Branching rules and Clebsch-Gordan series of semi-simple Lie algebras

Branching Rules and Clebsch-Gordan Series of Semi-Simple Lie Algebras

by **N. Straumann**
CERN – Geneva

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Abstract. We derive explicit formulae (branching rules) for the decomposition of an irreducible representation of a semi-simple Lie algebra L relative to semi-simple subalgebras L'. These formulae are valid for a large class of subalgebras L'. As an special case, we obtain an explicit formula for the Clebsch-Gordan series of semi-simple Lie algebras.

Introduction

Since the success of the octet model of Gell-Mann and Ne'eman (SU_3/Z_3) higher symmetry groups have come to play an increasingly important role in particle physics. Like the isospin invariance these higher symmetries are broken symmetries. This means that only some dominant part of the strong interactions is invariant under such a higher symmetry group while some weaker part is not. (In the case of isospin the electromagnetic interaction breaks the isospin symmetry but still leaves invariant the subgroup U_1 of "rotations around the third axis" in isospin space.)

Let $H = H_0 + H'$ be the corresponding splitting of the Hamilton operator, where H_0 is invariant with respect to some higher symmetry group G. This means that there exists a unitary representation of G in the Hilbert space of states which commutes with H_0. The Hilbert space can now be decomposed into a direct sum of irreducible (finite dimensional for compact G) subspaces in which the energy H_0 is constant (supermultiplets). The symmetry breaking interaction H' which is only invariant with respect to a subgroup $G' \subset G$ splits the supermultiplets up into multiplets of G' corresponding to the decomposition into irreducible constituents for G'. For instance in the octet model the spin $^1/_2$ octet of baryons splits under the medium strong interactions into the charge multiplets N, Λ, Σ, Ξ.

This is typical of broken symmetry schemes: one always has to decompose the irreducible representations of some higher symmetry group into irreducible constituents with respect to some subgroup. It is this general problem which we intend to study in this paper. More precisely we shall discuss the following question.

Given an irreducible module $\mathfrak{M}_\Lambda$ of a semi-simple Lie algebra $\mathfrak{L}$ (corresponding to a higher symmetry group) with the highest weight Λ and a semi-simple subalgebra $\mathfrak{L}' \subset \mathfrak{L}$. $\mathfrak{M}_\Lambda$, considered as a module for $\mathfrak{L}'$, is completely reducible. Let $\mathfrak{M}_\Lambda = \bigoplus_\lambda m_\lambda \mathfrak{N}_\lambda$ be the corresponding decomposition into irreducible constituents with the highest weights λ of $\mathfrak{L}'$ and multiplicities m_λ. One is then interested in m_λ.

In some special cases the solution of this 'branching problem' is well known[1]).

In this paper we shall derive an explicit formula for m_λ, which is valid for arbitrary semi-simple $\mathfrak{L}$ and a large class of subalgebras $\mathfrak{L}'$. As a special case we obtain a formula for the Clebsch-Gordan series of semi-simple Lie algebras. In Section I we repeat briefly the theory of characters for Lie algebras. This is useful, we feel, since there exists a highly developed purely algebraic theory of characters which is possibly not as well-known among physicists as it deserves to be. With the help of this theory we then solve in Section II the 'branching problem', and illustrate our result by several simple examples. In Section III the Clebsch-Gordan series of semi-simple Lie algebras is discussed and illustrated by examples.

The main interest of our formulae lies in their general validity. For 'big' algebras and large dimensional representations the numerical evaluation becomes, however, tedious. This happens to be also the case for STEINBERG's formula[2]) for the Clebsch-Gordan series of semi-simple Lie algebras or, for instance, for KONSTANT's[8]) formula for the weights and their multiplicities.

To obtain more useful formulae it seems to be necessary to restrict oneself to rather special situations (compare [3])[4]). Nevertheless, it is of considerable interest to examine the general situation which we shall study in this paper.

I. Theory of Characters for Lie Algebras

In this paragraph we give those parts of the theory of characters for Lie algebras which we need for the solution of the 'branching problem'. For details and proofs we refer to references[5])[6]).

To fix the notation we repeat first some well-known concepts and theorems.

Let $\mathfrak{L}$ be a semi-simple Lie algebra over a field ϕ of characteristic 0. If the field is not algebraically closed, we require in the following always that $\mathfrak{L}$ is 'split'. This means that $\mathfrak{L}$ has a Cartan sub-algebra $\mathfrak{H}$ with the property that the characteristic roots of $ad(h)$ for all $h \in \mathfrak{H}$ are in ϕ ($ad(h)$ is the linear transformation of $\mathfrak{L}$, which transforms every $x \in \mathfrak{L}$ into the vector $[h\,x]$). In $\mathfrak{H}$ we introduce a canonical basis. To do this let $\pi = (\alpha_1, \dots, \alpha_l)$ be a simple system of roots. The characteristic property of π is that every root $\alpha = \Sigma\, k_i\, \alpha_i$, $\alpha_i \in \pi$, where the k_i are all either non-negative or non-positive integers. Because the restriction of the Killing form $(x, y) = \mathrm{Tr}\,(adx \cdot a\,dy)$ on $\mathfrak{H}$ is non-degenerate, we can associate with the linear function $\alpha_i \in \mathfrak{H}^*$ ($\mathfrak{H}^*$ = dual space of $\mathfrak{H}$) a unique vector $h_{\alpha_i} \in \mathfrak{H}$ by $\alpha_i(h) = (h_{\alpha_i}, h)$ for all $h \in \mathfrak{H}$. The canonical base elements of $\mathfrak{H}$ are now defined as $h_i = 2\, h_{\alpha_i}/(\alpha_i, \alpha_i)$ [(α, β) is the bilinear form in $\mathfrak{H}^*$ which is induced by the Killing form].

One calls $M \in \mathfrak{H}^*$ an integral function on $\mathfrak{H}$ if the $M(h_i)$, $i = 1, \dots, l$, are integers. Let J denote the set of integral linear functions. A linear function $\Lambda \in J$, with $\Lambda(h_i)$ non-negative integers, is called dominant. The integral linear functions form a lattice with the fundamental dominant weights, defined by the property $\lambda_i(h_j) = \delta_{ij}$, as a basis. There is a 1:1 correspondence between the isomorphism classes of finite dimensional irreducible modules for $\mathfrak{L}$ and the set J_+ of dominant linear functions in which to corresponds an irreducible module with the highest weight Λ.

The set J is of course an Abelian additive group. We introduce now a multiplicative group G with the elements $\{e(M) \mid M \in J\}$ in 1:1 correspondence with J and the every $\Lambda \in J_+$ multiplication table

$$e(M)\, e(M') = e\,(M + M')\,. \tag{1}$$

Let A be the group algebra over ϕ of G. If A_+ is the sub-algebra of A which is generated by the elements $e(M)$, $M \in J_+$, then A_+ is, as one can easily see, the polynomial algebra $\phi[x_1, \dots, x_l]$ of the $x_i = e(\lambda_i)$. Hence A_+ is an integral domain. The algebra A is the quotient algebra of A_+ relative to the multiplicative system of the $e(M)$, $M \in J_+$. It follows from this that A is a commutative integral domain. With each element S in the Weyl group W we associate the automorphism in A such that $S\, e(M) = e(S\, M)$.

Now let $\mathfrak{M}$ be a finite dimensional module for $\mathfrak{L}$. If $M \in J$ we define the multiplicity n_M of M in $\mathfrak{M}$ to be 0 if M is not a weight and, if it is a weight then we define $n_M = \dim \mathfrak{M}_M$, where $\mathfrak{M}_M$ is the weight space in $\mathfrak{M}$ corresponding to the weight M. The character χ of $\mathfrak{M}$ is now defined to be

$$\chi = \sum_{M \in J} n_M\, e(M)\,. \tag{2}$$

One calls a character of a finite dimensional irreducible representation of $\mathfrak{L}$ a primitive character. Such a character has the form ($n_\Lambda = 1$)

$$\chi_\Lambda = e(\Lambda) + \sum_{M < \Lambda} n_M\, e(M)\,, \tag{3}$$

where Λ is the highest weight and the summation is taken over $M < \Lambda$ in the usual ordering*).

Because the $e(M)$ form a basis of A it follows from (3) that $\chi_{\Lambda_1} = \chi_{\Lambda_2}$ if and only if $\Lambda_1 = \Lambda_2$, i.e., $\mathfrak{M}_{\Lambda_1}$ and $\mathfrak{M}_{\Lambda_2}$ are isomorphic. It is also clear from (3) that distinct primitive characters are linearly independent. Hence, the decomposition of a character χ of a finite dimensional module $\mathfrak{M}$ into primitive characters

$$\chi = \sum_i m_{\Lambda_i}\, \chi_{\Lambda_i} \tag{4}$$

is unique. This decomposition gives therefore the irreducible constituents of $\mathfrak{M}$ together with their multiplicities.

$$\mathfrak{M} = \bigoplus_i m_{\Lambda_i}\, \mathfrak{M}_{\Lambda_i}\,.$$

The following theorem gives an explicit formula for a primitive character.

Weyl's Theorem

Let $\mathfrak{M}_\Lambda$ be the irreducible module for $\mathfrak{L}$ with highest weight Λ. Then the character χ_Λ of $\mathfrak{L}$ in $\mathfrak{M}_\Lambda$ is given by the formula

$$\chi_\Lambda = \frac{\sum_{S \in W} \det S\, e\,(S(\Lambda + \delta))}{\sum_{S \in W} \det S\, e(S\delta)}\,, \tag{5}$$

where $\delta = 1/2 \sum_{\alpha > 0} \alpha$, α a root.

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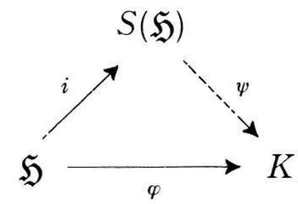
*) In the subspace $\mathfrak{H}_0^* \subset \mathfrak{H}^*$ over the rationals $\mathfrak{Q}$ and the basis $\alpha_1, \dots \alpha_l$, $\alpha_i \in \pi$, we define the ordering: $M = \Sigma k_i \alpha_i > 0$ if $k_1 = \cdots = k_h = 0$, $k_{h+1} > 0$, $h < l$. $M_1 > M_2$ if $M_1 - M_2 > 0$. The simple roots then cannot be written as sums of positive roots. The restriction of the bilinear form (α, β) on $\mathfrak{H}_0^*$ is positive definite.

We come now to another point which is important for the solution of the branching problem. A Lie module $\mathfrak{M}$ for $\mathfrak{L}$ can always be considered in one and only one way as a (left) module for the associative universal enveloping algebra $U(\mathfrak{L})$ (compare Appendix A). Let us denote with $\varrho(x)$ the linear transformation in $\mathfrak{M}$ which is associated to $x \in U(\mathfrak{L})$. We want to calculate $\operatorname{Tr} \varrho(h)$ for $h \in U(\mathfrak{H})$ *). Since $\mathfrak{H}$ is commutative, $U(\mathfrak{H})$ is the quotient algebra of the tensor algebra $T(\mathfrak{L})$ relative to the ideal which is generated by the elements $x \otimes y - y \otimes x$, $x, y \in \mathfrak{H}$ (compare Appendix A), i.e., $U(\mathfrak{H})$ is the symmetric algebra $S(\mathfrak{H})$. This algebra has the following universal property **): if φ is any linear mapping of $\mathfrak{H}$ (considered as a vector space) into a commutative associative algebra K, then there exists a unique homomorphism χ of $S(\mathfrak{H})$ into K such that the following diagram is commutative (i is the natural injection):



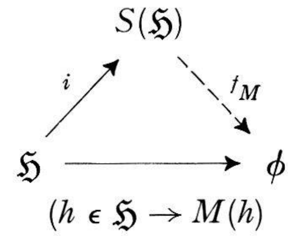
In a weight space

$$\mathfrak{M}_M = \{x \in \mathfrak{M} \mid \varrho(h)\, x = M(h)\, x\,; \quad h \in \mathfrak{H}\}$$

of a module $\mathfrak{M}$, $\varrho(h)$ for $h \in S(\mathfrak{H})$ is a multiple of the unit transformation

$$\varrho(h)\, x = f_M(h)\, x\,; \quad x \in \mathfrak{M}_M\,, \ h \in S(\mathfrak{H})$$

f_M is the uniquely existing extension of M (replace in the above diagram K by the field ϕ).



Hence

$$\operatorname{Tr} \varrho(h) = \sum_{M \in J} n_M f_M(h)\,. \tag{6}$$

f_M is an element of the dual space $S^*(\mathfrak{H})$ of $S(\mathfrak{H})$. We next make some remarks concerning this space.

$S(\mathfrak{H})$ is a graded algebra since this is the case for the tensor algebra and $S(\mathfrak{H})$ is the quotient algebra of the tensor algebra with a homogeneous ideal ***). In particular, $S(\mathfrak{H})$ is a direct sum $S(\mathfrak{H}) = \bigoplus_{p=0}^{\infty} S^p(\mathfrak{H})$ of vector spaces $S^p(\mathfrak{H})$. A direct sum is universal in the following sense: let there be given a family of θ_p of linear mappings of $S^p(\mathfrak{H})$ in a

*) The universal enveloping algebra of $\mathfrak{H}$ is the subalgebra in $U(\mathfrak{L})$ which is generated by $\mathfrak{H}$.

**) In this paper we use several times 'universal definitions' for algebraic objects. The reader, who is not familiar with this notion is referred to [7]).

***) An algebra A is graded if $A = \bigoplus_i A_i$ where A_i is a subspace $A_i A_j \subseteq A_{i+j}$. An ideal B is homogeneous if $B = \bigoplus_i B \cap A_i$.

vector space $\mathfrak{N}$ (over ϕ). Then there exists a unique linear mapping θ of $S(\mathfrak{H})$ into $\mathfrak{N}$ such that the following diagram is commutative (τ_p = natural injection):

$$\begin{array}{ccc} \{ S^p(\mathfrak{H}) \} & \xrightarrow{\tau_p} & S(\mathfrak{H}) \\ \| & & \downarrow \theta \\ \{ S^p(\mathfrak{H}) \} & \xrightarrow{\theta_p} & \mathfrak{N} \end{array}$$

Therefore, $S^*(\mathfrak{H}) = \operatorname{Hom}(S(\mathfrak{H}), \phi)$ is the direct product of the $S^{p*}(\mathfrak{H})$

$$S^*(\mathfrak{H}) = \prod_{p=0}^{\infty} S^{p*}(\mathfrak{H}) .$$

Thus, let $\varphi \in S^*(\mathfrak{H})$. By restricting φ to $S^p(\mathfrak{H})$ we can associate to every φ a family $\varphi_p \in S^{p*}(\mathfrak{H})$ and conversely there corresponds, according to the diagram (if one replaces $\mathfrak{N}$ by ϕ), uniquely to every family φ_p a $\varphi \in S^*(\mathfrak{H})$ such that

$$\varphi(h) = \sum_{p=0}^{\infty} \varphi_p(h_p) ; \quad h = \sum h_p, \quad h_p \in S^p(\mathfrak{H}) .$$

We now introduce a bilinear function over $S^p(\mathfrak{H})$ and $S^p(\mathfrak{H}^*)$. To do this we denote the elements of $\mathfrak{H}^*$ as h^* and write $\langle h, h^* \rangle$ for the value of the linear function h^* on h. $\langle h, h^* \rangle$ is then a non-degenerate bilinear form. Now, we define the following bilinear form over $S^p(\mathfrak{H})$ and $S^p(\mathfrak{H}^*)$

$$\left\langle h_1 h_2 \ldots h_p, h_1^* \ldots h_p^* \right\rangle = \sum_{\sigma \in \mathfrak{S}_p} \prod_{i=1}^{p} \left\langle h_i, h_{\sigma(i)}^* \right\rangle \quad h_i \in \mathfrak{H}, \; h_j^* \in \mathfrak{H}^* \tag{7}$$

The sum extends over the symmetric group $\mathfrak{S}_p$. It is not difficult to show that this bilinear form is non-degenerate. Hence $S^p(\mathfrak{H})$ and $S^p(\mathfrak{H}^*)$ are dual relative to this bilinear form and we have

$$S^*(\mathfrak{H}) \cong \prod_{p=0}^{\infty} S^p(\mathfrak{H}^*) \tag{8}$$

Explicitly (8) means: between the elements $\varphi \in S^*(\mathfrak{H})$ and the sequences $\{h_p^*\}$, $h_p^* \in S^p(\mathfrak{H}^*)$ there is a 1:1 correspondence such that

$$\varphi(h) = \sum_p \left\langle h_p, h_p^* \right\rangle ; \; h = \sum h_p, \; h_p \in S^p(\mathfrak{H})$$

with the scalar product (7). From the definition (7) follows immediately the following formula

$$\left\langle h_1 \ldots h_p, (h^*)^p \right\rangle = p! \prod_{i=1}^{p} \left\langle h_i, h^* \right\rangle . \tag{9}$$

With the help of this formula it is not difficult to show that the earlier defined f_M becomes

$$f_M(h) = \sum_{p=0}^{\infty} \left\langle h_p, (p!)^{-1} M^p \right\rangle . \tag{10}$$

Now we put

$$\prod_{p=0}^{\infty} S^p(\mathfrak{H}^*) = \overline{S}(\mathfrak{H}^*)$$

and write instead of infinite sequences formal series. Thus, the elements of $\overline{S}(\mathfrak{H}^*)$ are expressions $\Sigma\, a_p$, $a_p \in S^p(\mathfrak{H}^*)$, such that $\Sigma\, a_i = \Sigma\, b_i$ if and only if $a_i = b_i$, $i = 0, 1, \ldots$ With the following multiplication $(\Sigma\, a_p)\,(\Sigma\, b_p) = \Sigma\, c_p$, where

$$c_p = a_p\, b_0 + a_{p-1}\, b_1 + \cdots + a_0\, b_p\,.$$

$\overline{S}$ becomes an algebra. The subset of $\overline{S}$ of finite sums is a sub-algebra which we can identify with $S(\mathfrak{H}^*)$.

We define a valuation in $\overline{S}$ by setting

$$|0| = 0, \qquad |a| = z^{-p}$$

if in $a = \Sigma\, a_p$, a_p is the first non-vanishing element.

Then, we have the following properties:

i) $|a| \geqslant 0\,, \quad |a| = 0$ if and only if $a = 0$

ii) $|a\, b| \leqslant |a|\,|b|$

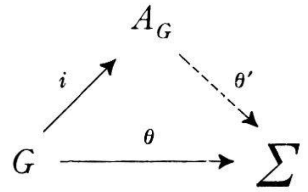
iii) $|a + b| \leqslant \max\,(|a|, |b|)\,.$

This valuation makes $\overline{S}$ a topological algebra. The non-archimedean property iii) of the valuation implies the very simple criterion that $\Sigma\, x_i$ converges if $|x_i| \to 0$ for $i \to \infty$. $S(\mathfrak{H}^*)$ is dense in $\overline{S}(\mathfrak{H}^*)$. Especially, e^a is well defined if in $a = \Sigma\, a_p$, $a_0 = 0$. Furthermore, we have the property $e^a\, e^b = e^{a+b}$ *).

The formula (10) can now be written in the following way

$$f_M(h) = \langle h, e^M \rangle\,. \tag{11}$$

Now, we consider in $\overline{S}$ the sub-algebra $\overline{A}$ which is generated by e^M, $M \in J$. This algebra is now isomorphic to the earlier introduced group algebra A. This can easily be seen with the following universal characterization of the group algebra of a group G. The pair (A_G, i), where A_G is an associative algebra and i a (monoid) homomorphism of G into A_G is a group algebra of G if the following holds: if Σ is any algebra and θ a (monoid) homomorphism of G into Σ, then there exists a unique homomorphism θ' of A_G into Σ such that the following diagram is commutative



With (11) and (6) we have now the following important formula

$$\mathrm{Tr}\, \varrho(h) = \langle h, \chi \rangle\,. \tag{12}$$

*) It is evident that the transition of $S(\mathfrak{H}^*)$ to $\overline{S}(\mathfrak{H}^*)$ can be made for every graded algebra.

II. Decomposition of an Irreducible $\mathfrak{L}$-Module into Irreducible Constituents Relative to a Subalgebra

In this paragraph we solve the branching problem for a very general situation. We formulate the main result in the following.

Theorem 1 (*branching rule*)

Let $\mathfrak{L}$ be a split semi-simple Lie algebra over a field of characteristic 0, $\mathfrak{H}$ a Cartan subalgebra and π a simple system of roots of $\mathfrak{L}$. Let $\mathfrak{L}'$ be a split semi-simple subalgebra of $\mathfrak{L}$ with the property that there exists a Cartan subalgebra $\mathfrak{H}'$ of $\mathfrak{L}'$ which is contained in $\mathfrak{H}$. We furthermore assume that the restriction of every fundamental dominant weight λ_i of $\mathfrak{L}$ on $\mathfrak{H}'$ is in J' (J' = integral linear functions on $\mathfrak{H}'$). Let

$$\mathfrak{M}_\Lambda = \bigoplus_\lambda m_\lambda \mathfrak{N}_\lambda \tag{13}$$

be the decomposition of an irreducible module $\mathfrak{M}_\Lambda$ for $\mathfrak{L}$ with the highest weight Λ relative to $\mathfrak{L}'$. Then the multiplicities m_λ are given by the following formula ($2\,\delta'$ = sum of the positive roots of $\mathfrak{L}'$):

$$m_\lambda = \sum_{S' \in W'} \det S' \, N(\Lambda; \lambda + \delta' - S'\delta') \,. \tag{14}$$

The sum in (14) extends over the Weyl group W' of $\mathfrak{L}'$. $N(\Lambda; M')$, $M' \in J'$ is given by

$$N(\Lambda; M') = \sum_{M \mid \mathfrak{H}' = M'} n_M \tag{15}$$

where the sum extends over those weights of $\mathfrak{M}_\Lambda$ whose restrictions on $\mathfrak{H}'$ are M'.

Before we prove this theorem we make some *remarks*:

i) if one adds to the formulae (13), (14), and (15) Konstant's formula for n_M[8]) *)

$$n_M = \sum_{S \in W} \det S \, P\,[S(\Lambda + \delta) - (M + \delta)] \,, \tag{16}$$

then one has in (14) a general explicit formula for m_λ

$$m_\lambda = \sum_{\substack{S' \in W' \\ S \in W}} \det S' \det S \left(\sum_{M \mid \mathfrak{H}' = \lambda + \delta' - S'\delta'} P\,[S(\Lambda + \delta) - (M + \delta)] \right) . \tag{14'}$$

For $\mathfrak{L} = A_l$ one has in the Gelfand diagrams[9]) a very useful tool to determine n_M (compare Section III). Unfortunately, we were not able to derive more compact formulae for $N(\Lambda; M')$. This may very well be possible, at least for special situations. This is one reason why the evaluation of our formulae becomes tedious for 'big' algebras and large dimensional representations.

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*) $\delta = 1/2 \sum_{\alpha > 0} \alpha$. $P[M]$ is the number of solutions of $\sum_{\alpha > 0} k_\alpha \alpha = M$, where the k_α are non-negative integers. From this definition follows that $P[M]$ is different from zero only if $M \in J$.

ii) the assumptions which we made in theorem 1 are not very explicit. In the following theorem we get an explicit description of a large class of $\mathfrak{L}'$ which satisfies the assumptions of theorem 1.

Theorem 2

Let $\mathfrak{L}$ be split semi-simple, $\pi = (\alpha_1, \dots, \alpha_l)$ a simple system of roots, h_i, e_i, f_i, $i = 1, 2, \dots, l$ the associated canonical generators of $\mathfrak{L}$*) and R a subset of $(1, 2, \dots, l)$ with the property that the matrix $(A^R)_{ij} \overset{\text{Def.}}{=} A_{ij}$, $i, j \in R$, is the Cartan matrix of a split semi-simple Lie algebra. Let $\mathfrak{L}^R$ be the subalgebra of $\mathfrak{L}$ which is generated by $h_j, e_j, f_j, j \in R$. Then $\mathfrak{L}^R$ is split semi-simple with the Cartan matrix A^R and h_j, e_j, f_j, $j \in R$ as canonical generators.

The somewhat involved proof of this theorem is given in Appendix B.

From theorem 2 we get the following sufficient *criterion*.

With the notations of theorem 2, $\mathfrak{L}^R$ is a subalgebra of $\mathfrak{L}$ which satisfies the assumptions of theorem 1.

The condition of theorem 2 that A^R is the Cartan matrix of a split semi-simple Lie algebra can easily be checked. We give an *algebraic* and a *graphical criterion*.

a) A^R is the Cartan matrix of a split semi-simple Lie algebra if $\det A^R \neq 0$.

Proof: A matrix A_{ij}, $i, j = 1, \dots, l$ is the Cartan matrix of a split semi-simple Lie algebra if the following conditions are satisfied (compare Ref. [10])).

i) $A_{ii} = 2$, $A_{ij} \leqslant 0$ for $i \neq j$; $A_{ij} = 0$ implies $A_{ji} = 0$.

ii) $\det A \neq 0$.

iii) If $(\alpha_1, \dots, \alpha_l)$ is a basis for an l dimensional vector space $\mathfrak{H}_0^*$ over the rationals $\mathfrak{Q}$, then the group W generated by the l linear transformations S_{α_i} defined by

$$S_{\alpha_i} \alpha_j = \alpha_j - A_{ij} \alpha_i, \; j = 1, \dots l$$

is a finite group.

Now the way in which A^R is obtained from A implies that i) and iii) are alwaysfulfilled for A^R, which proves criterion a).

b) The Cartan matrices are in 1:1 correspondence to the Dynkin diagrams. From the Dynkin diagram for $\mathfrak{L}$, choose those points which are labelled with numbers from R. Furthermore, draw only those lines which connect directly these points R in the original diagram. If in the resulting diagram the connected parts are Dynkin diagrams of simple Lie algebras, then A^R is the Cartan matrix of the semi-simple Lie algebra $\mathfrak{L}^R$. $\mathfrak{L}^R$ is then the direct sum of the simple Lie algebras corresponding to the connected parts.

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*) $e_i = e_{\alpha_i}, f_i = 2\, e_{-\alpha_i}/(\alpha_i, \alpha_i)$, where e_α is an element of the root space of $\mathfrak{L}$ corresponding to the root α. $e_{-\alpha}$ must be chosen such that $(e_\alpha, e_{-\alpha}) = -1$.

As an example consider the Lie algebra A_l of SU_{l+1} (Dynkin diagram: $\circ - \circ \ldots \circ - \circ$). Our criterion b) shows that every R is allowed. This can also be seen with criterion a) from the Cartan matrix which is

$$(A_{ij}) = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

$\mathfrak{L}^R$ is in this example a direct sum of certain $A_{l'}$. For $R = 1, 2, \ldots, l'), l' < l$ we have of course $\mathfrak{L}^R = A_{l'}$.

For applications we also mention that the currently interesting case $SU_m \times SU_n \subseteq SU_{mn}$ also satisfies the assumptions of theorem 1. It is, however, not contained in the class described by theorem 2.

Proof of theorem 1

Since by assumption $\mathfrak{H}' \subset \mathfrak{H}$, the universal enveloping algebra $U(\mathfrak{H}')$ is the sub-algebra in $U(\mathfrak{H})$ which is generated by $\mathfrak{H}'$.

The idea of the proof is now the following: we calculate $\operatorname{Tr} \varrho(h')$, $h' \in U(\mathfrak{H}')$ in two different ways. First, we get $\operatorname{Tr} \varrho(h')$ by restricting Equation (6) on the sub-algebra $U(\mathfrak{H}')$ and second by using the decomposition (13). In this way, we get an equation in $S^*(\mathfrak{H}')$ which contains the multiplicities m_λ. With the help of the isomorphism (8) we get an equation in $\bar{A}' \subset \bar{S}(\mathfrak{H}')$ and because of $\bar{A}' \cong A'$ finally an equation in the group algebra A'. WEYL's formula (5) gives the key to get from this an explicit expression for m_λ.

We now carry this out in detail. From (6) we get for $h' \in U(\mathfrak{H}')$

$$\operatorname{Tr} \varrho(h') = \sum_{M \in J} n_M f_M(h') = \sum_{M' \in J'} \sum_{M \mid \mathfrak{H}' = M'} n_M f_M(h') = \sum_{M' \in J'} f_{M'}(h') \sum_{M \mid \mathfrak{H}' = M'} n_M . \tag{17}$$

The second sum in (17) extends over those $M \in J$, whose restrictions on $\mathfrak{H}'$ coincide with $M' \in J'$. $f_{M'}$ is the unique extension on $U(\mathfrak{H}')$ of M' (compare the diagram above formula (6)).

With (11) and the isomorphism of $\bar{A}'$ and A' we get, using the definition (15),

$$\operatorname{Tr} \varrho(h') = \left\langle h', \sum_{M' \in J'} N(\Lambda; M')\, e(M') \right\rangle . \tag{18}$$

On the other hand, according to (13) and (12) we have

$$\operatorname{Tr} \varrho(h') = \left\langle h', \sum_\lambda m_\lambda \chi'_\lambda \right\rangle ,$$

where χ'_λ is the primitive character for the $\mathfrak{L}'$ module $\mathfrak{N}_\lambda$. Hence, we get the following equation in the group algebra A' of the group $\{e(M') \mid M' \in J'\}$:

$$\sum_{M' \in J'} N(\Lambda; M')\, e(M') = \sum m_\lambda \chi'_\lambda . \tag{19}$$

If we insert in (19) the Weyl formula (5) and multiply by the common denominator in (5) we get

$$\sum_{M' \in J'} N(\Lambda; M')\, e(M') \sum_{S' \in W'} \det S'\, e(S' \delta') = \sum_{\lambda} m_\lambda \sum_{S' \in W'} \det S'\, e\left(S'(\lambda + \delta')\right). \tag{20}$$

We first transform the right-hand side of Equation (20). The summation can be taken for all $\lambda \in J'$ if we define $m_\lambda = 0$ for non-dominant λ. Now we get

$$\begin{aligned}\sum_{M' \in J'} m_{M'} \sum_{S' \in W'} \det S'\, e\left(S'(M' + \delta')\right) &= \sum_{\lambda \in J'} \Bigg(\sum_{\substack{S' \in W' \\ M' \in J';\, S'(M'+\delta') = \lambda + \delta'}} \det S'\, m_{M'} \Bigg) e(\lambda + \delta') \\ &= \sum_{\lambda \in J'} \Bigg(\sum_{S' \in W'} \det S'\, m_{S'^{-1}(\lambda + \delta') - \delta'} \Bigg) e(\lambda + \delta') \\ &= \sum_{\lambda \in J'} \Bigg(\sum_{S' \in W'} \det S'\, m_{S'(\lambda + \delta') - \delta'} \Bigg) e(\lambda + \delta').\end{aligned}$$

For the left-hand side wie obtain, if we put $M' + S'\delta' = \lambda + \delta'$,

$$\begin{aligned}&\sum_{M' \in J'} N(\Lambda; M') \sum_{S' \in W'} \det S'\, e(M' + S'\delta') \\ &\qquad = \sum_{M' \in J'} \sum_{S' \in W'} \det S'\, N(\Lambda; \lambda + \delta' - S'\delta')\, e(\lambda + \delta').\end{aligned}$$

Since the $e(M')$ are linear independent, we get the following equation

$$\sum_{S' \in W'} \det S'\, N(\Lambda; \lambda + \delta' - S'\delta') = m_\lambda + \sum_{\substack{S' \in W' \\ S' \neq 1}} \det S'\, m_{S'(\lambda + \delta') - \delta'}. \tag{21}$$

The following lemma shows that the second term on the right in (21) vanishes for dominant λ. We thus get formula (14).

We still have to prove the following *lemma*.

For dominant λ and $S' \neq 1$, $S'(\lambda + \delta') - \delta'$ is not dominant.

Proof

From formula (22) below we get $\delta'(h_i) = 1$ for the canonical generators h_i in the Cartan algebra $\mathfrak{H}'$. Thus, for a reflection S_i' at the simple root $\alpha_i \in \pi'$ we have

$$(S_i'\delta', \alpha_i) = (\delta' - \alpha_i, \alpha_i) = (\delta', S_i'\alpha_i) = (\delta', -\alpha_i)$$

hence

$$\delta'(h_i) = \frac{2(\delta', \alpha_i)}{(\alpha_i, \alpha_i)} = 1.$$

If we now apply S_i' on $\lambda + \delta'$ we get

$$S_i'(\lambda + \delta') = \lambda + \delta' - (\lambda + \delta')(h_i)\, \alpha_i.$$

This equation applied to h_i gives $(\alpha_i(h_i) = 2)$

$$S_i'(\lambda + \delta')(h_i) = (\lambda + \delta')(h_i)\,[1 - \alpha_i(h_i)] < 0$$

for dominant λ. With this and $\delta'(h_i) = 1$ the assertion of the lemma follows.

For applications it is important to know the expressions $S'\delta'$, $S' \in W'$ in the argument of (14). Concerning this we make some remarks.

The Weyl group W' is generated by the reflections at the simple roots $\alpha_i \in \pi'$. Such a reflection is given by

$$S'_i \alpha_j = \alpha_j - A'_{ij} \alpha_i ,$$

where A'_{ij} is the Cartan matrix of $\mathfrak{L}'$

$$A'_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} ; \quad \alpha_i, \alpha_j \in \pi' .$$

We now calculate $S'_i \delta'$. Since $S'_i \alpha > 0$ for $\alpha > 0$, except for $\alpha = \alpha_i$[11]), where of course $S'_i \alpha_i = -\alpha_i$, it follows

$$S'_i \delta' = 1/2 \sum_{\substack{\alpha > 0 \\ \alpha \neq \alpha_i}} \alpha - 1/2\, \alpha_i = \delta' - \alpha_i . \tag{22}$$

Therefore, in concrete examples one only has to express the elements S' of the Weyl group as products of reflections S'_i at simple roots in order to calculate $S'\delta'$. For example, the Weyl group of the Lie algebra A_2 consists of the following elements:

$$W = \{1, S_1, S_2, S_1 S_2, S_1 S_2 S_1, (S_1 S_2)^2\} . \tag{23}$$

In order to illustrate our results for a non-trivial example, we take for $\mathfrak{L}$ the algebra A_l of the group $S\,U_{l+1}$ and for $\mathfrak{L}'$ the algebra A_2 with $R = (1, 2)$ (compare theorem 2). From (22) and (23) we obtain

$$\begin{aligned} m_\lambda = {} & N(\Lambda; \lambda) - N(\Lambda; \lambda + \alpha_1) - N(\Lambda; \lambda + \alpha_2) \\ & + N(\Lambda; \lambda + \alpha_1 + 2\alpha_2) - N(\Lambda; \lambda + 2\alpha_1 + 2\alpha_2) \\ & + N(\Lambda; \lambda + 2\alpha_1 + \alpha_2) . \end{aligned} \tag{24}$$

Let us take as an example for Λ the adjoint representation. In this case, the weights are the roots. These are

$$M = 0 \text{ with } n_M = l; \quad M = \alpha_i + \alpha_{i+1} + \ldots \alpha_j,\ 1 \leqq i \leqq j \leqq l \text{ with } n_M = 1 .$$

Except for $\lambda = [0, 0]$, only the first term contributes in (23). One easily obtains

$$m_{[0,0]} = (l-2)^2, \quad m_{[1,0]} = l - 2,\ m_{[0,1]} = l - 2, \quad m_{[1,1]} = 1 .$$

The sum of the dimensions is $l^2 + 2l$, which happens to be also the dimension of the adjoint representation.

III. Clebsch-Gordan Series of Semi-Simple Lie Algebras

Our branching rule (theorem 1) gives us also a formula for the Clebsch-Gordan series of semi-simple Lie algebras. Thus, let $\mathfrak{G}$ be a semi-simple Lie algebra (split always assumed) and $\mathfrak{M}_{\Lambda_1}$, $\mathfrak{M}_{\Lambda_2}$ two irreducible modules for $\mathfrak{G}$. If we associate to each element $(g_1, g_2) \in \mathfrak{L} \overset{\text{Def.}}{=} \mathfrak{G} \oplus \mathfrak{G}$ the transformation

$$(g_1, g_2)\,(x \otimes y) = g_1 x \otimes y + x \otimes g_2 y$$

in $\mathfrak{M}_{\Lambda_1} \otimes \mathfrak{M}_{\Lambda_2}$, then $\mathfrak{M}_{\Lambda_1} \otimes \mathfrak{M}_{\Lambda_2}$ becomes an irreducible $\mathfrak{L}$ module. The restriction of this representation to the sub-algebra $\mathfrak{L}'$, consisting of all pairs (g, g), is the tensor product representation of $\mathfrak{G}$. Since $\mathfrak{L}'$ is isomorphic to $\mathfrak{G}$, the decomposition of $\mathfrak{M}_{\Lambda_1} \otimes \mathfrak{M}_{\Lambda_2}$ relative to $\mathfrak{L}'$ gives the Clebsch-Gordan series for $\mathfrak{G}$.

With this remark it is easy to derive from (14) the following formula for the multiplicities m_Λ in

$$\mathfrak{M}_{\Lambda_1} \otimes \mathfrak{M}_{\Lambda_2} = \bigoplus_{\Lambda} m_\Lambda \mathfrak{M}_\Lambda . \tag{25}$$

One gets

$$m_\Lambda = \sum_{S \in W} \det S \; Q[(\Lambda_1, \Lambda_2); \Lambda + \delta - S\,\delta] , \tag{26}$$

where

$$Q[(\Lambda_1, \Lambda_2); M] = \sum_{M = M_1 + M_2} n^{(1)}_{M_1} n^{(2)}_{M_2} . \tag{27}$$

In (27) the $n^{(i)}_{M_i}$, $i = 1, 2$ are the dimensions of the weights M_i in $\mathfrak{M}_{\Lambda_1}$. The evaluation of this formula is in many cases simpler than for STEINBERG's formula[2]). For rank 2 algebras one can evaluate it graphically.

We illustrate formula (25) for a non-trivial example. For $\mathfrak{G}$ we take A_l, $l \geqslant 2$, and consider the tensor product of the adjoint representation with itself. The highest weight of the adjoint representation is $\lambda_1 + \lambda_l = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. According to a general theorem[2]) the highest weights in (25) are necessarily of the form

$$\Lambda = \Lambda_1 + \Lambda_2 - \sum_{j=1}^{l} n_j \, \alpha_j , \tag{28}$$

with non-negative integers n_j. Hence, in our case the possible Λ's are:

$$2\,(\lambda_1 + \lambda_l),\; 2\,(\lambda_1 + \lambda_l) - \alpha_1 = \lambda_2 + 2\,\lambda_l,\; 2\,(\lambda_1 + \lambda_l) - \alpha_l$$
$$= 2\,\lambda_1 + \lambda_{l-1},\; 2\,(\lambda_1 + \lambda_l) - (\alpha_1 + \alpha_l) = \lambda_2 + \lambda_{l-1},\; \lambda_1 + \lambda_l,\; 0 .$$

The multiplicity of $2\,(\lambda_1 + \lambda_l)$ is of course 1. For $\Lambda = 2\,(\lambda_1 + \lambda_l) - \alpha_1$ one gets

$$m_\Lambda = Q\left[\alpha_1 + 2\,\alpha_2 + 2\,\alpha_3 + \cdots + 2\,\alpha_l\right]$$
$$- Q\left[2\,\alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_l\right] = 1 .$$

The same multiplicity is obtained for $\Lambda = 2\,(\lambda_1 + \lambda_l) - \alpha_l$. For $\Lambda = 2\,(\lambda_1 + \lambda_l) - (\alpha_1 + \alpha_l)$ we get

$$m_\Lambda = Q\,[\alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_{l-1} + \alpha_l]$$
$$- Q\,[2\,\alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_{l-1} + \alpha_l]$$
$$- Q\,[\alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_l]$$
$$+ Q\,[2\,\alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_l]$$
$$= 4 - 2 - 2 + 1 = 1 .$$

The multiplicity of the adjoint representation is

$$m_\Lambda = Q\,[\alpha_1 + \cdots + \alpha_l] - \sum_{i=1}^{l} Q\,[\alpha_1 + \cdots + \alpha_{i-1} + 2\,\alpha_i + \alpha_{i+1} + \cdots + \alpha_l]$$
$$= [2\,l + 2\,(l-1)] - 4\,(l-1) = 2\,.$$

In the tensor calculus this corresponds to the two possible ways of forming simple contractions in a product of two tensors (with one upper and one lower index).

For the one-dimensional representation many terms contribute to (26). But its multiplicity is, of course, one (corresponding to complete contraction). Our result is therefore in a shorthand notation:

$$[\lambda_1 + \lambda_l] \otimes [\lambda_1 + \lambda_l] = [2\,(\lambda_1 + \lambda_l)] \oplus [\lambda_2 + 2\,\lambda_l]$$
$$\oplus\,[2\,\lambda_1 + \lambda_{l-1}] \oplus [\lambda_2 + \lambda_{l-1}] \oplus 2\,[\lambda_1 + \lambda_l] \oplus [0]\,. \tag{29}$$

(For $l = 2$ the fourth term drops out.) We also give the dimensions of the various terms in the right-hand side of (28).

$$\dim[2\,(\lambda_1 + \lambda_l)] = 1/4\; l\,(l+1)^2\,(l+4)$$
$$\dim[\lambda_2 + 2\,\lambda_l] = \dim\,[2\,\lambda_1 + \lambda_{l-1}] = 1/4\,(l-1)\,l\,(l+2)\,(l+3)$$
$$\dim[\lambda_2 + \lambda_{l-1}] = 1/4\,(l-2)\,(l+1)^2\,(l+2)$$
$$\dim[\lambda_1 + \lambda_l] = l\,(l+2)\,, \qquad \dim\,[0] = 1\,.$$

The comparison of the dimensions in (29) checks.

For A_l one has in the Gelfand diagrams[9]) a very useful tool to calculate $Q[(\Lambda', \Lambda''), M]$ in (27). We illustrate this for A_2. The Gelfand states are in 1:1 correspondence to the Gelfand diagrams.

$$(m) = \begin{pmatrix} m_{13} & & m_{23} & & 0 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix} \tag{30}$$

where $\Lambda = (m_{13} - m_{23}, m_{23})$ is the highest weight. The other number can take independently all values between the two numbers standing to the left and to the right in the next higher row; for instance, $m_{13} \leqslant m_{12} \leqslant m_{23}$. The eigenvalues of the canonical elements h_1, h_2 for a Gelfand state are

$$h_1 \to 2\,m_{11} - (m_{12} + m_{22})$$
$$h_2 \to 2\,(m_{12} + m_{22}) - (m_{11} + m_{13} + m_{23}) \tag{31}$$

If we put $M(h_i) = \mu_i$, then $Q[(\Lambda', \Lambda''), M]$ is the numbers of pairs of Gelfand diagrams (m'), (m'') who belong to Λ' and Λ'' with the property that

$$\mu_1 = (2\,m'_{11} - m'_{12} - m'_{22}) + (2\,m''_{11} - m''_{12} - m''_{22})$$
$$\mu_2 = 2\,(m'_{12} + m'_{22}) - (m'_{11} + m'_{13} + m'_{23}) + 2\,(m''_{12} + m''_{22}) - (m''_{11} + m''_{13} + m''_{23})\,.$$

If we put

$$N' = m'_{13} + m'_{23}; \qquad N'' = m''_{13} + m''_{23}$$

then $Q[(\Lambda', \Lambda''); M]$ *is the number of pairs*

$$\begin{pmatrix} m'_{12}, & m'_{22} \\ & m'_{11} & \end{pmatrix}, \quad \begin{pmatrix} m''_{12}, & m''_{22} \\ & m''_{11} & \end{pmatrix}$$

with

$$m'_{11} + m''_{11} = 1/3\,[2\,\mu_1 + \mu_2 + N' + N'']$$

$$(m'_{12} + m'_{22}) + (m''_{12} + m''_{22}) = 1/3\,[\mu_1 + 2\,(\mu_2 + N' + N'')]\,. \qquad (32)$$

In (32) the right-hand side is given for fixed Λ', Λ'' and M. In concrete examples Q can easily be evaluated. This example can immediately be generalized to A_l.

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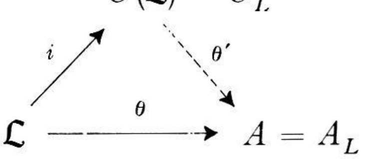
Appendix A

Since the universal enveloping algebra is an essential tool in this paper, we give here its definition and certain of its most important properties (compare Ref. [5]).

Definition

Let $\mathfrak{L}$ be a Lie algebra of arbitrary dimension and characteristic. A pair (U, i) where U is an associative algebra (with a unit element) and i a homomorphism of $\mathfrak{L}$ into U_L ($U_L = U$, considered as Lie algebra) is a universal enveloping algebra of $\mathfrak{L}$ if the following holds: if A is any algebra (with a unit element) and θ is a homomorphism of $\mathfrak{L}$ into A_L, then there exists a *unique* homomorphism θ' of U into A, such that the following diagram is commutative:



This 'universal' definition fixes $U(\mathfrak{L})$ up to an isomorphism, as one can easily see. Furthermore, $i(\mathfrak{L})$ generates the algebra $U(\mathfrak{L})$ (these two properties always hold for universal definitions). A universal enveloping algebra can be constructed as follows: take the tensor algebra $T(\mathfrak{L})$ over $\mathfrak{L}$ and consider the ideal J which is generated by the elements $J = x \otimes y - y \otimes x - [x\, y]$, $x, y \in \mathfrak{L}$. Let π be the canonical projection $\pi\colon T(\mathfrak{L}) \to T(\mathfrak{L})/J \overset{\text{Def.}}{=} U(\mathfrak{L})$. If i is the restriction of π to $\mathfrak{L}$, then the pair (U, i) is a universal enveloping algebra.

From the fundamental theorem of POINCARÉ, BIRKHOFF, WITT one can conclude that i is an injection.

We also remark that it follows from this theorem that the associated graded algebra $G\,U$ of the filtered algebra U (with the induced filtration of T) is a symmetric algebra over $\mathfrak{L}$, i.e., a polynomial algebra over a basis of $\mathfrak{L}$. Hence, this algebra is an integral domain, which is by the Hilbert basis theorem for finite dimensional $\mathfrak{L}$ also Noetherian. From these two conditions one can conclude quite generally that U has no zero divisors and satisfies the ascending chain condition for left and right ideals, i.e., U is left and right Noetherian.

Appendix B

In this appendix we prove theorem 2 of Section II.

We begin with a remark. According to [12]) one can represent every root β of $\mathfrak{L}$ as $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_m}$ for all $m \leqslant k$ is a root. To every root β we can associate in this way a sequence $(i_1, \ldots, i_k)$ and the elements*)

$$h_i, \quad [e_{i_1} \cdots e_{i_k}], \quad [f_{i_1} \cdots f_{i_k}] \tag{B.1}$$

form a basis of $\mathfrak{L}$.

We next give a transparent description of $\mathfrak{L}^R$ (instead of e_i, $i \in R$ we write e_i^R, etc.).

I. The elements h_i^R, $[e_{i_1}^R, \ldots, e_{i_k}^R]$, $[f_{i_1}^R, \ldots, f_{i_k}^R]$ of the basis (B.1) for $\mathfrak{L}$ form a basis for $\mathfrak{L}^R$.

Proof: We denote the vector space which is spanned by these elements as $\overline{\mathfrak{L}}^R$. We show that $\overline{\mathfrak{L}}^R$ is a sub-algebra from which I. follows.

i)
$$[h_i^R, h_j^R] = 0, \quad [[e_{i_1}^R \cdots e_{i_k}^R]\, h_j^R] = \sum_m A_{j\,i_m}^R\, [e_{i_1}^R \cdots e_{i_k}^R] \in \overline{\mathfrak{L}}^R$$
$$[[f_{i_1}^R \cdots f_{i_k}^R]\, h_j^R] = -\sum_m A_{j\,i_m}^R\, [f_{i_1}^R \cdots f_{i_k}^R] \in \overline{\mathfrak{L}}^R\,.$$

ii) Since
$$ad[e_{i_1} \cdots e_{i_k}] = [\ \cdots [ad\, e_{i_1}\; a\,d\, e_{i_2}] \cdots ad\, e_{i_k}]\,,$$
we have
$$[x[e_{i_1} \cdots e_{i_k}]] = x[\ \cdots [ad\, e_{i_1}\; ad\, e_{i_2}] \cdots]\,,$$
which shows that we have to apply a non-commutative polynomial in the $a\,d\, e_{i_m}$ on x. From this and a similar remark for the f's follows that it suffices to consider the following products.

$$[[e_{i_1}^R \cdots e_{i_k}^R]\, e_j^R], \quad [[f_{i_1}^R \cdots f_{i_k}^R]\, e_j^R]$$
$$[[e_{i_1}^R \cdots e_{i_k}^R]\, f_j^R], \quad [[f_{i_1}^R \cdots f_{i_k}^R]\, f_j^R].$$

*) Instead of $[\ldots [x_1\, x_2] \ldots x_r]$ we often write $[x_1, \ldots, x_r]$.

The argument is the same for the last two products as for the first two so we consider only the first two.

iii) For the product $[[e^R_{i_1}, \dots, e^R_{i_k}]\, e^R_j]$ we have to distinguish the following two cases:

a) $\beta + \alpha_j$ ($\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$) is no root. Then the product is zero.

b) $\beta + \alpha_j$ is a root. In this case the product is a multiple of the e-basis element which belongs to $\gamma = \beta + \alpha_j$; i.e., the product is an element of $\bar{\mathfrak{L}}^R$.

iv) We consider now the product

$$[[f^R_{i_1} \cdots f^R_{i_k}]\, e^R_j] \,.$$

For $k = 1$ we have $[\dots] = 0$ for $i_1 \neq j$ and $[\dots] = -h_j$ for $i_1 = j$.
For $k = 2$ we have $[\dots] = 0$ if $i_1 \neq j$ and $i_2 \neq j$
For $i_2 = j$ we have $[[f^R_{i_1} f^R_{i_2}]\, e^R_j] = [f^R_{i_1} [f^R_j\, e^R_j]] + [[f^R_{i_1}\, e^R_j] f^R_j] = -[f^R_{i_1}\, h^R_j] = A^R_{j i_1} f^R_{i_1}$.
For $k \geqslant 2$ we prove by induction that: $[[f^R_{i_1}, \dots, f^R_{i_k}]\, e^R_j]$ is a linear combination of the f^R base elements. If no $i_r = j$, this product is zero. In the other case let i_{r+1} be the last index in $[f^R_{i_1}, \dots, f^R_{i_k}]$ which equals j. Then

$$\begin{aligned}[][[f^R_{i_1} \cdots f^R_{i_r} f^R_j \cdots f^R_{i_k}]\, e^R_j] &= [[[[f^R_{i_1} \cdots f^R_{i_r} f^R_j]\, e^R_j]\, f^R_{r+2} \cdots f^R_{i_k}] \\ &= -[[[[f^R_{i_1} \cdots f^R_{i_r}]\, h_j] \cdots f^R_{i_k}] + [[[[f^R_{i_1} \cdots f^R_{i_r}]\, e^R_j]\, f^R_j \cdots f^R_{i_k}] \qquad \text{(B.2)}\end{aligned}$$

The first term after the last equality sign certainly is in $\bar{\mathfrak{L}}^R$ (compare i)). The induction hypothesis establishes the same claim for the second term. This proves I.

Now we consider a semi-simple Lie algebra $\mathfrak{L}'$ with the canonical generators $h'_j, e'_j, f'_j, j \in R$, the Cartan matrix A^R and the simple system of roots $\pi' = \{\alpha'_j\}$. Then, of course, $\alpha'_i(h_j) = A^R_{ji}$. $\mathfrak{L}'$ certainly exists and is uniquely defined by these requirements up to an isomorphism (compare Ref. [10])).

We shall now show:

II. $\beta' = \Sigma\, k_i\, \alpha'_i$ is a root if and only if $\beta^R = \Sigma\, k_i\, \alpha^R_j$ is a root.

Proof: For a linear function $\beta = \Sigma\, k_i\, \alpha_i$ (k_i integers) we define the level $\beta = \Sigma\, |k_i|$. We prove II by induction on levels. It suffices to consider the case when all k_i are non-negative integers. For level $n = 1$ II. is, of course, true. Now suppose that II. is true for levels $\leqslant n$. Let β' have level $n + 1$. Then, it is necessarily of the form $\beta' = \alpha' + \alpha'_j$, $\alpha' = \Sigma\, k_i\, \alpha'_i$. $\alpha^R = \Sigma\, k_i\, \alpha^R$ is a root (of level n) by the induction hypothesis. We have to show that $\beta' = \alpha' + \alpha'_j$ is exactly a root if $\beta^R = \alpha^R + \alpha^R_j$ is a root. To do this consider the elements of the form

$$\alpha' - \alpha'_j,\ \alpha' - 2\,\alpha'_j \dots \text{ resp. } \alpha^R - \alpha^R_j,\ \alpha^R - 2\,\alpha^R_j \dots$$

which are roots. These are positive and of level $< n$. By induction hypothesis these two sequences stop at the same position. Consider now the following strings of roots (the star stands for R or $'$, respectively):

$$\alpha^* - r^*\,\alpha^*_j,\ \alpha^* - (r^* - 1)\,\alpha^*_j,\ \dots,\ \alpha^* + q^*\,\alpha^*_j\,,$$

where r^* and q^* are chosen maximally, then, according to a well-known theorem

$$q^* = r^* - \frac{2\,(\alpha^*, \alpha_j^*)}{(\alpha_j^*, \alpha_j^*)} = r^* - \sum k_i\, A_{ji}^R$$

As we just showed $r' = r^R$, this formula implies $q' = q^R$, which proves II.

Hence, we know that the roots β' of $\mathfrak{L}'$ are in 1:1 correspondence to the roots β^R of $\mathfrak{L}^R$. To each β^R we associated earlier a sequence $(i_1, \ldots, i_k)$. This correspondence we carry over to the β'. Then, the following elements form a basis of $\mathfrak{L}'$ (compare the remark at the beginning of this appendix):

$$h_i', [e_{i_1}' \ldots e_{i_k}'], \quad [f_{i_1}' \ldots f_{i_k}'] .$$

Hence, according to I., $\mathfrak{L}'$ and $\mathfrak{L}^R$ are isomorphic as vector spaces, where the bases

$$h_i^*, [e_{i_1}^* \ldots e_{i_k}^*], [f_{i_1}^* \ldots f_{i_k}^*]$$

correspond to each other.

Now we claim:

III. The multiplication table is the same for both bases. If this is proved, we have shown that $\mathfrak{L}'$ and $\mathfrak{L}^R$ are isomorphic (as Lie algebras) and hence we have proved theorem 2.

Proof: The remarks which were made in the proof of I. show that it is sufficient to prove III. for the following products:

$$[[e_{i_1}^* \ldots e_{i_k}^*]\, e_j^*], \quad [[f_{i_1}^* \ldots f_{i_k}^*]\, e_j^*] .$$

i) We study first the second product. For $k = 1, 2$ III. is correct as the formulae in the proof of I. show. For $k \geqslant 2$ we prove III. by induction on k. The formulae (B.2) (if one replaces there R by a star) show that the step from k to $k + 1$ can be done.

ii) For the first product compare iii) of I. We have the case a) either for both $\mathfrak{L}'$ and $\mathfrak{L}^R$ (according to II.) or for none of them. There remains case b). According to what we said about this case, the multiplication coefficients equal if the following holds:

IV. Let β^* be a positive root, $(i_1, \ldots, i_k)$ the associated sequence. Let $1', \ldots, k'$ be a permutation of $1, 2, \ldots, k$. Then

$$[e_{i_{1'}}^* \ldots e_{i_{k'}}^*] = p^* [e_{i_1}^* \ldots e_{i_k}^*]$$

with $p' = p^R$.

Proof: For $k = 1$ this is trivial. We make induction on k. Let us distinguish the following two cases.

i) $i_k = i_k' = j$. In this case, we can assume that the sequence which is associated to the root $\beta^* - \alpha_j^*$ is $i_1, \ldots, i_{k-1}$. By induction hypothesis

$$[e_{i_{1'}}^* \ldots e_{i_{(k-1)'}}^*] = t\, [e_{i_1}^* \ldots e_{i_{k-1}}^*]$$

where t is for both cases the same number. From this IV. immediately follows.

ii) $j \equiv i_k \neq i'_k$. Then

$$\left[e^*_{i_1'} \dots e^*_{i_k'}\right] = \left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_j \dots e^*_{i_k'}\right],$$

where the displayed e^*_j is the last one occurring in the expression.

If any one of the $\alpha^*_{i_1'} + \dots + \alpha^*_{i_m'}$ is no root, then $[e^*_{i_1'}, \dots, e^*_{i_k'}] = 0$. According to II. this is for both cases simultaneously so or not so. If now every $\alpha^*_{i_1'} + \dots + \alpha^*_{i_m'}$ is a root, then $[e^*_{i_1'}, \dots, e^*_{i_k'}] \neq 0$. We certainly have the equation

$$\left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_j \dots e^*_{i_k'} f^*_j\right] = \left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_j\, f^*_j \dots e^*_{i_k'}\right]$$

But according to [13])

$$\left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_j\, f^*_j\right] = -\,q^*\,(r^* + 1)\left[e^*_{i_1'} \dots e^*_{i_r'}\right],$$

q^* and r^* are determined by the α^*_j string which contains $\beta = \alpha^*_{i_1'} + \dots + \alpha^*_{i_r'}$. II. implies

$$r' = r^R \equiv r \geqslant 0\,, \quad q' = q^R \equiv q > 0\,.$$

An analogous argument shows

$$\left[e^*_{i_1'} \dots e^*_{i_k'} f^*_j\, e^*_j\right] = s\left[e^*_{i_1'} \dots e^*_{i_k'}\right],$$

with a constant non-vanishing s. Hence,

$$s\left[e^*_{i_1'} \dots e^*_{i_k'}\right] = \left[e^*_{i_1'} \dots e^*_{i_k'} f^*_j\, e^*_j\right] = -\,q\,(r + 1)\left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_{i_{(r+2)'}} \dots e^*_{i_k'}\, e^*_j\right].$$

Therefore

$$\left[e^*_{i_1'} \dots e^*_{i_k'}\right] = t\left[e^*_{i_1'} \dots e^*_{i_r'}\, e^*_{i_{(r+2)'}} \dots e^*_{i_k'}\, e^*_j\right]$$

again with constant t. This reduces the discussion to the first case and hence IV. is proved.

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