s \in S \qquad s\Delta = \delta_s \Delta \qquad s\chi = \chi$$

It may be noted that as a function of the φ^k, $|\Delta|^2$ is also the Jacobian under the group integral.

The fundamental theorem concerning irreducible representations then says (in STIEFEL's formulation[3]))

- Every characteristic X is an alternating sum (with respect to S) of exponentials of equivalent lattice vectors. (4)
- Every alternating sum of this kind is a characteristic. (5)

---

*) The nomenclature is not uniform in the literature. We follow here STIEFEL who derived formula (5) in the frame of HOPF's global theory [3]). WEYL had used the infinitesimal method.

In other words: Every vector $K_0 \in D_0$ (strictly inside D_0, not on its boundary) with integer coefficients, i.e., lying on g^c, defines the characteristic of an irreducible representation of G, and every characteristic can be obtained in this way.

The character therefore has the following from:

$$\chi = \sum \gamma_M \, e^{i(\boldsymbol{M} \cdot \varphi)} \qquad \gamma_M = \text{positive integer} . \tag{6}$$

2. We can now easily relate the figures and diagrams which we used with the corresponding expressions in the last section.

(a) *The lattice* (our geometric pattern), where we plotted the dimensions of the representations is the lattice g^c contained in the universal covering group of the maximal toroid. The dimension was plotted at the point (vector) $\boldsymbol{K}_0$ (cf. [4])).

For this only one domain D_0 was used; the complete diagram of, e.g., SU_3 would look like this:

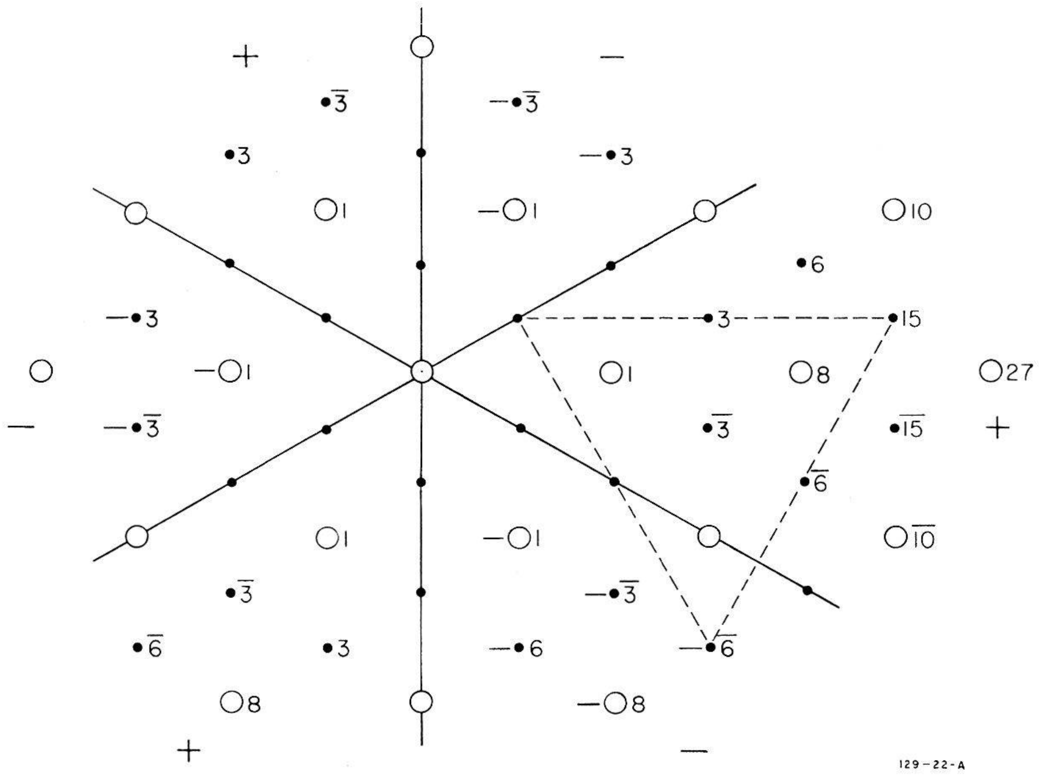


Fig. 22

The assertion that every representation is contained in this diagram, is an expression of theorems (2) and (4); theorem (5) says that in turn also to every point belongs a representation.

In Ref. [6]) it is explained in detail how g^c in the general case is composed of several equivalent sublattices. Representations attached to the same sublattice belong to the same class, and these classes form a (finite) group isomorphic with the center of the group to which g^c belongs.

The *characteristic* is represented by the set of vectors $s_q K_0$ (with signs + or −) which have all the same length, and form a convex polyhedron. Two adjacent corners are symmetric with respect to a hyperplane I_i; therefore the edge joining them is parallel to the corresponding root α_i.

(b) *The weight diagrams* (WD's) represent the characters as given by (10) *). The vectors $\boldsymbol{M} = (M_1 \ldots M_l)$ $\boldsymbol{M} \in g$, unlike the $\boldsymbol{K}$ associated with X are not all of the same length, since χ is not homogeneous They are the *weights* of the representation, the integers γ_M their *multiplicity*. One sees from (4) that the highest weight of the representation is the vector $\boldsymbol{L}_0 = \boldsymbol{K}_0 + \boldsymbol{R}_0$. Already CARTAN had proved that the highest weight of every representation is simple.

The term $e^{i(\boldsymbol{P} \cdot \varphi)}$ is geometrically represented in T by the point $\boldsymbol{P} \in g^c$. Thus the character is represented by the set of points $\boldsymbol{M} \in g^c$ of (6), which form the *weight diagram*. Also the WD is invariant under the operations of S.

Thus one sees: the geometric calculus presented in Sec. 1 is based on the theory of characters from which it can be derived easily enough. On the other hand: *the construction of a WD is the computation of a character*. The character, however, contains all the informations on representations needed.

Ref. 6, I and II are devoted to the determination of the multiplicities of the weights, and WEYL's formula is transformed in an m-fold sum where

$$2\,m = n - l = \text{dimension - rank}$$

Again (2) and (4) say that our "build up" prescription furnishes all WD's (if also in a complicated way; the work done in Ref. 6 makes this task easiei), and (5) shows that all WD's obtained in this way belong to a representation. That all WD's finally can be obtained by starting from l only expresses a well-known fundamental theorem due to CARTAN.

(c) *The decomposition of direct products*. The general procedure for reducing a direct product can be stated as follows: In order to find all irreducible parts of D_K contained in a direct product,

$$D_1 \otimes D_2 = \sum_{\otimes k} D_K \,, \tag{7}$$

- take the WD attached to D_1.
- put its center at the lattice point $\boldsymbol{K}$ of X (D_2).
- read off all lattice points on which a weight of the WD (D_1) falls, taking properly into account, of course, the multiplicities of the weights.
- cancel all pairs of D_K's attached to points equivalent under the group S if they have *opposite* signs.
- omit weights falling on boundaiies of the domains D.
- then remaining weights give exactly the D_K on the r.h.s. of Equation (7).

In order to prove the assertion, it is enough to recall that the WD's represent the characters $\chi(D)$ and the lattice points in D_0 (together with their equivalents the characteristics $X(D)$. From (7) follows

$$\chi(D_1) \cdot \chi(D_2) = \sum \chi(D_k) \tag{8}$$

and multiplying this equation with Δ:

$$\chi(D_1)\; X(D_2) = \sum X(D_K) \,. \tag{9}$$

*) In the infinitesimal approach, the weights were eigenvectors.

This equation expresses directly the rule formulated above, for it is not difficult to see that restriction to one term of X yields the same result as if one would take into account all of them (see also Ref. 14).

(d) The procedure for obtaining the *dimensions* of all representations is a special case of this rule since:

$$\dim D = \chi(e) \,. \tag{10}$$

As an example, Figure 22 shows the decomposition:

$$\begin{aligned} 6 \otimes \bar{3} &= 0 \oplus 3 \oplus 15 \\ & \quad\; 0 \oplus \bar{6} \\ & \quad\; \ominus \bar{6} \\ &= 3 \oplus 15 \,. \end{aligned}$$

III. A Curious Connection between Lie Groups of Rank 2 and the Theory of Complex Variables

The fact that the value of the discrete dimension function of A_2 and B_2 in Figures is the arithmetical mean of its closest neighbors is reminiscent of harmonic functions. It might be called, therefore, "a harmonic lattice function". Can it be interpolated by a harmonic function? For convenience we rotate the lattice g^c of Figure 8 such that the p-axis coincides with the x-axis of the complex planes.

One then verifies easily that the dimension of the representation assigned to a lattice point is equal to the imaginary part of the function z^3 at this point:

$$\dim D(p\, q) = \frac{1}{2}\, p\, q(p+q) = \frac{1}{3\sqrt{3}} \operatorname{Im} z^3 \,,$$

Likewise one finds for B_2

$$\dim D(p, q) = \frac{1}{6}\, p\, q(p+q)\,(2p+q) = \operatorname{Im} \frac{z^4}{6}$$

and for G_2:

$$\dim D(p, q) = \frac{1}{120}\, p \cdot q(p+q)\,(p+2q)\,(p+3q)\,(2p+3q) = \operatorname{Im} \frac{z^6}{180\sqrt{3}} \,.$$

These formulae are remarkably simple. What then is the significance of the real parts of the functions? They are the eigenvalues of the second Casimir-Racah operators (of order m) if suitably normalized, or a rational function of eigenvalues of both C.R. operators.

$$A_2\colon \operatorname{Re} z^3 = (p-q)\,(2p+q)\,(p+2q) = C_3$$

$$B_2\colon \operatorname{Re} z^4 = -\,(8\, C_4 - C_2^2)$$

$$G_2\colon \operatorname{Re} z^6 = -\left(\frac{27}{2}\, C_6 - C_2^3\right).$$

Normalization and sign of the operators C_4 and C_6 are to some extent arbitrary: Here we follow RACAH[5]).

The Casimir operator of rank 2 is given up to inessential additional constants by the expression $z\,\bar{z} = x^2 + y^2$

$$A_2\colon\ C_2 = p^2 + p\,q + q^2$$

$$B_2\colon\ C_2 = p^2 + p\,q + \frac{1}{2}\,q^2$$

$$G_2\colon\ C_2 = p^2 + 3\,p\,q + 3\,q^2\,.$$

I suspect that polynomials of several complex variables can be connected in much the same way to groups of rank > 2.

Appendix A

Appendix A is inserted only in order to be able to denote the other appendices logically. The fundamental representations of the groups A_l present no problem since all their weights are simple, cf. [6]).

These groups are best treated with the help of the projective trick: work in a $l + 1$ dim space and impose the subsidiary condition $\Sigma\, x^i = 0$.

The appendix may be closed with PASCAL's triangle which, as was shown, is the key which provides the easiest answer to all these questions.

									1							
								1		1						
A_1							1		2		1					
A_2	$B_1 = C_1$					1	◇	3		3		1				
A_3	D_2				1		4		6		4		1			
A_4	$B_2 = C_2$			1	◇	5	◇	10		10		5		1		
A_5	D_3		1		6		15		20		15		6		1	
A_6	B_3, C_3	1	◇	7	◇	21	◇	35		35		21		7		1

At the edges the groups are listed, the fundamental representations of which can be found in the corresponding row. The signs ◇ mark the differences which give the dimensions of fundamental representation of C_l. See Appendix C.

C_1	2			
C_2	4	5		
C_3	6	14	14′	
C_4	8	27	48	42 .

Appendix B

The Weight-diagrams of the Fundamental Representation of B_l

The following tables contain the weight diagrams of the groups $B_1 - B_5$ in such a way that they can be continued for $l > 5$ without any brain breaking computations. In order to make the laws which underlie these tables completely transparent they begin with a fictitious B_0.

Consider again the example of B_2

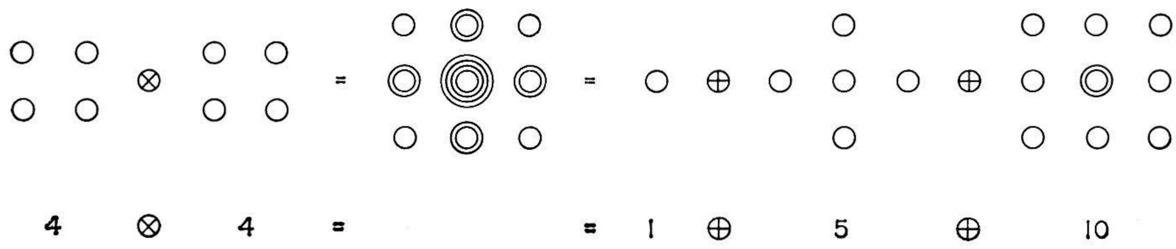


Fig. 15

The composite 2 dim WD in the middle has 4.1 weight at the summits (C_0), 4·2 weight at the center of the edges (C_1), and 1·4 weight at the center of the surface (C_2) of the figure.

In Table I this is tabulated on the 4th row.

Making similar computations for the product of cubes of higher dimension more dimensional cubes we find the following composite WD:

Table B-I

l	Numbers of Weights	C_0	C_1	C_2	C_3	C_4	C_5
0	1·1	1·1					
1	2·2	2·1	1·2				
2	4·4	4·1	4·2	1·4			
3	8·8	8·1	12·2	6·4	1·8		
4	16·16	16·1	32·2	24·4	8·8	1·16	
5	32·32	32·1	80·2	80·4	40·8	10·16	1·32

It is easy to continue this table beyond $l = 5$. The first factor of each product indicates the number of homologous places in the WD. This number is given in Table III. The second factor is the multiplicity, i.e., the number of weights at each place. This factor is such that the product (total number of all homologous weights) is $2^l(\nu^l)$, i.e., 2^l times the corresponding number of PASCAL's triangle.

Tables II(a–e) contains the multiplicities of all weights of fundamental representation. The first factor in each product is the number of homologous weights, the second the multiplicity.

Table B-II (a–e)

(a)	B_1	n	C_0	C_1	
		1		1	cf. Figure 1
		3	2·1	1	

(b)	B_2	n	C_0	C_1	C_2	
		1			1	
		5		4·1	1	cf. Figure 15
		10	4·1	4·1	2	

(c)	B_3	n	C_0	C_1	C_2	C_3	
		1				1	
		7			6·1	1	cf. Figure 18
		21		12·1	6·1	3	
		35	8·1	12·1	6·2	3	

(d)	B_4	n	C_0	C_1	C_2	C_3	C_4
		1					1
		9				8·1	1
		36			24·1	8·1	4
		84		32·1	24·1	8·3	4
		126	16·1	32·1	24·2	8·3	6

(e)	B_5	n	C_0	C_1	C_2	C_3	C_4	C_5
		1						1
		11					10·1	1
		55				40·1	10·1	5
		165			80·1	40·1	10·4	5
		330		80·1	80·1	40·3	10·4	10
		462	32·1	80·1	80·2	40·3	10·6	10

Again the law underlying these tables is transparent. The first factor (numbers of homologous weights) is taken from Table 1, the second factors are again PASCAL's triangle, but the numbers of the same row are now in the same column and ordered from top to bottom according to size.

Example: B_3 the coordinates $(x\, y\, z)$ of the weights are as follows:

1	(scalar)	(000)			
7	(vector)	(000)	(100)	(010)	(001)
			(–100)	(0–10)	(00–1)
21	(skew-tensor)	3 × (000)	(100)	(010)	(001)
	"root diagram"		(110)	(101)	(011)
			(1–10)	(10–1)	(01–1)
			(–110)	(–101)	(0–11)
			(–1–10)	(–10–1)	(0–1–1)

35	$3\times(000)$	(100)	etc.		
		(110)	etc.		
		(111)	(11–1)	(1–11)	(–111)
		(–1–1–1)	(–1–11)	(–11–1)	(1–1–1)

Note: The first $\left(\binom{n}{l}\right)$ and the last scalar representation of each table are strictly speaking not fundamental ones. They have been added for completeness and in order to make the table transparent. The lattice (points of maximal intersection of Γ) of all representations has as basic vectors the spinor and the $l-1$ representations of the table next after the scalar. The l representations (next after the scalar) span the lattice of all so-called single valued representations (i.e., those which represent faithfully the adjoint group). The character of the vector representation, e.g., then is:

$$\chi(V) = 1 + e^{i(+1\varphi_x)} + e^{i(-1\varphi_x)} + e^{i(+1\varphi_y)} + e^{i(-1\varphi_y)} + e^{i(+1\varphi_z)} + e^{i(-1\varphi_z)}$$

$$= 1 + 2\cos\varphi_x + 2\cos\varphi_y + 2\cos\varphi_z\,.$$

The numbers in Tables B-I and B-II may be derived from Table B-III which contains the number of summits, edges, surfaces, etc., of an l-dim. cube.

Table B-III

l/ν	C_0	C_1	C_2	C_3	C_4	C_5
C_0	1·1					
C_1	2·1	1·1				
C_2	4·1	2·2	1·1			
C_3	8·1	4·3	2·3	1·1		
C_4	16·1	8·4	4·6	2·4	1·1	
C_5	32·1	16·5	8·10	4·10	2·5	1·1

Again the law of the table is transparent: the first factor is $2^{l-\nu}$, the second factors are again PASCAL's triangle. The product is $2^{l-\nu}\binom{l}{\nu}$. Note the following amusing *theorem*: the number of summits, edges, surfaces, volumes, etc., of a n-hypercube is 3^n! *)

Appendix C

What are the rules which govern the groups C_l? Unlike cases B and D, the spinor trick is not available. Therefore it seems that we cannot avoid going through all the trouble of the antisymmetrization.

But before we resign ourselves we list the fundamental representations of C_2 and C_3.

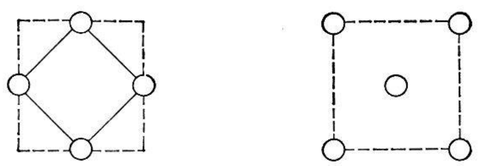


Fig. 20

*) I still feel embarrassed when remembering how STUECKELBERG proved this relation which had puzzled me, off the cuff. The reader, no doubt, will do like STUECKELBERG.

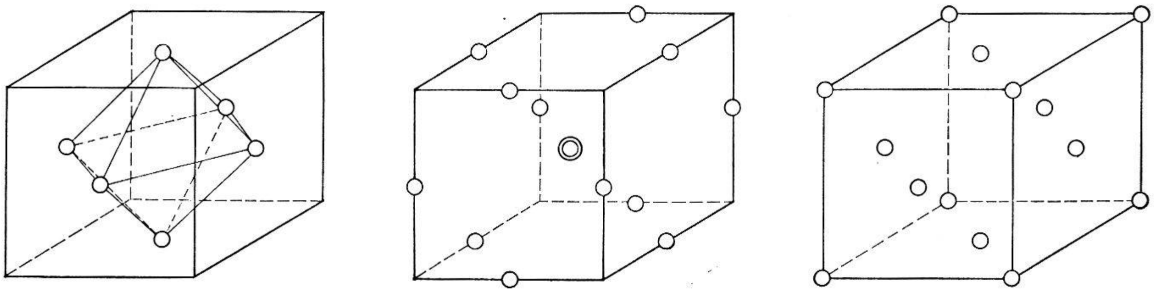


Fig. 21

We list the number of weights at summits, center of edges, etc., into which the octahedron is inscribed.

C_2:	n	C_0	C_1	C_2
S	1			1
F_1	4		4·1	
F_2	5	4·1		1

C_3:	n	C_0	C_1	C_2	C_3
S	1				1
F_1	6			6·1	
F_2	14		12·1		2
F_3	14	8·1		6·1	

The numbers occur in alternating columns. This corresponds to the fact that F_1, F_3, etc., represent faithfully the universal covering group (double-valued representations), S, F_2, F_4, etc., (single-valued representation) only the adjoint group; they belong therefore to different sublattices.

Comparison of these tables with those for B_2 and B_3 respectively reveals now a remarkable and curious connection:

The numbers listed in the Table C_l at a certain place is the difference between the corresponding number in Table B_l and the number immediately above the latter one!

From Tables B_4 and B_5 we find in this way:

C_4	C_0	C_1	C_2	C_3	C_4	n
S					1	1
F_1				8·1		8
F_2			24·1		3	27
F_3		32·1		8·2		48
F_4	16·1		24·1		2	42

C_5	C_0	C_1	C_2	C_3	C_4	C_5	n
S						1	1
F_1					10·1		10
F_2				40·1		4	44
F_3			80·1		10·3		110
F_4		80·1		40·2		5	165
F_5	32·1		80·1		10·2		132

The characters then can be written down at once as was shown in detail in Appendix B.

Pressure to complete this work in time for the Stueckelberg Festschrift prevents a proof, but there is little doubt about its general validity.

This remarkable connection is an aspect of the duality between groups B_l (body centered cubic lattice) and C_l (face centered cubic lattice) cf. [13]).

Appendix D

The Weight Diagrams of the Fundamental Representations of D_l

The following tables correspond to the tables of Appendix 1. The significance is the same. D_2 although not simple, is included since the laws of composition are the same.

Table D-I

l	Number of Weights	C_0	C_1	C_2	C_3	C_4
2	$2 \otimes 2'$		$4 \cdot 1$			
	$2 \otimes 2$	$2 \cdot 1$		$1 \cdot 2$		
3	$4 \otimes 4'$		$12 \cdot 1$		$1 \cdot 4$	
	$4 \otimes 4$	$4 \cdot 1$		$6 \cdot 2$		
4	$8 \otimes 8'$		$32 \cdot 1$		$8 \cdot 4$	
	$8 \otimes 8$	$8 \cdot 1$		$24 \cdot 2$		$1 \cdot 8$

Table D-II

(a)	D_2	n	C_0	C_1	C_2
	$2 \otimes 2'$	4		$4 \cdot 1$	
	$2 \otimes 2$	1			1
	$2' \otimes 2'$	3	$2 \cdot 1$		1 (+1)

(b)	D_3	n	C_0	C_1	C_2	C_3
	$4 \otimes 4'$	1				1
		15		$12 \cdot 1$		3
	$4 \otimes 4$	6			$6 \cdot 1$	
	$4' \otimes 4'$	10 (10′)	$4 \cdot 1$		$6 \cdot 1$ ($+6 \cdot 1'$)	

(c)	D_4	n	C_0	C_1	C_2	C_3	C_4
	$8 \otimes 8'$	8				$8 \cdot 1$	
		56		$32 \cdot 1$		$8 \cdot 3$	
	$8 \otimes 8$	1					
	$8' \otimes 8'$	28			$24 \cdot 1$		4
		35 (35′)	$8 \cdot 1$		$24 \cdot 1$		3 (+3′)

(d)	D_5	n	C_0	C_1	C_2	C_3	C_4	C_5
	$16 \otimes 16'$	1						1
		45				$40 \cdot 1$		5
		210		$80 \cdot 1$		$40 \cdot 3$		20
	$16 \otimes 16$	10					$10 \cdot 1$	
	$16' \otimes 16'$	120			$80 \cdot 1$		$10 \cdot 4$	
		126 (126′)	$16 \cdot 1$		$80 \cdot 1$		$10 \cdot 3$ ($+10 \cdot 3'$)	

The numbers in parenthesis in the last column indicate that half of the weights belong to each of the two "half tensors".

Both tables are to be used as the analogous ones in Appendix B, and here too it is clear how the tables must be continued.

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