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On the Representations of Haag Fields

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Abstract. It is shown that the algebra of *quasi-local operators* of a *Haag field* (with certain reasonable properties) is *simple*. Consequently all representations of such Haag fields are *faithful*. The intuitive notion of the "physical equivalence" of two local fields is formulated mathematically. It is then proved that all (faithful) representations of a given local Haag field are "physically equivalent".

§ 1. Introduction

In recent years several mathematical frames have been proposed which are expected to embody the essential features (such as: locality and relativistic invariance) of quantized field theories. One of these was proposed by WIGHTMAN[1]). This frame (which will be referred to as the *Wightman field*) is specified by a Hilbert space and a linear and weakly continuous mapping $f(x) \rightarrow A(f)$ from a suitable test-function space into the set of closed linear operators in $\mathcal{H}$. Locality and relativistic invariance of the theory are then expressed by appropriate properties of this mapping. A somewhat different scheme was proposed by HAAG[2])[3])[4])[5]). In this method the mathematical structure (which will be called the *Haag field*) is specified by a Hilbert space $\mathcal{H}$ and a correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ between (bounded) open space-time domains Δ and *von Neumann algebras* $\mathfrak{R}(\Delta)$ in $\mathcal{H}$. Such features of the theory as locality and relativistic invariance are now reflected by appropriate properties of the correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ (see definitions (1) and (6) of § 2).

Unfortunately the exact relationships of these two descriptions (of Wightman field and Haag field) have not been fully explored. One might expect, at first sight, that a given local Wightman field $f \rightarrow A(f)$ will correspond to a local Haag field if one establishes the correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$; where $\mathfrak{R}(\Delta)$ is the von Neumann algebra *generated*[5]) by all field operators $A(f)$ with f vanishing outside the space-time domain Δ. The correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ thus obtained may be called the *local correspondence* generated by the given Wightman field. Two questions arise naturally in this connection:

(1) When does the local correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ generated by a Wightman field define a local Haag field?

(2) When is the correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ of a given local Haag field the *local correspondence* generated by a Wightman field?

At present, the first of these questions has received only a partial answer[6]), whereas the second question remains completely open.

The present paper is not directly concerned with the questions just mentioned. Nevertheless the main result of this paper will enable us to comment on some aspects of the second question of the previous paragraph.

The principal questions discussed in this paper and the main conclusions arrived at may be described roughly as follows: Let the (separable) Hilbert space $\mathcal{H}$ and the correspondence $\Delta \rightarrow \mathfrak{R}(\Delta)$ characterize a given local Haag field. The set-theoretic union of all $\mathfrak{R}(\Delta)$'s corresponding to all *bounded* and open domains Δ will be a *--algebra Q of bounded linear operators in $\mathcal{H}$. Taking the completion of Q, in the topology determined by the operator norm, we shall obtain a so-called C^*-algebra $\overline{Q}$ which will be called the algebra of *quasi-local operators* of the given Haag field. The first important problem is to study the structure of the C^*-algebra $\overline{Q}$. Section 3 of this paper is devoted to this problem. It will be shown there (under some additional mild assumptions about the local Haag fields) that the algebra $\overline{Q}$ as well as the incomplete algebra Q of a local Haag field are *simple*. (For a precise statement of this result see theorem 1.)

Let us now consider a faithful, separable representation of the algebra $\overline{Q}$ of the given Haag field. In other words we consider a *-isomorphism (say h) of the *-algebra $\overline{Q}$ onto a *-algebra (say $\overline{Q}_1$) of bounded operators of a *separable Hilbert space* $\mathcal{H}_1$. The isomorphism h will map a local ring $\mathfrak{R}(\Delta)$ of the original Haag field onto a subalgebra $h(\mathfrak{R}(\Delta)) \equiv \mathfrak{R}_1(\Delta)$ of $\overline{Q}_1$. We then expect that the Hilbert space $\mathcal{H}_1$, and the correspondence $\Delta \rightarrow \mathfrak{R}_1(\Delta)$ will define a local Haag field. In fact, *isotony* and *local commutativity* (see definition 1 of § 2) of the given Haag field imply that the correspondence $\Delta \rightarrow \mathfrak{R}_1(\Delta)$ will have the corresponding properties. Therefore, the only question that needs further investigation is, as to whether; the algebras $\mathfrak{R}_1(\Delta)$ are again von Neumann algebras. This question will be discussed elsewhere[7]). It will be shown there (with some additional assumptions about the given Haag field) that the algebras $\mathfrak{R}_1(\Delta)$ are indeed von Neumann algebras. Anticipating the result of reference 7 we may then conclude that every *separable* (faithful) representation of the algebra $\overline{Q}$ of a given local Haag field defines (in the manner just described) a local Haag field which will be called a *representation of the given Haag field*.

Two local Haag fields are said to be isomorphic if they are faithful representations of each other. Evidently the set of all Haag fields can be divided into *equivalence classes* of isomorphic Haag fields.

We shall study in this paper the relationship existing between isomorphic Haag fields. A very satisfying result in this direction would have been that all isomorphic (irreducible) Haag fields are unitarily equivalent. This, however, is not true. We shall therefore introduce a weaker notion than that of unitary equivalence (viz. the notion of local unitary equivalence) and show that all isomorphic Haag are locally unitarily equivalent (Theorem 2 of § 4). This mathematical theorem may be interpreted physically to say the following: two isomorphic Haag fields will predict the same outcome for all experiments carried out in an arbitrarily chosen but finite space-time region. In other words, if a given Haag field provides an adequate theory for a group of physical phenomena which can be explored by experiments performed in an arbitrarily chosen finite space-time domain then any (faithful) representation of the given Haag field will also be adequate as the theory of the same group of phenomena. Since all physical measurements are carried out in a finite space-time region, it seems reasonable to conclude that all isomorphic Haag fields are *physically equivalent*[8]).

We now mention some of the implications of the above conclusion. First of all it is clear that unitary equivalence is not a *necessary* condition for physical equivalence.

In fact there exist unitarily inequivalent Haag fields which are isomorphic (and thus physically equivalent).

Furthermore since all isomorphic Haag fields are physically equivalent it suggests that only abstract algebraic properties (i.e. those properties which are invariant under algebraic isomorphism) of the algebra $\overline{Q}$ of quasi-local operators and those of the local correspondence $\Delta \to \mathfrak{R}(\Delta)$ are physically relevant. It is thus both natural and possible to develop a purely algebraic framework for quantum field theory[9]).

Finally in appendix (1) of this paper we shall exploit theorem 2 for answering the following question: suppose that the local correspondence of one of the Haag fields of a given equivalence class (of isomorphic Haag fields) is generated by a Wightman field. Does it then follow that the local correspondence of every Haag field in the given equivalence class is generated by a Wightman field?

§ 2. Some Basic Definitions

In the introduction we have described some of the fundamental concepts (such as those of a Haag field, isomorphic representations of Haag fields etc.) in an informal manner. For the sake of ready reference we shall now state the precise definitions of these concepts.

Definition 1

A separable Hilbert space $\mathcal{H}$ and a correspondence $\Delta \to \mathfrak{R}(\Delta)$ between open space-time domains Δ and algebras $\mathfrak{R}(\Delta)$ of bounded linear operators in $\mathcal{H}$ are said to define a (*local*) *Haag field* if the following conditions are fulfilled:

(a) The algebras $\mathfrak{R}(\Delta)$'s are *von Neumann algebras*[10]).

(b) $\Delta_1 \subseteqq \Delta_2$ implies that $\mathfrak{R}(\Delta_1) \subseteqq \mathfrak{R}(\Delta_2)$ (*Isotony*).

(c) If the space-time domains Δ_1, and Δ_2 are totally space-like with respect to each other then $\mathfrak{R}(\Delta_1) \subseteqq \mathfrak{R}'(\Delta_2)$ (*Local commutativity*). Here $\mathfrak{R}'(\Delta_2)$ (called the commutant of $\mathfrak{R}(\Delta_2)$) denotes the set of all bounded linear operators of $\mathcal{H}$ which commute with every operator of $\mathfrak{R}(\Delta_2)$.

Let Q denote the set-theoretical union of all $\mathfrak{R}(\Delta)$'s which correspond to bounded open domains Δ. Then the completion of Q in the uniform topology will be denoted by $\overline{Q}$ and it is called the *algebra of quasi-local operators* of the given Haag field. A Haag field is said to be *irreducible* if its algebra of quasi-local operators $\overline{Q}$ is irreducible (i.e. if $\overline{Q}' = \{\lambda I\}$).

Invariance of a Haag field under a given symmetry group G is expressed by the existence of a suitable representation of G in the automorphism group of the algebra of quasi-local operators. We shall therefore state now the definitions of relevant mathematical concepts.

Definition 2

Let R_1 and R_2 be two *-algebras. A mapping ϕ: $T \to \phi(T)$ from R_1 onto R_2 is called a *-homomorphism of R_1 onto R_2 if the following relations hold for all T_1, T_2 in R_1 and complex co-efficients α, β:

$$\phi(\alpha\, T_1 + \beta\, T_2) = \alpha\, \phi(T_1) + \beta\, \phi(T_2) \tag{1}$$

$$\phi(T_1\, T_2) = \phi(T_1)\, \phi(T_2) \tag{2}$$

$$\phi(T_1^*) = (\phi(T_1))^* \tag{3}$$

If the mapping ϕ is, in addition, one-one (bijective) then one speaks of **-isomorphism* of *-algebra R_1 onto the *-algebra R_2.

Definition 3

A *-isomorphism ϕ: $T \to \phi(T)$ of the algebra R onto itself is called an *automorphism* of R. The class of all automorphisms of the algebra R forms a group (denoted by $A(R)$) when the product operation, inverse operation, and the identity are defined as follows:

The product of two autmorphisms ϕ_1 and ϕ_2 is defined to be the automorphism given by the mapping $\phi_1\, \phi_2$: $T \to \phi_1(\phi_2(T))$.

The inverse of an automorphism ϕ: $T \to \phi(T)$ is the inverse mapping ϕ^{-1} from R onto itself. The identity of the automorphism group $A(R)$ is now the identity mapping $T \to T$ of R onto itself.

Definition 4

A group G is said to have a *representation in the automorphism group* $A(R)$ of the algebra R if there exists a mapping h: $g \to h(g)$ from G into $A(R)$ such that

(a) $h(g_1\, g_2) = h(g_1)\, h(g_2)$ and

(b) the identity of the group G is mapped into the identity of the group $A(R)$.

Definition 5

Let G be a topological group[11]) and a suitable topology be defined on $A(R)$ so that it becomes a topological group also. Then a continuous mapping from G into $A(R)$ which satisfies conditions (a) and (b) of the previous definition is called a *continuous representation* of G in the automorphism group $A(R)$. Usually one speaks of representations of groups in more restricted sense. For example, one speaks of *unitary representations* of a group G when one considers mappings h: $g \to h(g)$ from G into the group of unitary operators of a Hilberts space, which satisfy conditions (a) and (b) of definition 4. It may be noted here that the concept of representation (in the sense of definition 4) coincides with that of unitary representation when one specializes the algebra R to be the algebra of all bounded operators in a Hilbert space[12]).

We are now prepared to express the invariance of a Haag field under a given symmetry group.

Definition 6

Let the symmetry group G be a group of transformations of the space-time manifold. (For example G may be the 4-translation group, Euclidean group or inhomogeneous Lorentz group etc.) A Haag field, characterized by the Hilbert space $\mathcal{H}$, the algebra of quasi-local operators $\overline{Q}$ and the correspondence $\Delta \to \mathfrak{N}(\Delta)$, is said to be *invariant under* G if there exists a representation $g \to \phi_g$ of the group G in the automorphism group of $\overline{Q}$ such that

$$\phi_g \mathfrak{N}((\Delta)) = \mathfrak{N}(g\, \Delta) \tag{4}$$

for all g in G and all open and bounded space-time domains Δ.

Here $\phi_g(\mathfrak{N}(\Delta))$ denotes the set of all operators $\phi_g(T)$ with $T \in \mathfrak{N}(\Delta)$ and $g\Delta$ is the space-time domain into which Δ is mapped by the transformation g.

We may note that definition 6 ignores the topological structure of the symmetry group G. It is of course desirable, both for mathematical and physical reasons, to introduce an appropriate topology in the automorphism group $A(\overline{Q})$ and express the G-invariance of a Haag field by the existence of a *continuous representation* of G in $A(\overline{Q})$ which fulfils the condition (1) of definition 6. However, we shall not discuss here the details of the construction of an appropriate topology in $A(\overline{Q})$.

Instead of definition 6, one often adopts the following more restrictive definition of G-invariance.

Definilion 7

A Haag field

$$\{\mathcal{H}, \overline{Q}, \Delta \to \mathfrak{N}(\Delta)\}$$

is said to be invariant under the symmetry group G if there exists a *continuous unitary representation* $g \to U g$ of G (in $\mathcal{H}$) such that

$$U_g\, \mathfrak{N}(\Delta)\, U_g^{-1} = \mathfrak{N}(g\, \Delta)$$

for all bounded and open space-time domains Δ and every $g \in G$.

Definition 8

A Haag field

$$\{\mathcal{H}_2, \overline{Q}_2, \Delta \to \mathfrak{N}_2(\Delta)\}$$

is said to be a *representation* of the given Haag field

$$\{\mathcal{H}_1, \overline{Q}_1, \Delta \to \mathfrak{N}_1(\Delta)\}$$

if there exists a *-homomorphism h: $T \to h(T)$ from the *-algebra $\overline{Q_1}$ on to the *-algebra $\overline{Q}_2$ such that

$$h(\mathfrak{N}_1(\Delta)) = \mathfrak{N}_2(\Delta)\ ,$$

for all bounded and open domains Δ.

If the homomorphism h is a *-isomorphism then the Haag field

$$\{\mathcal{H}_2, \overline{Q}_2, \Delta \to \mathfrak{N}_2(\Delta)\}$$

is said to be a *faithful* (*or isomorphic*) *representation* of the given Haag field.

Definition 9

Two Haag fields

$$\{\mathcal{H}_1, \overline{Q}_1, \Delta \to \mathfrak{N}_1(\Delta)\};\ \text{and}\ \{\mathcal{H}_2, \overline{Q}_2, \Delta \to \mathfrak{N}_2(\Delta)\}$$

are said to be *unitarily equivalent* if there exists a unitary transformation U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$U\, \mathfrak{N}_1(\Delta)\, U^{-1} = \mathfrak{N}_2(\Delta)$$

for all open space-time domains Δ.

Evidently two unitarily equivalent Haag fields are isomorphic (i.e. are faithful representations of each other); but two isomorphic Haag fields need not be unitarily equivalent. However, it will be shown (Theorem 2) that two isomorphic Haag fields are always *locally unitarily equivalent* in the following sense:

Definition 10

Two Haag fields

$$\{\mathcal{H}_1, \overline{Q}_1, \Delta \rightarrow \mathfrak{R}_1(\Delta)\}; \text{ and } \{\mathcal{H}_2, \overline{Q}_2, \Delta \rightarrow \mathfrak{R}_2(\Delta)\}$$

are said to be *locally unitarily equivalent* if for any given bounded space-time domain Δ (no matter how large) there exists a unitary transformation $U(\Delta)$ from $\mathcal{H}_1$, onto $\mathcal{H}_2$, such that $U(\Delta)\, \mathfrak{R}_1(\Delta_0)\, U^{-1}(\Delta) = \mathfrak{R}_2(\Delta_0)$ for all $\Delta_0 \subseteqq \Delta$.

Finally we give a mathematical formulation of the notion of *physical equivalence* of two Haag fields:

Definition 11

Two Haag fields

$$\{\mathcal{H}_1, \overline{Q}_1, \Delta \rightarrow \mathfrak{R}_1(\Delta)\}; \text{ and } \{\mathcal{H}_2, \overline{Q}_2, \Delta \rightarrow \mathfrak{R}_2(\Delta)\}$$

are said to be physically equivalent if the following two conditions are fulfilled:

(1) There exists a*-isomorphism $h\colon T \rightarrow h(T)$ from $\overline{Q_1}$ onto $\overline{Q_2}$ such that

$$h(\mathfrak{R}_1(\Delta)) = \mathfrak{R}_2(\Delta).$$

for all bounded and open domains Δ.

(2) For any given bounded domain Δ there exists a unitary transformation $U(\Delta)$ from $\mathcal{H}_1$, onto $\mathcal{H}_2$, such that

$$U(\Delta)\; T\; U^{-1}(\Delta) = h(T) \text{ for all } T \in \mathfrak{R}_1(\Delta). \tag{2a}$$

In other words, physical equivalence of two Haag fields implies that the two Haag fields, are isomorphic, as well as locally unitarily equivalent; the local unitary transformations $U(\Delta)$ being such that relation (2a) holds.

§3. The Structure of the Algebra of Quasi-Local Operators

It will be shown in this section that the algebra of quasi-local operators of a Haag field with certain reasonable properties, is simple[13]). For giving a more precise formulation of this result we first introduce the

Definition 12

A Haag field

$$\{\mathcal{H}, \overline{Q}, \Delta \rightarrow \mathfrak{R}(\Delta)\}$$

is said to have the property F ("F" for factor) if for any bounded open space-time domain Δ_1, there exists a *bounded* and open domain Δ_2 such that $\Delta_1 \subseteqq \Delta_2$ and the von Neumann algebra $\mathfrak{R}(\Delta_2)$ is a *factor*[14]).

The result, alluded to earlier, can now be stated more precisely as follows:

Theorem 1

Let the Hilbert space $\mathcal{H}$ and the correspondence $\Delta \to \mathfrak{R}(\Delta)$ characterize a translation invariant local Haag field (see definition 1 and 6) which has the property F. Then the algebra $Q = \underset{\Delta,\ \text{bounded}}{U}\ \mathfrak{R}(\Delta)$, as well as its completion $\overline{Q}$ in the uniform topology are *simple*.

Before proving theorem 1, we should of course, make sure that we are not talking about empty set. In other words we should ascertain the existence of at least one translation invariant local Haag field with the property F. It is known[15]) that the Haag field generated by free Boson field has the property F and it is of course translation invariant. It is not known whether the property F is true for every Haag field which is generated by (not necessarily free) Wightman field. Nevertheless it is plausible that property F is true for a wide class of Haag fields. Thus the range of applicability of theorem 1 seems to be rather general.

After these preliminary remarks we now turn to the proof of theorem 1. We shall first prove that the algebra Q which is the set-theoretical union of all $\mathfrak{R}(\Delta)$'s corresponding to bounded and open domains Δ is simple. For this purpose we shall need the following lemmata:

Lemma 1

Every non-trivial (left) ideal of a von Neumann algebra $\mathfrak{R}$ contains at least one non-zero projection operator.

A proof of lemma 1, under the additional assumption that $\mathfrak{R}$ is the algebra of all bounded operators of a Hilbert space, can be found in reference 16 (pp. 291–292). In appendix 2 we shall give a proof of this lemma (without making the extra assumption just mentioned).

Lemma 2

Let the Hilbert space $\mathcal{H}$ and the correspondence $\Delta \to \mathfrak{R}(\Delta)$ define a translation-invariant local Haag field with the property F. Let Δ_1, be a *bounded* and open domain such that $\mathfrak{R}(\Delta_1)$ is an *infinite factor*[17])[21]). Then there exists a *bounded* and open domain Δ such that:

(a) $\mathfrak{R}(\Delta_1) \subseteqq \mathfrak{R}(\Delta)$;
(b) $\mathfrak{R}(\Delta)$ is a factor; and
(c) Every non-zero projection of $\mathfrak{R}(\Delta_1)$ is *infinite relative to* $\mathfrak{R}(\Delta)$.

Proof of lemma 2

It is clear that by translating (spatially) the bounded domain Δ_1, we can obtain a domain Δ_2 which is totally space-like with respect to Δ_1. The property F of the Haag field guarantees the existence of a bounded and open domain Δ which covers Δ_1 and Δ_2 and such that $\mathfrak{R}(\Delta)$ is a factor. It is evident that $\mathfrak{R}(\Delta_1) \subseteqq \mathfrak{R}(\Delta)$ (isotony). Thus lemma 2 will be stablished if we show that every non-zero projection of $\mathfrak{R}(\Delta_1)$ is infinite relative to $\mathfrak{R}(\Delta)$.

For this purpose it is sufficient to prove that if P ($\neq 0$) is a projection of $\mathfrak{R}(\Delta_1)$ then the reduction[18]) $(\mathfrak{R}(\Delta))_P$ of $\mathfrak{R}(\Delta)$ to the range M of P is an infinite factor[19]). Towards this end we first prove the existence of a *-isomorphism from $\mathfrak{R}(\Delta_2)$ into $(\mathfrak{R}(\Delta)_P$.

As a preliminary to the construction of the desired isomorphism, we recall that the domain Δ_2 is constructed to be totally space-like with respect to Δ_1, and therefore

$$\mathfrak{R}(\Delta_2) \subseteqq \mathfrak{R}'(\Delta_1) \text{ (Local commutativity).}$$

Consequently every operator T of $\mathfrak{R}(\Delta_2)$ is reduced by the subspace m (i.e. the range of the projection $P \in \mathfrak{R}(\Delta_1)$). Hence if $T \in \mathfrak{R}(\Delta_2)$ then its restriction $T_{(m)}$ to the subspace m will belong to $(\mathfrak{R}(\Delta_2))_P$; (and therefore a fortiori to $(\mathfrak{R}(\Delta))_P$). Thus the mapping $T \to T_{(m)}$ (with $T \in \mathfrak{R}(\Delta_2)$) is a mapping from $\mathfrak{R}(\Delta_2)$ into $(\mathfrak{R}(\Delta))_P$. It is now claimed that this mapping provides the desired isomorphism. In fact it can be easily verified that if $S \to S_{(m)}$, and $T \to T_{(m)}$ (with $S, T \in \mathfrak{R}(\Delta_2)$) then $S + T \to S_{(m)} + T_{(m)}$; $\alpha S \to \alpha S_{(m)}$; $S T \to S_{(m)} T_{(m)}$ and $T^* \to (T_{(m)})^*$.

It only remains to show that the mapping $T \to T_{(m)}$ (with $T \in \mathfrak{R}(\Delta_2)$) is one-one. In other words we have to show that if S and T are in $\mathfrak{R}(\Delta_2)$ and if $S_{(m)} = T_{(m)}$ then $S = T$. As a matter of fact, $S_{(m)} = T_{(m)}$ implies that $S P = T P$ or $(S - T) P = 0$. The projection $P (\neq 0)$ belongs to the factor $\mathfrak{R}(\Delta_1)$ and $(S - T)$ belongs to $\mathfrak{R}(\Delta_2)$ which is contained in the commutant of $\mathfrak{R}(\Delta_1)$. Therefore $(S - T) P = 0$ if and only if $S - T = 0$[20]). Hence $S_{(m)} = T_{(m)}$ (with $S, T \in \mathfrak{R}(\Delta_2)$) implies that $S = T$. We have thus proved that the mapping $T \to T_{(m)}$ is a *-isomorphism from $\mathfrak{R}(\Delta_2)$ into $(\mathfrak{R}(\Delta))_P$.

In order to complete the proof of lemma 2 we recall that the domain Δ_2 is obtained by translating the domain Δ_1. Therefore translation invariance of the Haag field entails that $\mathfrak{R}(\Delta_2)$ is *-isomorphic to $\mathfrak{R}(\Delta_1)$. Since $\mathfrak{R}(\Delta_1)$ is an infinite factor, $\mathfrak{R}(\Delta_2)$ will be an infinite factor too. Therefore there exists a projection, say $P^{(1)}$, in $\mathfrak{R}(\Delta_2)$ which is infinite relative to $\mathfrak{R}(\Delta_2)$. Since the mapping $T \to T_{(m)}$ (with $T \in \mathfrak{R}(\Delta_2)$) is a *-isomorphism of $\mathfrak{R}(\Delta_2)$ into $(\mathfrak{R}(\Delta))_P$, it follows that $P^{(1)}_{(m)}$ is an infinite projection relative to $(\mathfrak{R}(\Delta))_P$. Thus the factor $(\mathfrak{R}(\Delta)_P)$ is infinite. This establishes the lemma 2.

It may be useful to mention here the following variant of lemma 2.

Lemma 2a

Let the correspondence $\Delta \to \mathfrak{R}(\Delta)$ define a local Haag field, such that the algebras $\mathfrak{R}(\Delta)$'s of local observables corresponding to bounded and open space-time domains are all *infinite factors*[21]). If, now, Δ_0 is a non-empty open space-time domain (no matter how small) that is totally space-like with respect to the open space-time domain Δ_1, then every (non-zero) projection of $\mathfrak{R}(\Delta_1)$ is infinite relative to $\mathfrak{R}(\Delta_1 \cup \Delta_0)$.

The proof of this lemma is the same as proof of lemma 2.

We now prove that Q is simple. Suppose, to the contrary of this assertion, that $\mathfrak{J}$ is a non-trivial (two-sided) ideal of Q. We shall show that this hypothesis leads to a contradiction.

Let T $(\neq 0)$ be an operator of $\mathfrak{J}$. It follows from the definition of Q that there exists a bounded domain Δ_0 such that $T \in \mathfrak{R}(\Delta_0)$. The property F of the Haag field says that there is a bounded domain Δ_1 which covers Δ_0 and such that $\mathfrak{R}(\Delta_1)$ is a factor. It can be easily verified that the set $\mathfrak{J} \cap \mathfrak{R}(\Delta_1) \equiv \mathfrak{J}_1$ is a two-sided ideal of $\mathfrak{R}(\Delta_1)$; and since $\mathfrak{J}_1$ contains at least the operator T $(\neq 0)$, it is a non-trivial two-sided ideal.

Now the factor $\mathfrak{R}(\Delta_1)$ can either be a finite factor or an infinite factor[21]). In the case that it is a finite factor, the desired contradiction results immediately. For, it is known that a finite factor has no non-trivial two-sided ideals[22]).

We now assume the only other possibility that $\mathfrak{N}(\Delta_1)$ is infinite. Since $\mathcal{J}_1$ is a non-trivial two-sided ideal of $\mathfrak{N}(\Delta_1)$, it follows according to lemma 1 that there exists a non-zero projection, say P, in $\mathcal{J}_1$. Now the projection P is either finite or infinite relative to $\mathfrak{N}(\Delta_1)$.

In any case there exists (according to lemma 2) a bounded domain Δ such that $\Delta_1 \subseteq \Delta$ $\mathfrak{N}(\Delta)$ is a factor and P is infinite relative to $\mathfrak{N}(\Delta)$. It is evident that the set $\mathcal{J} \cap \mathfrak{N}(\Delta) \equiv \mathcal{J}_0$ is a non-trivial two-sided ideal of $\mathfrak{N}(\Delta)$ and it contains P. However, P is infinite relative to $\mathfrak{N}(\Delta)$. Therefore there exist operators Ω and Ω^* in $\mathfrak{N}(\Delta)$ such that

$$\Omega^* \, \Omega = P \text{ and } \Omega \, \Omega^* = I$$ [23])

Hence $P \in \mathcal{J}_0$ implies that $\Omega \, P \, \Omega^* = \Omega \, \Omega^* \, \Omega \, \Omega^* = I$ belongs to $\mathcal{J}_0$. This would mean that $\mathcal{J} = Q$ which is not possible.

This proves the assertion that Q *is simple.*

We now show that "simplicity" of Q entails the "simplicity" of its closure $\overline{Q}$ in uniform topology. This will be accomplished with the aid of the following lemma:

Lemma 3

If the mapping $T \to A_T$ from Q into the set of all bounded operators in a Hilbert space (say $\mathcal{H}_1$) is a *-homomorphism (i.e. a *-representation) of Q then

$$||T|| = ||A_T|| \text{ for all } T \in Q\,.$$

Here $|| \, T \, ||$ and $|| \, A_T \, ||$ denote respectively the norm of the operator T (of $\mathcal{H}$) and A_T (of $\mathcal{H}_1$).

Proof of Lemma 3

First, it may be noted that since Q is simple, every *-homomorphism of Q is automatically a *-isomorphism.

Thus we may consider a *-isomorphism $T \to A_T$ of Q. Let T be any given operator of Q. We have to show that $|| \, T \, || = || \, A_T \, ||$. Towards this end we remark that there exists a bounded domain Δ such that $T \in \mathfrak{N}(\Delta)$. The *-isomorphism of Q into the algebra $\mathcal{B}(\mathcal{H}_1)$ of bounded operators in $\mathcal{H}_1$ automatically gives (by restriction to $\mathfrak{N}(\Delta)$) a *-isomorphism of $\mathfrak{N}(\Delta)$ into $\mathcal{B}(\mathcal{H}_1)$. The algebras $\mathfrak{N}(\Delta)$ and $\mathcal{B}(\mathcal{H}_1)$ can both be considered as *C^*-algebras*[24]), (*the norms in these algebras being the corresponding operator norms*).

It is well-known that every *-isomorphism of a C^*-algebra into another C^*-algebra preserves the norm[25]). Hence it follows that the norm of element T of the C^*-algebra $\mathfrak{N}(\Delta)$ is the same as the norm of the element A_T in the C^*-algebra $\mathcal{B}(\mathcal{H}_1)$. However, as mentioned earlier these norms coincide with the operator norms. Thus $|| \, T \, || = || \, A_T \, ||$.

Lemma 4

A C^*-algebra R is simple if and only if all *-representations of R (i.e. *-homomorphisms of R into the algebra of bounded operators in some Hilbert *space*) *are faithful.*

The proof of this mathematical lemma is for convenience deferred to appendix 2.

We now show that $\overline{Q}$ is simple. According to lemma 4 it will be sufficient to prove that every *-representation of $\overline{Q}$ is faithful (observe that $\overline{Q}$ is a C^*-algebra).

Let the mapping $T \to A_T$ from $\overline{Q}$ into the algebra of bounded operators of some Hilbert space be an arbitrary *-representation of $\overline{Q}$. It will be shown that it is faithful. In other words, it will be proved that if $T \in \overline{Q}$ and $A_T = 0$ then $T = 0$.

Since $T \in \overline{Q}$, there exists a sequence $\{T_n\}$ $(n = 1, 2, \ldots)$ of operators *in* Q such that $|| T_n - T || \to 0$ as $n \to \infty$. We now have:

$$(A)\ ||A_{(T_n - T)}|| = ||A_{T_n} - A_T|| = ||A_{T_n}|| \leqslant ||T_n - T||$$

(Here the first equality follows from the linearity of the mapping $T \to A_T$; the second from the hypothesis that $A_T = 0$ and the last inequality follows from a well-known result[26])).

Evidently the mapping $T \to A_T$ when restricted to Q gives a *-homomorphism of Q which maps T_n into A_{T_n}. Thus according to lemma 3 $|| A_{T_n} || = || T_n ||$.

Now, $|| T_n - T || \to 0$ as $n \to \infty$; Hence it follows from relation (A) that $|| A_{T_n} || = || T_n || \to 0$ as $n \to \infty$. But $|| T_n - T || \to 0$ as $n \to \infty$ and $|| T_n || \to 0$ as $n \to \infty$ imply that $T = 0$.

This completes the proof of theorem 1.

Before ending this section we mention some immediate corollaries:

Corollary 1

All representations of a translation invariant local Haag field with the property F are faithful.

Corollary 2

The algebra of quasi-local operators of a translation invariant local Haag field with the property F does not contain any completely continuous (compact) operator (other than 0). In fact every compact operator which belongs to $\overline{Q}$ will generate a non-trivial two-sided ideal of $\overline{Q}$.

§ 4. Physical Equivalence of Isomorphic Haag Fields

The main result of this section is that all (faithful) representations of a given Haag field (or, in other words, all isomorphic Haag fields) are physically equivalent. This result would of course be redundant if it so happened that all irreducible isomorphic Haag fields were unitarily equivalent.

It can, however, be shown that Haag fields, with physically reasonable properties, have unitarily inequivalent, irreducible representations[27]). We may therefore proceed to prove the theorem on physical equivalence, without much fear of redundancy.

Theorem 2

Let $\{\mathcal{H}, \overline{Q}, \Delta \to \mathfrak{N}(\Delta)\}$ be a local Haag field with the following property:

The von Neumann algebra $\mathfrak{N}(\Delta)$ of any open domain Δ is infinite[28]).

Then all faithful representations of the field $\{\mathcal{H}, \overline{Q}, \Delta \to \mathfrak{N}(\Delta)\}$ are physically equivalent (in the sense of definition 11 (of § 2)).

The proof of theorem 2 is based upon the following mathematical lemmata:

Lemma 1

Every infinite von Neumann algebra of a separable Hilbert space has a cyclic vector[29]).

Proof of lemma 1

It is known that if $\mathfrak{R}$ is an infinite von Neumann algebra then its commutant $\mathfrak{R}'$ has a *separating vector*[30]). On the other hand every separating vector of a von Neumann algebra is a cyclic vector for its commutant[31]). It thus follows that every infinite von Neumann algebra of separable Hilbert space possesses a cyclic vector.

Lemma 2

Let $\mathcal{H}$ and $\mathcal{H}_1$ be two separable Hilbert spaces and let $\mathfrak{R}$ and $\mathfrak{R}_1$ be von Neumann algebras in $\mathcal{H}$ and $\mathcal{H}_1$ respectively. Assume further that $\mathfrak{R}$, $\mathfrak{R}_1$ as well as their (respective) commutants $\mathfrak{R}'$ and $\mathfrak{R}'_1$ are infinite von Neumann algebras. Then for every *-isomorphism $h: T \to h(T)$ from $\mathfrak{R}$ onto $\mathfrak{R}_1$ there exists a unitary transformation U from $\mathcal{H}$ onto $\mathcal{H}_1$ such that

$$U T U^{-1} = h(T) \text{ for all } T \in \mathfrak{R} .$$

Proof of lemma 2

Since $\mathfrak{R}$ is infinite, it has a cyclic vector, say ψ. Similarly $\mathfrak{R}'$ has also a cyclic vector, say ϕ. Since the vector ϕ is cyclic for $\mathfrak{R}'$, it is separating for $\mathfrak{R}'' = \mathfrak{R}$. Thus $\mathfrak{R}$ has a cyclic vector and also a separating vector. It now follows from a well-known result[32]) that there exists a vector which is *both* cyclic and separating for $\mathfrak{R}$. Similarly one can prove the existence of a vector (in $\mathcal{H}_1$) which is *both* cyclic and separating for $\mathfrak{R}_1$.

The proof of lemma 2 will now follow immediately from the following well-known.

Theorem[33])

Let $\mathfrak{R}$ be a von Neumann algebra (in a Hilbert space $\mathcal{H}$) such that there exists a vector (in $\mathcal{H}$) which is both cyclic and separating for $\mathfrak{R}$. Let $\mathfrak{R}_1$ be another von Neumann algebra (in a possibly different Hilbert space $\mathcal{H}_1$). Assume again that there exists a vector in $\mathcal{H}_1$ which is cyclic as well as separating for $\mathfrak{R}_1$. Then for every *-isomorphism $h: T \to h(T)$ from $\mathfrak{R}$ onto $\mathfrak{R}_1$ there exists an unitary transformation U from $\mathcal{H}$ onto $\mathcal{H}_1$ such that

$$U T U^{-1} = h(T) \text{ for all } T \in \mathfrak{R} .$$

Theorem 2 can now be proved easily as follows:

Let the Haag fields

$$\{\mathcal{H}_1, \overline{Q}_1, \Delta \to \mathfrak{R}_1(\Delta)\} \text{ and } \{\mathcal{H}_2, \overline{Q}_2, \Delta \to \mathfrak{R}_2(\Delta)\}$$

be any two faithful representations of the given field

$$\{\mathcal{H}, \overline{Q}, \Delta \to \mathfrak{R}(\Delta)\} .$$

Hence there exist *-isomorphisms $h_1: T \to h_1(T)$ from $\bar{Q}$ onto $\bar{Q}_1$ and $h_2: T \to h_2(T)$ from $\bar{Q}$ onto $\bar{Q}_2$ such that

$$h_1(\mathfrak{R}(\Delta)) = \mathfrak{R}_1(\Delta) \text{ and}$$
$$h_2(\mathfrak{R}(\Delta)) = \mathfrak{R}_2(\Delta)$$

for every open space-time domain Δ. It can be verified easily that the mapping $h_2\, h_1^{-1} \equiv h$ is a *-isomorphism from $\bar{Q}_1$ onto $\bar{Q}_2$ such that

$$h(\mathfrak{R}_1(\Delta)) = \mathfrak{R}_2(\Delta)\ ,$$

for all open domains Δ. The physical equivalence of the two Haag fields

$$\{\mathcal{H}_1,\ \bar{Q}_1,\ \Delta \to \mathfrak{R}_1(\Delta)\} \text{ and } \{\mathcal{H}_2,\ \bar{Q}_2,\ \Delta \to \mathfrak{R}_2(\Delta)\}$$

will be established if we show that for every given bounded space-time domain Δ_1 there exists an unitary transformation $U(\Delta_1)$ such that

$$(A)\ U(\Delta_1)\ T\ U^{-1}(\Delta_1) = h(T) \text{ for all } T \in \mathfrak{R}_1(\Delta_1)\ .$$

Towards this end we first remark that the von Neumann algebras $\mathfrak{R}_1(\Delta)$ and $\mathfrak{R}_2(\Delta)$ are *-isomorphic to the von Neumann algebra $\mathfrak{R}(\Delta)$ which is assumed to be infinite (cf. footnote 28). Thus for any open domain Δ, the von Neumann algebras $\mathfrak{R}_1(\Delta)$ and $\mathfrak{R}_2(\Delta)$ are infinite too.

We now consider an arbitrarily chosen bounded domain Δ_1 and prove the existence of an unitary transformation $U(\Delta_1)$ with property (A). Since Δ_1 is bounded there evidently exists a non-empty open domain, say Δ_2 which is totally space-like with respect to Δ_1. It follows from the foregoing remark that the von Neumann algebras $\mathfrak{R}_1(\Delta_1)$, $\mathfrak{R}_2(\Delta_1)$, $\mathfrak{R}_1(\Delta_2)$ and $\mathfrak{R}_2(\Delta_2)$ are all infinite. Furthermore local-commutativity implies that the infinite von Neumann algebra $\mathfrak{R}_1(\Delta_2)$ is a subalgebra of $\mathfrak{R}'_1(\Delta_1)$; and thus $\mathfrak{R}'_1(\Delta_1)$ must be infinite too.

Similarly one sees that $\mathfrak{R}'_2(\Delta_1)$ is infinite.

We have therefore the following situation: The von Neumann algebras $\mathfrak{R}_1(\Delta_1)$, $\mathfrak{R}_2(\Delta_1)$ as well as their respective commutants $\mathfrak{R}'_1(\Delta_1)$ and $\mathfrak{R}'_2(\Delta_2)$ are all infinite and h is a *-isomorphism from $\mathfrak{R}_1(\Delta_1)$ onto $\mathfrak{R}_2(\Delta_1)$. Hence one can apply lemma 2 and establish the existence of a unitary transformation $U(\Delta_1)$ from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$U(\Delta_1)\ T\ U^{-1}(\Delta_1) = h(T) \text{ for all } T \in \mathfrak{R}_1(\Delta_1)\ .$$

This establishes theorem 2. Since physical equivalence of two Haag fields implies their local unitary equivalence, this argument establishes also that all isomorphic Haag fields are locally unitarily equivalent.

The physical interpretation and some consequences of theorem 2 have already been mentioned in the introduction. We shall discuss only one further consequence of theorem 2 in appendix 1.

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Appendix 1

The problem of finding sufficient (and necessary) criteria so that a local Wightman field may be *associated* with a given Haag field (in the sense that the correspondence $\Delta \to \mathfrak{R}(\Delta)$ of the given Haag field is the local correspondence generated by the associated Wightman field) is, at present, completely unsolved. An answer to the question, which is formulated at the end of the introduction (§ 1) will provide some hints as to where to look for such criteria. For example, if that question has an affirmative answer, it would follow that only the representation independent properties of the Haag field are relevant for the formulation of the desired criteria.

We show now (with some reservations) that the question mentioned at the end of § 1 has affirmative answer. Let us consider a local Haag field $\{\mathcal{H}, \bar{Q}, \Delta \to \mathfrak{R}(\Delta)\}$ such that it is associated with a local Wightman field $\{\mathcal{H}, f \to \phi(f)\}$. Now let the Haag field $\{\mathcal{H}_1, \bar{Q}_1, \Delta \to \mathfrak{R}_1(\Delta)\}$ be a faithful representation of $\{\mathcal{H}, \bar{Q}, \Delta \to \mathfrak{R}(\Delta)\}$. In other words, let there exist a *-isomorphism $h\colon T \to h(T)$ from $\bar{Q}$ onto $\bar{Q}_1$ such that $h(\mathfrak{R}(\Delta)) = \mathfrak{R}_1(\Delta)$ for every bounded space-time domain Δ. We want to establish the existence of a Wightman field $\{\mathcal{H}_1, f \to \phi_1(f)\}$ such that the algebra $\mathfrak{R}_1(\Delta)$ corresponding to any bounded domain Δ is the von Neumann algebra generated by all operators $\phi_1(f)$ with f vanishing outside the space-time region Δ.

The construction of the field $f \to \phi_1(f)$ can be carried out as follows: Let f be any given test-function in the space $\mathcal{D}$, let Δ be a bounded and open space-time domain such that the support of f is contained in Δ. (Such bounded space-time domains exist because the supports of functions in $\mathcal{D}$ are compact). Let $U(\Delta)$ be a unitary operator from $\mathcal{H}$ onto $\mathcal{H}_1$ such that:

$$U(\Delta)\ T\ U^{-1}(\Delta) = h(T) \text{ for all } T \in \mathfrak{R}(\Delta)\ . \tag{1}$$

(The existence of such unitary transformations is guaranteed by theorem 2 of § 4.)

We may then define $\phi_1(f)$ by the equation:

$$\phi_1(f) = U(\Delta)\ \phi(f)\ U^{-1}(\Delta)\ . \tag{2}$$

We have to verify that definition 2 is unambiguous. Ambiguity may arise due to two reasons. First of all the choice of the domain Δ is not unique. (The only requirement on Δ being that it contains the support of the given test-function). The other possible cause of ambiguity is that even when Δ is chosen the unitary transformation $U(\Delta)$ satisfying relation 1, is not unique.

We now verify that none of these will lead to an ambiguity in the definition of $\phi_1(f)$.

Let Δ_1 and Δ_2 be any two bounded and open domains which contain the support of the given test-function f and let $U(\Delta_1)$ and $U(\Delta_2)$ be corresponding unitary transformations. It will be shown that

$$U(\Delta_1)\ \phi(f)\ U^{-1}(\Delta_1) = U(\Delta_2)\ \phi(f)\ U^{-1}(\Delta_2) = \phi_1(f)\ . \tag{3}$$

To this end we first observe that the set $\Delta_0 \equiv \Delta_1 \cap \Delta_2$ is also a bounded and open domain containing the support of f. Furthermore

$$h(T) = U(\Delta_1)\ T\ U^{-1}(\Delta_1) = U(\Delta_2)\ T\ U^{-1}(\Delta_2)$$

for all

$$T \in \mathfrak{R}(\Delta_0) \subseteqq (\mathfrak{R}(\Delta_1) \cap \mathfrak{R}(\Delta_2)) .$$

Hence

$$U^{-1}(\Delta_2)\, U(\Delta_1)\, T\, U^{-1}(\Delta_1)\, U(\Delta_2) = T$$

for all $T \in \mathfrak{R}(\Delta_0)$. In other words the unitary operator $U^{-1}(\Delta_2)\, U(\Delta_1) \equiv U_0$ belongs to $\mathfrak{R}'(\Delta_0)$. Since the support of f is within Δ_0, the operator $\phi(f)$ is (by hypothesis) affiliated with $\mathfrak{R}(\Delta_0)$. It thus follows that $U_0\, \phi(f)\, U_0^{-1} = \phi(f)$. Relation 3 can now be proved easily. In fact

$$U(\Delta_1)\, \phi(f)\, U^{-1}(\Delta_1) = U(\Delta_2)\, U_0\, \phi(f)\, U_0^{-1}\, U^{-1}(\Delta_2) = U(\Delta_2)\, \phi(f)\, U^{-1}(\Delta_2) .$$

We have thus verified that definition 2 is unambiguous.

The assertion that $\mathfrak{R}_1(\Delta)$ is the von Neumann algebra generated by all operators $\phi_1(f)$ which correspond to test-functions f vanishing outside the region Δ now follows immediately from the definition 2. In fact (according to our hypothesis)

$$\mathfrak{R}(\Delta) = \{\phi(f) \mid f, \text{ vanishing outside } \Delta\}''.$$

Therefore

$$\mathfrak{R}_1(\Delta) = h(\mathfrak{R}(\Delta)) = U(\Delta)\, \mathfrak{R}(\Delta)\, U^{-1}(\Delta) =$$
$$\{U(\Delta)\, \phi(f)\, U^{-1}(\Delta) \equiv \phi_1(f) \mid f, \text{ vanishing outside } \Delta\}''.$$

It thus remains to show that the mapping $f \to \phi_1(f)$ defines a local Wightman field. For this purpose we have to verify the following properties of the mapping $f \to \phi_1(f)$:

(a) There exists a dense linear manifold D_1 in $\mathcal{H}_1$ which is a common domain of definition of all $\phi_1(f)$ (with $f \in \mathcal{D}$) and such that

$$\phi_1(f)\, D_1 \subseteqq D_1 \text{ and } \phi^*(f)\, D_1 \subseteqq D_1$$

(b) If f_1 and f_2 are any two test-functions in $\mathcal{D}$ then

$$\phi_1(\alpha f_1 + \beta f_2)\, \psi = \alpha\, \phi(f_1)\, \psi + \beta\, \phi(f_2)\, \psi$$

for any $\psi \in D_1$ and complex α, β.

(c) If $f_n \to 0$ (in the topology of $\mathcal{D}$) then $(\psi_1, \phi_1(f_n)\, \psi_2) \to 0$ for any $\psi_1, \psi_2 \in D_1$.

(d) If the supports of the test-functions f and g are completely space-like with respect to each other then

$$[\phi_1(f), \phi_1(g)] = [\phi_1(f), \phi_1^*(g)] = 0 \text{ on } D_1 .$$

We can not at present prove the property (a) of the mapping $f \to \phi_1(f)$. It can, however, be proved easily that for every given bounded space-time domain Δ (*no matter how large*) there exists a dense linear manifold $D_1(\Delta)$ such that all operators $\phi_1(f)$ (*with f vanishing outside* Δ) are defined on $D_1(\Delta)$; $\phi_1(f)\, D_1(\Delta) \subseteqq D_1(\Delta)$ and $\phi_1^*(f)\, D_1(\Delta) \subseteqq D_1(\Delta)$. In fact, it follows from the definition of $\phi_1(f)$ that the set $D_1(\Delta) \equiv U(\Delta)\, D$ is such a domain, where D is a common invariant dense domain of the field operators $\phi(f)$. (Here $U(\Delta)\, D$ denotes the set of all vectors of the form $U(\Delta)\, \psi$ with $\psi \in D$). It is also quite likely that $\bigcap_{\Delta,\text{ bounded}} (U(\Delta)\, D)$ is still dense in $\mathcal{H}_1$. In this case property (a) will hold strictly. However, we leave this question open for the present.

As for the properties (b), (c) and (d), they can be proved easily from the corresponding properties of $\phi(f)$ and definition of $\phi_1(f)$ if D_1 is replaced by $D_1(\Delta) = U(\Delta)\, D$ for suitably large Δ.

For example in (b) we replace D_1 by $D_1(\Delta)$ where Δ contains the supports of f_1 and f_2. Similarly in (c) we replace D_1 by $D_1(\Delta)$ where Δ is any *bounded* domain which contains the support of all test functions f_n $(n = 1, 2, \ldots)$. (Such a bounded domain always exists for every convergent sequence $\{f_n\}$ of test functions in $\mathcal{D}$).

Finally we mention that the foregoing arguments cannot be carried through if it is required that Wightman fields be defined on test-function space $\mathcal{S}$ instead of on $\mathcal{D}$.

Appendix 2

In this appendix we shall supply the proofs of lemma 1 and lemma 4 of § 3.

Lemma 1

Every non-trivial (left) ideal of a von Neumann algebra contains at least one non-zero projection operator.

Proof of lemma 1

Let the operator T $(\neq 0)$ belong to the non-trivial (left) ideal $\mathfrak{J}$ of $\mathfrak{N}$. Then the operator $T^*\, T \equiv H$ is also in $\mathfrak{J}$. Let $E(\lambda)$ denote the spectral family of the self-adjoint operator H. Since H is positive definite, there exists an interval $L \equiv (\alpha, \beta)$ $(0 < \alpha < \beta < \infty)$ such that $E(L) = E(\beta) - E(\alpha) \neq 0$. We shall show that the projection $E(L)$ belongs to the ideal $\mathfrak{J}$. Since $H \in \mathfrak{N}$, the projection $E(L)$ also belongs to $\mathfrak{N}$. Furthermore $H\, E(L) = E(L)\, H$. We denote by $H_{(m)}$ the restriction of H to range of $E(L)$. Thus $H_{(m)}$ is a bounded, linear, self-adjoint operator of the space m. Because of the choice of the interval L the point 0 of the real line is outside the spectrum of $H_{(m)}$. Therefore there exists a bounded, linear operator $H_{(m)}^{-1}$ in m which is the inverse of $H_{(m)}$. We now consider the operator B which is defined by the equation:

$$B\,\psi = H_{(m)}^{-1}\,\psi \text{ if } \psi \in m$$

and

$$B\,\phi = 0 \text{ if } \phi$$

is orthogonal to m. The operator B, when extended to the entire space by linearity, is a bounded linear operator of the underlying Hilbert space $\mathcal{H}$. We may also write $B = H_{(m)}^{-1}\, E(L)$. We shall now show that the operator B, thus defined, belongs to $\mathfrak{N}$.

To this end, we consider the reduction $(\mathfrak{N})_{(m)}$ of the algebra $\mathfrak{N}$ to the subspace m. Evidently $H_{(m)} \in (\mathfrak{N})_{(m)}$. Therefore $H_{(m)}^{-1}$ belongs to $(\mathfrak{N}_{(m)})''$. In other words, any bounded linear operator (of the space m) which commutes with all operators in $\mathfrak{N}_{(m)}$ will commute with $H_{(m)}^{-1}$ also.

Furthermore one has

$$(\mathfrak{N}_{(m)})' = (\mathfrak{N}')_{(m)}\ ^{34}).$$

With the help of these facts we can now show that B commutes with every operator T of the commutant $\mathfrak{R}'$ of $\mathfrak{R}$. In fact, we have:

$$B\,T\,\psi = H_{(m)}^{-1}\,E(L)\,T\,\psi = H_{(m)}^{-1}\,T\,E(L)\,\psi = H_{(m)}^{-1}\,T_{(m)}\,\big(E(L)\,\psi\big)$$
$$= T_{(m)}\,H_{(m)}\,E(L)\,\psi = T\,H_{(m)}^{-1}\,E(L)\,\psi = T\,B\,\psi\,, \text{ for all } \psi \in \mathcal{H}\,.$$

Here the second equality is valid because $T \in \mathfrak{R}'$ and $E(L) \in \mathfrak{R}$ and the fourth because $H_{(m)}^{-1} \in (\mathfrak{R}_{(m)})''$ and $T_{(m)} \in \mathfrak{R}'_{(m)} = (\mathfrak{R}_{(m)})'$. Thus we have proved that $B \in \mathfrak{R}$. Since $H \in \mathfrak{J}$ and $B \in \mathfrak{R}$; the operator $B\,H = H_{(m)}^{-1}\,E(L)\,H = H_{(m)}^{-1}\,H\,E(L) = E(L)$ also belongs to $\mathfrak{J}$. This establishes lemma 1.

Lemma 4

A C^*-algebra R with identity is *simple* if and only if all (cyclic) *-representations of R are faithful.

Proof of lemma 4

Let $T \to A_T$ be any *-representation of the algebra R. It can be verified easily that the set $\mathfrak{J}$ which consists of all elements T of R such that $A_T = 0$; is a two-sided ideal of R. Hence, if R is simple, the representation $T \to A_T$ must be faithful (i.e. $A_T = 0$ if and only if $T = 0$). This proves the "only if" part of the lemma.

For proving the "if" part of the lemma we now show that if the C^*-algebra R has a non-trivial two-sided ideal then it has at least one (cyclic) *-representation which is not faithful.

Let us suppose that $\mathfrak{J}$ is a non-trivial t.s. ideal of R. Then the closure $\bar{\mathfrak{J}}$ of $\mathfrak{J}$ (in the topology defined by the norm on R) is a non-trivial two-sided ideal (and hence, a fortiori, a left ideal) of R[35]). An application of two well-known theorems[36]) then guarantees the existence of a *positive linear functional*[37]) $\phi(T)$ on R such that

$$\phi(I) = 1, \text{ and } \phi(T^*\,T) = 0$$

for every T in the two-sided ideal $\bar{\mathfrak{J}}$.

It can be easily verified that the Gelfand construction[38]) corresponding to the positive linear functional ϕ will yield a cyclic *-representation $T \to A_T$ of R such that $A_T = 0$ whenever $T \in \bar{\mathfrak{J}}$. In fact we may recall that the Gelfand representation $T \to A_T$ (corresponding to the functional ϕ) has the following property: There exists a correspondence $T \to \psi_T$ between the elements T of R and vectors ψ_T in the representation space $\mathcal{H}$ (i.e. the Hilbert space in which the operators A_T act) such that

(a) The set D of all vectors ψ_T (with $T \in R$) is a dense linear manifold of $\mathcal{H}$,

(b) $A_S\,\psi_T = \psi_{ST}$ for every S and T in R, and

(c) $(\psi_T, \psi_T) = \phi(T^*\,T)$ for all $T \in R$.

Nowl et $T \in \bar{\mathfrak{J}}$. We shall show that $A_T = 0$. If ψ_S is any vector in D then

$$\|A_T\,\psi_S\|^2 = (A_T\,\psi_S, A_T\,\psi_S) = (\psi_{TS}, \psi_{TS}) = \phi\big((T\,S)^*\,(T\,S)\big)\,.$$

Since $\overline{\mathfrak{J}}$ is a t.s. ideal (and hence a right ideal also) and $T \in \overline{\mathfrak{J}}$, it follows that $T S \in \overline{\mathfrak{J}}$. Therefore $\| A_T \psi_S \|^2 = 0$. In other words $A_T \psi_S = 0$ for all $\psi_S \in D$. Since D is dense in $\mathcal{H}$ it follows that $A_T = 0$.

This completes the proof of lemma 4.

References and Footnotes

1) A. S. Wightman, Phys. Rev. *101*, 860 (1956). See also Les Problèmes Mathématique de la Théorie Quantique des Champs (CNRS, Paris 1959).

2) R. Haag, *Les Problèmes Mathématiques de la Théorie Quantique des Champs* (CNRS, Paris 1959).

3) R. Haag and B. Schroer, J. Math. Phys. *3*, 248 (1962).

4) H. Araki, *Lecture notes* (ETH, Zürich 1962).

5) M. Guenin and B. Misra, Nuovo Cim. *30*, 1272 (1963).

6) H. J. Borchers and W. Zimmermann, Nuovo Cim. *31*, 1047 (1964).

7) M. Guenin and B. Misra, (in preparation).

8) The intuitive notion of "physical equivalence" is given a mathematical formulation in definition 11 (§2) of this paper. An alternative definition has been suggested and motivated by D. Kastler and R. Haag ("An algebraic approach to quantum field theory"; preprint (1964)). It may be mentioned here that the "physical equivalence" (in the sense of definition 11) of two fields implies their "physical equivalence" in the sense of Kastler and Haag. The converse is also true for an physically interesting Haag fields (of Theorem 2).

9) Cf. D. Kastler and R. Haag, An algebraic approach to quantum field theory (preprint 1964).

10) A *-algebra $\mathfrak{N}$ of bounded operators which contains the identity operator is called a von Neumann algebra if $\mathfrak{N}'' = \mathfrak{N}$. Here $\mathfrak{N}''$ denotes the set of all bounded operators which commute with every operator permutable with all operators of $\mathfrak{N}$. For a detailed discussion of von Neumann algebras see J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (Paris, 1957).

11) For the definition and properties of topological groups see L. Pontrjagin, *Topological groups* (Princeton, 1958).

12) This conclusion follows from the observation that every automorphism of the algebras of all bounded operators of a Hilbert space $\mathcal{H}$ can be implemented by an unitary operator of $\mathcal{H}$. Cf. J. Dixmier, l.c. (P. 256).

13) A set $\mathfrak{J}$ of elements of an algebra Q is said to be a *two-sided ideal* of Q if (1) $\mathfrak{J} \subsetneqq Q$, (2) $T_1 \in \mathfrak{J}$ and $T_2 \in \mathfrak{J}$ imply that $\alpha\, T_1 \in \mathfrak{J}$, $T_1 + T_2 \in \mathfrak{J}$ and (3) $T \in \mathfrak{J}$; $S \in Q$ imply that TS as well as ST are in $\mathfrak{J}$.
Evidently the set (0) consisting only of the zero element of the algebra is a t.s. ideal of Q. An algebra is called *simple* if it has not t.s. ideal other than the trivial one (0).

14) A von Neumann algebra $\mathfrak{N}$ is said to be a factor if its center $\mathfrak{N}' \cap \mathfrak{N}$ contains the multiples of identity operator only.

15) H. Araki, *von Neumann algebras of local observables for free scalar fields* (preprint).

16) M. A. Naimark, normed rings (Noordhoff 1959).

17) A projection P in a von Neumann algebra $\mathfrak{N}$ is said to be infinite relative to $\mathfrak{N}$ if there exists a projection P_0 and a partial isometry $\Omega \in \mathfrak{N}$ such that $P_0 \lneqq P$, $\Omega\,\Omega^* = P_0$ and $\Omega^*\,\Omega = P$. A factor $\mathfrak{N}$ is said to be infinite if it contains at least one projection which is infinite relative to $\mathfrak{N}$. Otherwise it is said to be finite.

18) Let $\mathfrak{N}$ be a von Neumann algebra and P be any projection. Then the reduction $(\mathfrak{N})_P$ of $\mathfrak{N}$ to the range m of P is constructed as follows: Take all the operators A of $\mathfrak{N}$ which are reduced by the subspace m and form the restriction $A_{(m)}$ of A to m. The set of all such operators $A_{(m)}$ is the reduction $(\mathfrak{N})_P$. It can be proved that if $\mathfrak{N}$ is a factor and $P \in \mathfrak{N}$ then $(\mathfrak{N})_P$ is a factor too (cf. reference 16, p. 477).

19) See M. A. Naimark, l.c. (proposition *V*, p. 479).

20) M. A. Naimark, i.c. (p. 451).

21) If the Haag field has certain physically reasonable properties then it can even be proved that the factors $\mathfrak{N}(\Delta)$ are *necessarily infinite*.
See R. V. Kadison, J. Math. Phys. *4*, 1511 (1963) and M. Guenin and B. Misra, Nuovo Cim. *30*, 1272 (1963).

[22]) J. DIXMIER, l.c. p. 275.

[23]) M. A. NAIMARK, l.c. (proposition VI, p. 457).

[24]) A *-algebra R is called a *normed *-algebra* if to every $x \in R$ there corresponds a non-negative number $\| x \|$ (called the *norm* of x) such that

$$\begin{aligned}
&\text{(a)} \quad \| x \| = 0 = \text{if and only if } x = 0 \\
&\text{(b)} \quad \| \alpha x \| = | \alpha | \, \| x \| \\
&\text{(c)} \quad \| x + y \| \leqslant \| x \| + \| y \| \\
&\text{(d)} \quad \| x y \| \leqslant \| x \| \, \| y \| \quad \text{and} \\
&\text{(e)} \quad \| x^* \| = \| x \| .
\end{aligned}$$

If the algebra R is complete in the topology defined by the norm then it is called a *Banach *-algebra*.

A Banach *-algebra R is called a (abstract) C^*-algebra if the norm on R satisfies the following additional condition:

$\| x^* x \| = \| x \|^2$ for every $x \in R$. For details on C^*-algebras see reference 16 (where they are called *complete completely regular rings*).

[25]) M. A. NAIMARK, l.c. (Theorem 3, p. 311).

[26]) M. A. NAIMARK, l.c. (Theorem 1, p. 241).

[27]) Cf. Reference 9, see also B. MISRA, Lecture notes (Institute of Theoretical Physics, Geneva).

[28]) The property that the von Neumann algebras $\mathfrak{N}(\Delta)$ are infinite, follows from the usual postulates of quantum field theory (see the references cited in footnote 21). Thus the range of applicability of theorem 2 is sufficiently wide.

[29]) Let $\mathfrak{N}$ be a set of operators of a Hilbert space $\mathcal{H}$. A vector ψ of $\mathcal{H}$ is called a *cyclic vector* of $\mathfrak{N}$ if the closed linear subspace spanned by the vectors of the form $T \psi$ (with $T \in \mathfrak{N}$) is the entire space $\mathcal{H}$. A vector ϕ on the other hand, is called a *separating vector* of $\mathfrak{N}$ if $T \in \mathfrak{N}$ and $T \phi = 0$ imply that $T = 0$.

[30]) J. DIXMIER, l.c. (p. 323).

[31]) J. DIXMIER, l.c. (p. 6).

[32]) J. DIXMIER, l.c. (p. 232).

[33]) J. DIXMIER, l.c. (Theorem 3, p. 233).

[34]) J. VON NEUMANN, collected works, vol. III (Pergamon Press, 1961) p. 76, lemma 11.3.2.

[35]) M. A. NAIMARK, l.c. (proposition II, p. 178).

[36]) M. A. NAIMARK, l.c. (Theorem 4, p. 313 and proposition 1, p. 302).

[37]) A mapping $T \rightarrow \phi(T)$ from a *-algebra R into the field of complex numbers is called a positive linear fonctional on R if

$$\phi(\alpha T) = \alpha \, \phi(T) \, ; \quad \phi \, (T_1 + T_2) = \phi(T_1) + \phi(T_2)$$

and $\phi(T^* T) \geqslant 0$ for all T, T_1, T_2 in R and every complex coefficient α, β.

[38]) See M. A. NAIMARK, l.c. (pp. 242–245).