On the Point Dipole Representation of a Uniformly Magnetised Cylinder

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Abstract. The validity of assumption of a dipole field, for that produced by a homogeneous circular cylinder is examined in terms of exact calculations of axial and radial fields at distances comparable to the dimensions of the cylinder. It is shown that for an optimum value of the ratio of length to diameter, a cylindrical specimen would behave almost as a small dipole in so far as the fields in axial and radial directions are concerned.

1. Introduction

A rock sample could be represented by a single dipole at its centre, provided the magnetisation is uniform and the shape almost spherical. However, for practical convenience, rocks are often cut in cylindrical form for investigation of their remanent magnetisation. Even though assuming a uniform magnetisation, a cylindrical sample, owing to its finite size and shape, can not be considered to act as a dipole except at large distances compared to its size. In practice, however, measurements of weak magnetisations often necessitate large specimens of dimensions comparable with the measurement distance and in such cases the validity of dipole assumption is quite questionable.

The magnetisation vector in a cylindrical specimen could be measured in terms of axial and radial fields produced in the two directions by the respective magnetisation components. However, the relative magnitude of the axial and radial fields would also depend on the relative geometry of the cylindrical specimen, i.e. on the ratio of length to its diameter. Thus by varying the relative geometry of the cylinder, it could be possible to make the two fields at outside points in axial and radial directions agree very closely with the corresponding fields produced by a dipole of equivalent moment, if placed at the centre of the cylinder. So, for practical purposes, at this optimum value of the ratio of length to diameter, a cylinder could be considered to act as a small dipole so far as the fields in axial and radial directions are concerned.

In the present investigation, exact calculations of the magnetic field of a uniform circular cylinder at points in axial and radial direction have been made for different values of the ratio of the length to diameter and the results compared graphically with those obtained on the single dipole assumption for the cylinder. In each case the appropriate distance range, within which the dipole assumption is fairly valid, is defined. It is shown that for a close approximation to a small dipole, the length of the cylinder should be a little smaller than its diameter.

2. Methods for Exact Field Calculation

In geophysics literature one finds frequent use of the Poisson's relation[1]) for calculating the magnetic fields of homogeneous bodies, if their gravity effects are known. According to this relation, the magnetic potential is proportional to the gravity component in the direction of magnetisation, i.e.

$$W = -\frac{I_k}{G\varrho} \frac{\partial U}{\partial k}, \tag{2.1}$$

where W is the magnetic potential, I_k the intensity of magnetisation in a direction k, G the constant of gravitation, ϱ the density of the body and U the gravitational potential.

The magnetic field of the body in any direction i is, therefore, given by

$$H_i = \frac{I_k}{G\varrho} \frac{\partial}{\partial i} \left(\frac{\partial U}{\partial k}\right), \tag{2.2}$$

An alternative approach sometimes found more convenient, for calculating magnetic effects at points outside a uniformly magnetised body, is to replace the volume magnetisation by a surface distribution of magnetisation by application of the well known Gauss theorem. This method enables the conversion of a triple volume integral into a set of surface integrals which are comparatively easier to evaluate.

An appropriate use of both methods is illustrated in the present case of a uniform cylinder for which the calculation of axial field is relatively easier by the first method and that of radial field by the second method.

3. Axial Field of the Cylinder

It is assumed that a vertical cylinder of length $2l$ and diameter $2r$ is uniformly magnetised parallel to its axis and the vertical field is to be calculated at an axial point say P outside the cylinder. The right handed system of coordinates (x_1, x_2, x_3) could be so chosen that the origin coincides with the point P and the axis x_3 is coincident with the vertical axis of the cylinder. The axial component of the magnetic field at point P, according to equation (2.2) is given by

$$H_3 = \frac{I_3}{G\varrho} \frac{\partial}{\partial s} \left(\frac{\partial U}{\partial s}\right), \tag{3.1}$$

where s is the vertical distance (along x_3 axis) of the point P from the upper face of the cylinder.

The gravitational attraction $-\partial U/\partial s$ of a uniform cylinder of length $2l$ and radius r at an axial point is shown by RAMSEY[2]) to be

$$-\frac{\partial U}{\partial s} = 2\pi G\varrho \left[2l - \sqrt{(s+2l)^2 + r^2} + \sqrt{s^2 + r^2}\right]. \tag{3.2}$$

Thus

$$\frac{\partial^2 U}{\partial s^2} = 2\pi G\varrho \left[\frac{s+2l}{\sqrt{r^2 + (s+2l)^2}} - \frac{s}{\sqrt{r^2 + s^2}}\right]. \tag{3.3}$$

Finally, substituting equation (3.3) in (3.1), the axial field H_3 is given by

$$H_3 = 2\pi I_3 \left[\frac{(d+l)}{\sqrt{r^2 + (d+l)^2}} - \frac{(d-l)}{\sqrt{r^2 + (d-l)^2}}\right], \tag{3.4}$$

where $d = (s + l)$ is the distance of point P from the centre of the cylinder.

Compared to this lengthy expression for H_3, the corresponding dipole field H_{D3} obtained on the representation of the magnetisation of the cylinder by a single dipole, would be, simply

$$H_{D_3} = \frac{2\,M}{d^3} = \frac{2\,I_3\,\pi\,r^2\,2\,l}{d^3}\,, \qquad (3.5)$$

where M is the equivalent dipole moment for the cylinder of volume $\pi\,r^2\,2\,l$.

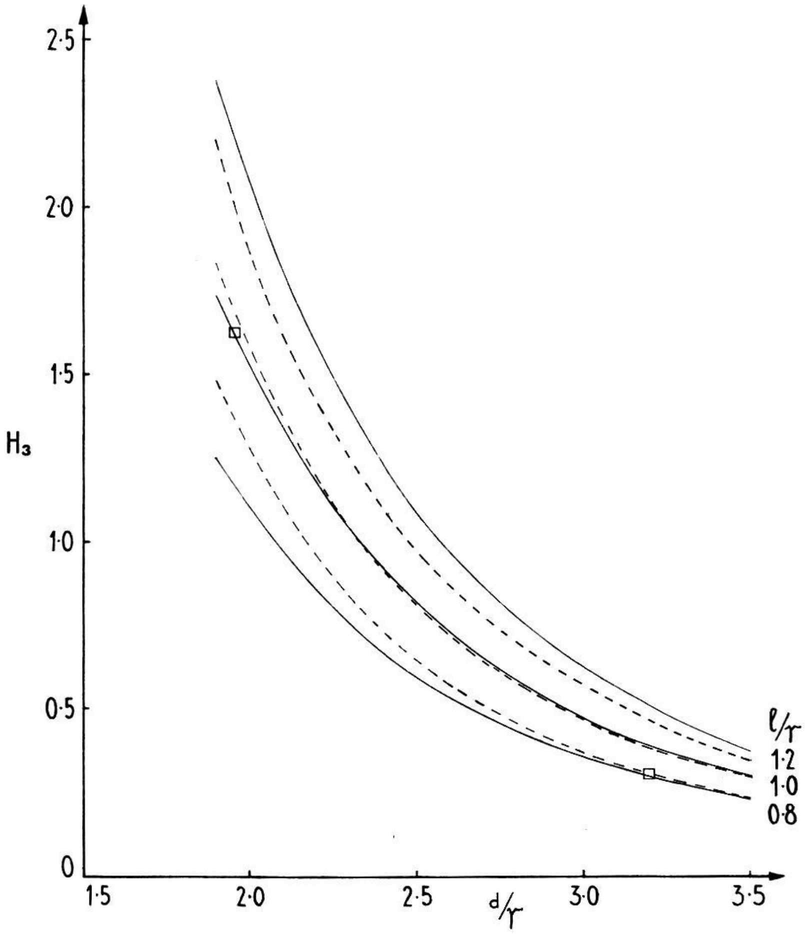

Figure 1

A closer examination of equation (3.4) would reveal that for a given value of d, H_3 would be maximum when $l \approx d$ and $r \approx 0$. In other words, at a given distance, the axial field will be larger for a long cylinder than for a disc of same volume and intensity of magnetisation. Figure 1 shows a plot of the axial field H_3 as a function of d/r for cylinders of different l/r ratio though of same volume ($= 16\,\pi$) and same intensity of magnetisation I_3 ($= 1$) in C.G.S. units. For comparison sake, the plot of H_{D3} is shown side by side for each case with a dashed curve. It would be seen that a comparatively closer agreement between H_3 and H_{D3} is obtained for a cylinder with the l/r ratio $= 1$. Thus, in this case the assumption of a dipole field for that produced by the cylinder is fairly valid even at closer distances say upto $d/r = 1.95$, where the relative deviation between H_3 and H_{D3} does not exceed 4%. Allowing for the same deviation of 4%, the corresponding lower distance limits are considerably larger in other cases as shown by the square mark on the curves. As expected, at large values of d/r say > 3.5, the agreement between H_3 and H_{D3} tends to be fairly close in all the three cases thereby testifying that the effect of relative dimensions of the specimen is practically insignificant at large distances.

4. Radial Field of the Cylinder

For calculating the radial field of a cylinder of length $2l$ and diameter $2r$ at an outside point P distant d from the centre of the cylinder in a radial direction (see Figure 2), let us first consider only the lower half cylinder of length l with a uniform magnetisation in a radial direction parallel to OP. In this case, it is convenient to choose the right handed system of coordinates so that the origin O coincides with the centre of the upper face of the lower half cylinder and the axis x_3 is coincident with the axis of the cylinder.

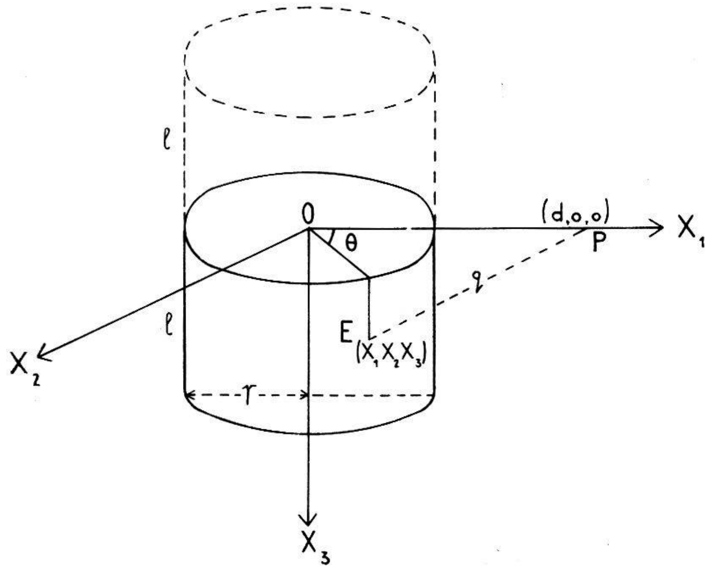

Figure 2

The magnetic field produced at the point P due to a small volume element at point E (x_1, x_2, x_3) distant q from E, is given by the well known formula

$$dH = I \operatorname{grad}^2 \left(\frac{1}{q}\right) dx_1\, dx_2\, dx_3 \,, \tag{4.1}$$

where I is the intensity of magnetisation.

Assuming a uniform magnetisation of the cylinder in a radial direction x_1, the radial field H_1 at point P is obtained by integrating (4.1) over the volume of the cylinder, i.e.

$$H_1 = I_1 \int\limits_V \frac{\partial}{\partial x_1} \left(\frac{d - x_1}{q^3}\right) dV \,. \tag{4.2}$$

By application of the Gauss theorem, the volume integral could be converted into a surface integral and so (4.2) becomes

$$H_1 = I_1 \int\limits_S \frac{d - x_1}{q^3} \cos (n\, x_1)\, dS \,, \tag{4.3}$$

where n is the outer normal and dS the surface element.

Now for both the upper and lower faces of the cylinder in Figure 2, $\cos (n\, x_1) = 0$ and for the curved surface of the cylinder $\cos (n\, x_1) = \cos\theta$. Further, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $dS = r\, dx_3\, d\theta$ and $q = \sqrt{(x_1 - d)^2 + x_2^2 + x_3^2}$.

On making these substitutions, equation (4.3) becomes

$$H_1 = I_1 \, r \int_{-\pi}^{\pi} \int_{0}^{l} \frac{(d - r \cos\theta) \cos\theta}{(x_3^2 + r^2 + d^2 - 2\, r\, d \cos\theta)^{3/2}} \, d\theta \, dx_3 \, . \tag{4.4}$$

Integrating (4.4) with respect to x_3 yields

$$H_1 = 2\, r\, l\, I_1\, w \tag{4.5}$$

where

$$w = \int_{0}^{\pi} \frac{(d - r \cos\theta) \cos\theta \, d\theta}{(r^2 + d^2 - 2\, r\, d \cos\theta) \sqrt{l^2 + r^2 + d^2 - 2\, r\, d \cos\theta}} \, . \tag{4.6a}$$

Evaluation of the above integral requires w being put in a more suitable form as below

$$w = \frac{1}{4\, d^2\, r} \int_{0}^{\pi} \left[(r^2 - d^2 + 2\, r\, d \cos\theta) + \frac{(d^4 - r^4)}{(r^2 + d^2 - 2\, r\, d \cos\theta)} \right] \frac{d\theta}{\sqrt{l^2 + r^2 + d^2 - 2\, r\, d \cos\theta}} \, . \tag{4.6b}$$

By substituting $(\pi - 2\,\phi)$ for θ and changing the limits (4.6b) could be put in the form of elliptic integrals as shown below

$$w = \frac{1}{2\, d^2\, r \sqrt{l^2 + (r + d)^2}} \left[\{ 2\, r^2 - (r + d)^2 \} F_1 + 4\, r\, d\, F_2 + \frac{d^4 - r^4}{(r + d)^2} F_3 \right] , \tag{4.6c}$$

where F_1, F_2, and F_3 are complete elliptic integrals. In terms of parameter k, $\pi/2$, and μ they could be written as

$$F_1 \left(k, \frac{\pi}{2} \right) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \, , \tag{4.7a}$$

$$F_2 \left(k, \frac{\pi}{2} \right) = \int_{0}^{\pi/2} \frac{\sin^2\phi \, d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \, , \tag{4.7b}$$

$$F_3 \left(k, \frac{\pi}{2}, \mu \right) = \int_{0}^{\pi/2} \frac{d\phi}{(1 - \mu^2 \sin^2\phi) \sqrt{1 - k^2 \sin^2\phi}} \, , \tag{4.7c}$$

where

$$\mu^2 = \frac{4\, r\, d}{(r + d)^2} \text{ and } k^2 = \frac{4\, r\, d}{l^2 + (r + d)^2} \, . \tag{4.7d}$$

Thus the problem of calculating H_1 is reduced to the evaluation of the above integrals. F_1 and F_2 are easy to evaluate in terms of complete elliptic integrals of the first and second kind for which tables exist in standard works on elliptic integrals. F_3, which is an elliptic integral of third kind could be reduced to the following form by making use of Heuman's lambda function[3])

$$F_3 \left(k, \frac{\pi}{2}, \mu \right) = F_1 \left(k, \frac{\pi}{2} \right) + \frac{\pi\, \mu\, [1 - \Lambda_0(\beta, k)]}{2 \sqrt{(\mu^2 - k^2)(1 - \mu^2)}} \, , \tag{4.8}$$

where

$$\beta = \sin^{-1} \frac{\sqrt{1-\mu^2}}{\sqrt{1-k^2}} .$$

The function $\Lambda_0(\beta, k)$ has been tabulated by HEUMAN[4]) in terms of parameter β and k for facilitating numerical calculations of F_3.

It is easy to see from Figure 2 that the radial field at point P due to another cylinder of length l and radius r placed in position as shown by dashed lines would be same as that given by equation (4.5). Thus, the radial field at point P due to the whole cylinder of length $2\,l$ and radius r would be just double of that calculated for the half cylinder, i.e.

$$H_1 = 4\, r\, l\, I_1\, w . \tag{4.9}$$

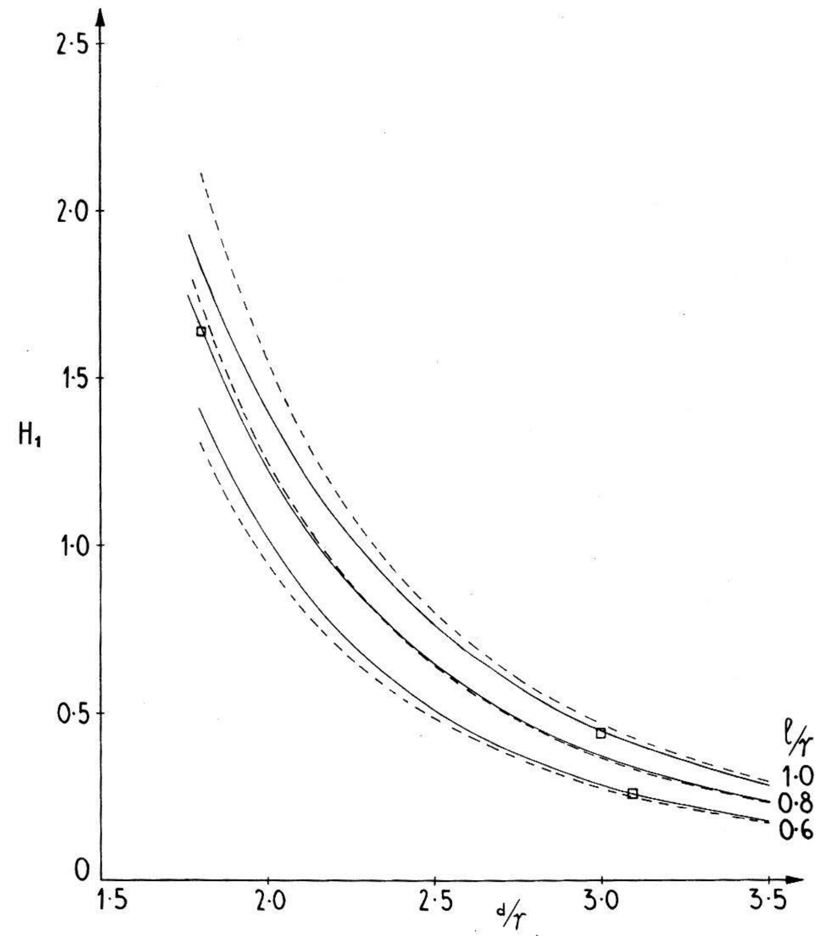

Figure 3

Figure 3 shows a plot of H_1 as a function of d/r for cylinders of same volume $(= 16\,\pi)$ and same intensity of magnetisation I_1 $(= 1)$ in C.G.S. units, but possessing different l/r ratio. Again for comparison sake, a plot of the corresponding dipole field H_{D1} $(= 2\, I_1\, \pi\, r^2\, 2\, l/d^3)$ assumed for the cylinder is shown side by side in dashed curve. Contrary to the case of axial field, it would be seen here that the radial field is larger for a broad cylinder than for a long cylinder. Here again at large values of d/r say > 3.5, there is a fairly good agreement between H_1 and H_{D1} values for all the three given cases. However, at smaller distances, a comparatively better agreement is obtained for the cylinder with the l/r ratio $= 0.8$. In this case, therefore, the assumption of a dipole field for the radial field of the cylinder is fairly valid even at smaller distances as close as $d/r = 1.8$, where the deviation of H_1 from H_{D1} does not exceed 4%. In other cases ($l/r = 0.6$, 1) the corresponding lower limit of distance allowing for the same deviation of 4%, is comparatively much higher as shown by the square mark on the curves.

5. Discussion and Conclusions

The curves in Figure 1 and Figure 3, although drawn for cylinders of a specified volume, could be applied to cylinders of other volumes as well. For it is easy to see that for a specified ratio l/r, the field of a cylinder of length $2\,l\,n$ and diameter $2\,r\,n$ at a distance $n\,d$ would be same as that of a cylinder of length $2\,l$ and diameter $2\,r$ at distance d. This could be verified by calculations from the equations (3.4), (3.5), and (4.9). It is, therefore, possible to draw some general conclusions from these curves. Assuming the axial and radial magnetisations to be of same strength, the axial field is greater than the radial field for a long cylinder, where as the radial field is greater than the axial field for a broad cylinder; thus, one increases at the expense of the other by variation of the ratio l/r. Further, so far as the field in axial direction is concerned, a cylinder with l/r ratio ≈ 1 would behave almost as a dipole, where as for the radial field a cylinder with l/r ratio ≈ 0.8 would approximate closely to a dipole even at distances comparable to the dimensions of the cylinder.

Taking both of these factors into consideration, it could be concluded that for a fair validity of the representation of a uniformly magnetised cylinder by a dipole (in terms of axial and radial fields), the ratio l/r should have an optimum value given by

$$0.8 < \frac{l}{r} < 1 \tag{5.1}$$

So, in practice where the cylindrical specimen is measured in the first Gauss position, in turn for both the axial and radial components of magnetisation, it is preferable to use a specimen of relative dimensions given by the equation (5.1). Since, besides producing optimum fields in both directions, it would make the dipole representation of the specimen also fairly reasonable. It is clear from both the figures (1 and 3) that the use of long cylinders or of thin discs would lead to a considerable reduction in the magnitude of one of the field components besides involving larger deviations from the dipole fields assumed for the cylinder in either case.

Acknowledgement

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References

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