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Gauge Fields of an Algebraic Hilbert Space

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Abstract. The theory of a spinor particle in the presence of a Yang-Mills type field provides an example of an algebraic Hilbert space, i.e., a vector space linear over quantities of a non-commutative algebra. A discussion of the field equations applicable when the gauge field belongs to a general finite algebra is given. It is shown, considering only the unquantized spinor field, that the Hamiltonian operator is not an observable. The system described by these field equations does not have, for example, a well-defined rest mass, and it consequently does not seem to correspond to the type of object one conventionally accepts as an elementary particle.

I. Introduction

The theory of a spinor particle in the presence of a Yang-Mills isotopic spin field [1][2]) provides an example of an algebraic Hilbert space [2], i.e., a vector space linear over quantities of a non-commutative algebra.

In the following, we shall show that a Dirac equation on the wave functions of an algebraic Hilbert space implies, through gauge invariance, the existence of a general algebraic Yang-Mills field with properties precisely analogous to those of the isotopic spin field. The problem of the second quantization of such a Hilbert space has not, so far, been formally investigated. Consequently, in this paper we shall not consider these spinor wave functions in the rôle of quantized fields (cf., however, Section IV).

Glashow and Gell-Mann [3] have discussed a very broad class of generalizations of the Yang-Mills idea, i.e., those with the infinitesimal transformation properties of the simple Lie algebras. Our model has well-defined transformation properties in the large (gauge transformations induce algebraic inner automorphisms) corresponding to the special class of Lie groups generated by the Lie algebras of finite associative algebraic systems. Hence, for example, all of the rotation or orthogonal groups are included (corresponding to the various Clifford algebras).

It has been pointed out [4] that a quantum mechanics represented by a Hilbert space over a finite algebra has intrinsic superselection rules when it is assumed that the observables are totally linear[3]) with respect to quantities of the algebra. Even though the Hamiltonian defining the time variation of the algebraic spinor wave function may be chosen to be totally linear in the absence of the Yang-Mills field, the Hamiltonian of the interacting system is not totally linear and hence, according to this

[1]) Work supported by the Swiss National Fund.

[2]) Numbers in brackets refer to References, page 154.

[3]) An operator on a wave function is *totally linear* if its operation may be applied, with the same result, before or after multiplication by any element of the algebra.

assumption, it is not an observable. We discuss explicitly in Section II some of the difficulties encountered.

If it were such objects as quarks [5, 6] which are the basic fermions and give rise to vector meson fields through the requirements of general gauge invariance, it would follow from our assumption that these basic fermions are not observable as particles in the usual sense (e.g., in states of definite energy). Alternatively, one might say that the observation of these 'quarks' is inhibited by the presence of superselection rules[4]).

It should be emphasized that the non-linearity referred to above is not the same as the well-known non-linearity of the equations of motion for the Yang-Mills field, although it is the same mechanism (the non-commutative nature of the algebra) which gives rise to both.

II. The Gauge Field

We start by assuming that the wave function satisfying the Dirac equation[5])

$$(\gamma^\mu \partial_\mu + i \varkappa) \psi(x) = 0 \tag{2.1}$$

is an element of the Hilbert space of state vectors describing a free particle. According to Reference [4], the state determined by ψ is given by

$$m_\psi(P) = \mathrm{tr}(\psi, P \psi) \tag{2.2}$$

as a function of the projections onto closed linear manifolds of the Hilbert space. If ψ' generates a pure state, then (with proper normalization) (ψ', ψ') is a primitive idempotent e_0 of our algebra $\mathfrak{A}$, i.e., $\psi' e_0 = \psi'$. Under the transformation $\psi = \psi' s$ for $s \in \mathfrak{A}$, the state (2.2) is invariant when

$$(s s^*)_{00} = \mathrm{tr}(s s^* e_0) = 1 , \tag{2.3}$$

i.e., for totally linear P (such that $P(\psi a) = (P \psi) a$ for any $a \in \mathfrak{A}$, corresponding to algebraically closed linear manifolds),

$$\mathrm{tr}(\psi' s, P \psi' s) = \mathrm{tr}\, s s^*(\psi', P \psi') = \mathrm{tr}(s s^* e_0(\psi', P \psi') e_0) = (s s^*)_{00} \mathrm{tr}(\psi', P \psi') . \tag{2.4}$$

Using a Frobenius basis (cf. Equation (2.12) of Reference [4], for example) for s, i.e., $s = \sum_{ij} \alpha_{ij} \varrho_{ij}$, we note that $\psi' s$ involves only

$$e_0 s = \sum_j \alpha_{0j} \varrho_{0j} \tag{2.5}$$

and our normalization implies that

$$\mathrm{tr}(s s^* e_0) = \sum_j | \alpha_{0j} |^2 = 1 . \tag{2.6}$$

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[4]) KATAYAMA *et al.* [7] have suggested superselection rules based on parastatistics [8], i.e., their algebra is that of a representation of the permutation group of dimensionality greater than one.

[5]) $\gamma^k = -\gamma^{k*}$, $\gamma^0 = \gamma^{0*}$; $\gamma^k = \beta \alpha^k$, $\beta = \gamma^0$. We assume that our algebra is defined over the base field of complex numbers, and therefore the complex unit 'i' may be used in the usual way. This is in distinction from the work of Reference [9] where an element not in the center of the algebra must be used and in fact directly gives rise to a theoretical structure analogous to what will be discussed in the later sections (cf. Equation (3.11) and the discussion following).

Hence, as pointed out in Reference [4] the minimal right ideals of the algebra $\mathfrak{A}$ play the role of *phases* on the wave functions ψ. Just as the complex phases may vary from point to point for the wave function $\psi(x)$ in the ordinary Hilbert space, the generalized phases s in the right ideals of $\mathfrak{A}$ may vary; the minimal right ideal itself may vary also (with $\psi(x)$).

In the following, we assume that ψ is a general, not necessarily pure (or primitive), wave function and that the gauge function $s(x)$ is a general member of $\mathfrak{A}$ with inverse. It will then be easy to discuss the pure states and ideals ('phase' transformations) systematically. For example, the transformation from ψ to $\psi\, s$ corresponds to an *independent* phase transformation on each of the primitive constituents of ψ since $\psi\, s = \sum_j (\psi\, e_j)\,(e_j\, s)$, where the e_j are primitive idempotents of $\mathfrak{A}$.

Equation (2.1) is not invariant under the transformation $\psi \to \psi'$ given by

$$\psi = \psi'\, s\,. \tag{2.7}$$

Following Yang and Mills [1], we therefore introduce the compensating gauge field b_μ. If the wave function satisfies

$$\gamma^\mu\, \partial_\mu\, \psi - i\, \gamma^\mu\, \psi\, b_\mu + i\, \varkappa\, \psi = 0\,, \tag{2.8}$$

then ψ' and b'_μ also satisfy (2.8) when

$$b'_\mu = s(x)\, b_\mu\, s^{-1}(x) + i\, \big(\partial_\mu\, s(x)\big)\, s^{-1}(x)\,. \tag{2.9}$$

Consequently, the quantity

$$F_{\mu\nu} = \partial_\mu\, b_\nu - \partial_\nu\, b_\mu + i\, (b_\mu\, b_\nu - b_\nu\, b_\mu) \tag{2.10}$$

transforms under (2.7) as

$$F'_{\mu\nu} = s\, F_{\mu\nu}\, s^{-1}\,. \tag{2.11}$$

Hence, according to (2.9) and (2.11), gauge transformations induce local algebraic inner automorphisms in addition to the usual additive gauge compensation. Constant gauge transformations introduce only an inner automorphism, and gauge invariance of this type is equivalent to invariance under coordinate independent algebraic inner automorphisms.

In concluding this section, we wish to emphasize two important qualitative differences between the free field Equation (2.1) and the interacting system (2.8).

The first of these concerns the *degeneracy* of ψ, i.e., multiplying (2.1) by each of the finite set of idempotents $\{e_i\}$ of $\mathfrak{A}$, we obtain the set of equations

$$\gamma^\mu\, \partial_\mu(\psi\, e_i) + i\, \varkappa(\psi\, e_i) = 0 \tag{2.12}$$

for the vectors $\psi\, e_i$ which generate pure states (minimal ideals). In the same way, from Equation (2.8) we obtain

$$\gamma^\mu\, \partial_\mu(\psi\, e_i) - \sum_j i\, \gamma^\mu\, (\psi\, e_j)\, (e_j\, b_\mu\, e_i) + i\, \varkappa(\psi\, e_i) = 0\,, \tag{2.13}$$

i.e., if b_μ is not in the center of $\mathfrak{A}$, it introduces a coupling among the previously degenerate minimal ideal functions $\psi\, e_i$ in a way analogous to the isotopic spin formalism in the description of the nucleon. The dimensions of the resulting multiplets are that of the irreducible representations of $\mathfrak{A}$, since b_μ can only connect primitive idempotents within the same irreducible representation.

For the second, we remark that the free Hamiltonian H_0, defined by (2.1), i.e.,

$$H_0 \psi = i \hbar \frac{\partial \psi}{\partial t} = - i \hbar c \{\boldsymbol{\alpha} \cdot \nabla + i \beta \varkappa\} \psi ,$$

is a totally linear operator in the sense used in Reference [4] [6]):

$$H_0(\psi a) = (H_0 \psi) a$$

for any $a \in \mathfrak{A}$. However, for the interacting system described by (2.8),

$$H \psi = - i \hbar c \{\boldsymbol{\alpha} \cdot \nabla + i \beta \varkappa\} \psi - \hbar c \gamma^0 \gamma^\mu \psi b_\mu . \tag{2.14}$$

Since $[a, b_\mu] \neq 0$ for $a \in \mathfrak{A}$, in general, it follows that

$$H(\psi a) \neq (H \psi) a \tag{2.15}$$

i.e., that H is not totally linear over the algebra $\mathfrak{A}$. We show in the following how this result affects the measurement of the properties of a 'particle' described by the wave function ψ.

The totally linear operator H_0 has a spectral resolution [2] of the form

$$H_0 = \int E \, dP_E \tag{2.16}$$

where the projection operators P_E are totally linear. Hence, with the help of (2.2) we may assign a value (expectation value) to H_0 in any state generated by ψ, i.e.,

$$m_\psi(H_0) = \int E \, d \left(\mathrm{tr}(\psi, P_E \psi)\right) . \tag{2.17}$$

By our previous arguments, the transformation $\psi \to \psi a$ leaves (2.17) invariant (when all quantities are properly normalized).

It is also true that H has a spectral resolution [2] (it is Hermitian in the weaker sense $\mathrm{tr}\,(\psi_1, H \psi_2) = \mathrm{tr}\,(H \psi_1, \psi_2)$), but in terms of projections defined for linear manifolds which are *not* closed under right multiplication by elements of $\mathfrak{A}$. Formally, we may still define

$$m_\psi(H) = \mathrm{tr}(\psi, H \psi)$$

but

$$m_{\psi a}(H) = \mathrm{tr}\, a^*(\psi, H(\psi a)) . \tag{2.18}$$

One finds from (2.14) that

$$H(\psi a) - (H \psi) a = \hbar c \gamma^0 \gamma^\mu \psi [b_\mu, a] \tag{2.19}$$

and therefore

$$m_{\psi a}(H) = \mathrm{tr}(\psi, H \psi) + \hbar c \,\mathrm{tr} \left(a^*(\psi, \gamma^0 \gamma^\mu \psi [b_\mu, a])\right) . \tag{2.20}$$

The basic hypothesis allowing us to perform gauge transformations, however, is that the transformation $\psi \to \psi a$ have no measurable consequences, and (2.20) violates this

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[6]) This follows (with our present convention of right multiplication of 'scalars') even when the Dirac algebra is included in $\mathfrak{A}$.

condition. If ψ corresponds to the rest state of a 'particle' described by (2.14), (2.20) implies that the rest mass will depend on the choice of phase, and one therefore could not hope to detect a particle with well-defined rest mass experimentally. We have not explored other particle-like manifestations, but it seems clear from what has already been discussed that (2.14) does not describe the type of object one conventionally accepts as an elementary particle.

We wish to conclude this section with some remarks concerning the interpretation of (2.19) and (2.20). If we consider the Hamiltonian as the generator of infinitesimal transformations in time, it follows from (2.14) (or (2.19)) that

$$\psi_{t+\delta t}\, a - (\psi\, a)_{t+\delta t} = i\,\delta\, t\, c\,\gamma^0\,\gamma^\mu\,\psi_t\,[b_\mu, a]\,. \tag{2.21}$$

The time evolution of ψ and $\psi\, a$ are therefore non-trivially different, and we therefore conclude that the elements of $\mathfrak{A}$ in their rôle as phases of the wave function ψ, *are not time independent*. Explicitly, we may define a_t such that $(\psi\, a)_{t+\delta t} = \psi_{t+\delta t}\, a_{t+\delta t}$, and hence (2.21) becomes (to first order in $\delta\, t$)

$$\psi_t\,(a_t - a_{t+\delta t}) = i\,\delta\, t\, c\,\gamma^0\,\gamma^\mu\,\psi_t\,[b_\mu, a_t]\,. \tag{2.22}$$

To obtain a somewhat simpler result, suppose that b_μ has only the component b_0 unequal to zero. Then one finds that, as a multiplier of ψ_t,

$$\dot{a}_t \equiv \frac{da_t}{dt} = -\,i\,c\,[b_0, a_t]\,, \tag{2.23}$$

i.e., the b_0 field acts as the generator of time transformations on the algebra of phases[7]). Similar considerations may also be applied to spacial variations through the use of the canonical (gauge invariant) momenta defined with the help of (3.12), but we shall not pursue this further here.

With the help of (2.22), (2.20) then implies that

$$m_{\psi a}(H) = \mathrm{tr}(\psi, H\,\psi) + i\,\hbar\,\mathrm{tr}\,(\psi, \psi\,\dot{a}_t\, a_t^*) \tag{2.24}$$

If a_t has harmonic time dependence, the extra term in (2.24) then provides a frequency-dependent shift in energy.

III. Currents and the Coupling of Spinor and Gauge Fields

According to our interpretation of $\psi(x)$ as a wave function the quantity $\mathrm{tr}\,(\psi^*(x)\,\psi(x))$, transforming like the 0^{th} component of a four vector, may be considered a particle density or probability density. The corresponding current is

$$J^\mu = \mathrm{tr}\,(\overline{\psi}(x)\,\gamma^\mu\,\psi(x)) \tag{3.1}$$

where

$$\overline{\psi}(x) = \psi^*(x)\,\gamma^0\,,$$

[7]) Note that $\hbar$ does not appear in (2.22) or (2.23). It is c and the coupling constant of the b_μ field which determines the scale of frequencies in the algebra of phases. Hence, like the Zitterbewegung phenomenon, the situation persists as $\hbar \to 0$.

and, as in (2.2), 'tr' is the algebraic trace. In case the ψ are second-quantized operators, we note that the algebraic properties of any operator A can be abstracted by means of the following identity:

$$A = \sum_{ij} K_{ij}(A)\, \varrho_{ij}$$

where

$$K_{ij} = \sum_k \varrho_{ki}\, A\, \varrho_{jk} \tag{3.2}$$

commutes with all elements of the algebra $\mathfrak{A}$. Hence, for example,

$$\operatorname{tr} A = \sum_i K_{ii}(A) = \sum_{ij} \varrho_{ji}\, A\, \varrho_{ij}\, . \tag{3.3}$$

We find, however, that it is not (3.1) that enters the field equations, but rather the *algebra-valued* current[8])

$$j^\mu = \overline{\psi}(x)\, \gamma^\mu\, \psi(x)\, . \tag{3.4}$$

With the help of (2.8) it follows that

$$\partial_\mu\, j^\mu = i\, [j^\mu, b_\mu]\, , \tag{3.5}$$

and j^μ is therefore not, in general, divergenceless [1]. The kinematical independence of the gauge field and the spinor field ψ may be expressed by requiring the lack of commutativity between j^μ and b_μ to rest entirely on their algebraic properties in the algebra $\mathfrak{A}$ (i.e., their 'kernels', in the sense of (3.2), commute); since we have supposed this to be a finite algebra, it follows that

$$\partial_\mu\, J^\mu = \operatorname{tr} \partial_\mu\, j^\mu = i \operatorname{tr} [j^\mu, b_\mu] = 0\, , \tag{3.6}$$

and hence the current (3.1) is conserved[9]).

According to (3.5), the positive-definite particle density $\psi^* \psi$ undergoes a time variation which, in a small spatial volume, is not entirely accounted for by the space divergence of currents. The additional variation is provided by transformations among the components of the current density matrix (in some representation of $\mathfrak{A}$, for example), analogous to some sort of circulation or exchange currents, induced by the b_μ-field, which leave the trace or spinor particle current invariant.

Under a gauge transformation of the form (2.7), the current j^μ becomes

$$j_\mu = s^*\, \overline{\psi}'\, \gamma^\mu\, \psi'\, s = s^*\, j^{\mu'}\, s\, . \tag{3.7}$$

We verify in the following that the expression (3.5) for the divergence of the algebraic current is gauge invariant. Under (2.7), b_μ becomes, with the help of (2.9),

$$b_\mu = s^{-1}\, b'_\mu\, s - i\, s^{-1}(\partial_\mu\, s) \tag{3.8}$$

and therefore (for $s^* = s^{-1}$)

$$i\, [j^\mu, b_\mu] = i\, s^*\, [j^{\mu'}, b'_\mu]\, s + s^*\, j^{\mu'}\, \partial_\mu\, s - s^{-1}\, (\partial_\mu\, s)\, s^*\, j^{\mu'}\, s\, .$$

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[8]) This form for the current is also obtained in the usual way from the variation (cf. for example, Reference [3]) of the Lagrangian introduced in Section IV.

[9]) We discuss this point further in Section IV.

However

$$\partial_\mu s^{-1} = - s^{-1} (\partial_\mu s) s^{-1}$$

and therefore

$$i [j^\mu, b_\mu] = i s^* [j^{\mu'}, b'_\mu] s + \partial_\mu (s^* j^{\mu'} s) - s^* (\partial_\mu j^{\mu'}) s . \tag{3.9}$$

For the left side of (3.5), we note that

$$\partial_\mu j^\mu = \partial_\mu (s^* j^{\mu'} s)$$

which cancels the second term of (3.9) and implies that (3.5) is valid for the transformed j^μ and b_μ as well.

We now turn to the question of an equation for b_μ. In case the gauge field is in the center of $\mathfrak{A}$, it seems reasonable to expect it to satisfy Maxwell's equations, i.e., to reduce to the electromagnetic field. However, the equation $\partial_\nu F^{\mu\nu} = j^\mu$ is not gauge invariant in the general case; the derivative produces a commutator:

$$\begin{aligned}\partial_\nu F^{\mu\nu} &= \partial_\nu(s^{-1} F^{\mu\nu'} s) = s^{-1} F^{\mu\nu'}(\partial_\nu s) - s^{-1}(\partial_\nu s) s^{-1} F^{\mu\nu'} s + s^{-1}(\partial_\nu F^{\mu\nu'}) s \\ &= [F^{\mu\nu}, s^{-1}(\partial_\nu s)] + s^{-1}(\partial_\nu F^{\mu\nu'}) s .\end{aligned}$$

Replacing $s^{-1}(\partial_\nu s)$ by $- i s^{-1} b'_\mu s + i b_\mu$ (as given by (3.8)), one finds

$$\partial_\nu F^{\mu\nu} - i [F^{\mu\nu}, b_\mu] = s^{-1} \{\partial_\nu F^{\mu\nu'} - i [F^{\mu\nu'}, b'_\mu]\} s . \tag{3.10}$$

The derivative must therefore (as in (3.5)) be supplemented with a commutator with the b_μ-field. The proper gauge invariant equation is therefore[10])

$$\partial_\nu F^{\mu\nu} = j^\mu + i [F^{\mu\nu}, b_\nu] , \tag{3.11}$$

and hence is non-linear (functionally) in the b_μ-field. The quantities tr $F^{\mu\nu}$, however, satisfy the usual equations with $J^\mu = \text{tr}\, j^\mu$ as source.

An analogous situation is discussed by FINKELSTEIN, JAUCH, SHIMINOVICH and SPEISER [9], where our gauge field may be associated with their 'Q-connection' and $F^{\mu\nu}$ with their 'Q-curvature', for the special case of a quaternion algebra. As discussed above in connection with our use of the complex field, their 'gauge transformations' arise from a somewhat different mechanism (although also induced by local algebraic automorphisms).

We have seen in the preceding, where quantities such as $F^{\mu\nu}$ and j^μ are bilinear in the gauge function, that the gradient is generally accompanied by a commutator with the gauge field. In the construction of the second-order equation for ψ, however, which is linear in the gauge function, the gauge field enters in much the same way as in the construction of the gauge invariant derivatives for the Dirac electron in interaction with the electromagnetic field.

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[10]) This result also follows from the variation of the Lagrangian given in Section IV. The relation (3.11) is of course not uniquely prescribed by gauge invariance (cf. footnote accompanying (4.2)); it appears, however, to be the simplest gauge invariant generalization of the electromagnetic analog in view of (3.10).

Let

$$D_\mu \psi \equiv \partial_\mu \psi - i \psi b_\mu \tag{3.12}$$

define the gauge invariant derivative of ψ, i.e.,

$$D_\mu(\psi' s) = (\partial_\mu \psi') s + \psi' \partial_\mu s - i \psi' s b_\mu = (\partial_\mu \psi') s + \psi' \partial_\mu s - i \psi' (b'_\mu s - i \partial_\mu s)$$
$$= (\partial_\mu \psi') s - i \psi' b'_\mu s = (D_\mu \psi') s , \tag{3.13}$$

where it is understood that if b_μ is appropriate to ψ in (3.12) then b'_μ is appropriate to ψ'. Hence, D_μ is linear with respect to gauge transformations, but it is, in general, *not linear* over the algebra $\mathfrak{A}$.
Using the relations among Dirac matrices

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \, (g^{kk} = -1, g^{00} = +1) ,$$

one obtains in the usual way

$$D^\mu D_\mu \psi + \varkappa^2 \psi - \frac{i}{2} \sigma^{\mu\nu} \psi F_{\mu\nu} = 0 , \tag{3.14}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) . \tag{3.15}$$

Since the D^μ are linear over the gauge transformations according to (3.13), and $F_{\mu\nu}$ transforms according to (2.11) under the transformation (2.7), (3.14) is obviously gauge invariant.

It is interesting to recall the form (2.10) of $F_{\mu\nu}$ in connection with (3.14), i.e., the commutator $[b_\mu, b_\nu]$, in addition to the electromagnetic-like crossed derivatives of the b_μ-field, is coupled to the 'spin' $\sigma^{\mu\nu}$.

We illustrate another connection between the b_μ-field and the spin in the following. The usual decomposition [10] of the relativistic current into a 'convection' current and a current corresponding to something like a distribution of magnetic dipoles in the absence of an electromagnetic field can easily be generalized to include the electromagnetic field by replacing the gradient operators by gauge invariant derivatives. With the help of (2.8) and noting that on $\bar{\psi}$, D_μ is defined as

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + i b_\mu \bar{\psi} , \tag{3.16}$$

we obtain for (3.4)

$$j^\mu = - \frac{i}{2\varkappa} \{\bar{\psi}(D^\mu \psi) - (D^\mu \bar{\psi}) \psi + \partial_\mu(\bar{\psi} \sigma^{\mu\nu} \psi)\} + \frac{1}{2\varkappa} [b_\mu, \bar{\psi} \sigma^{\mu\nu} \psi] . \tag{3.17}$$

The first bracketed term corresponds to what one would find if the b_μ-field were simply electromagnetic; the last term indicates that the commutator of the b_μ-field and the dipole density distribution provides another source of current.

IV. The Lagrangian Formalism

In this section we justify the remarks made above concerning the existence of a Lagrangian for which (2.8) and (3.11) are the Euler-Lagrange equations, and the j^μ of (3.4) is the appropriate current.

It is possible, without much additional effort, to work with the antisymmetrized Lagrangian more appropriate for application to quantized fields, and we therefore do so in the following. One obtains in this case a current

$$j^\mu = \frac{1}{2} [\overline{\psi}(x), \gamma^\mu \psi(x)] , \tag{4.1}$$

as is usual for the second quantized theory, but the bilinear gauge properties of j^μ (clearly specified by the variational result) require some discussion. Under the gauge transformation (2.7), we saw in (3.7) that $j^\mu = s^* j^{\mu'} s$. The first term of the commutator in (4.1) clearly has this property, but the second is of the form $-\gamma^\mu \psi \overline{\psi}$, which, under (2.7) becomes $-\gamma^\mu \psi' s s^* \overline{\psi}'$. Remembering that ψ is multiplied by quantities of $\mathfrak{A}$ on the right and $\overline{\psi}$ on the left, we will write this result as $-s^* \gamma^\mu \psi' \overline{\psi}' s$, understanding that, for example, in any representation of $\mathfrak{A}$, s^* has indices contiguous with $\overline{\psi}'$ and s with ψ'.

For our Lagrangian, we take[11])

$$\mathfrak{L} = \mathrm{tr}\left(\frac{i}{4} [\overline{\psi}, (\gamma^\mu \partial_\mu \psi - i \gamma^\mu \psi b_\mu + i \varkappa \psi)] + h.c.\right.$$
$$\left. + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \{F_{\mu\nu}, \partial^\mu b^\nu - \partial^\nu b^\mu + i [b^\mu, b^\nu]\}\right), \tag{4.2}$$

where 'tr' is the trace over $\mathfrak{A}$ as defined in (3.3).

We shall assume that the b_μ-field is the simplest possible generalization of the electromagnetic field, and, as for (3.6), assume that its non-commutativity with the spinor field rests entirely on its properties in $\mathfrak{A}$. With the help of the usual anti-commutation relations for the spinor field variations, we then may use the following identities necessary for simplifying the variation of (4.2):

$$\mathrm{tr}\,(\gamma^\mu \delta \psi \partial_\mu \overline{\psi}) = - \mathrm{tr}\,(\partial_\mu \overline{\psi} \gamma^\mu \delta \psi)$$
$$\mathrm{tr}\,(\overline{\psi} \gamma^\mu \delta \psi b_\mu) = \mathrm{tr}\,(b_\mu \overline{\psi} \gamma^\mu \delta \psi)$$
$$\mathrm{tr}\,(\gamma^\mu \delta \psi b_\mu \overline{\psi}) = - \mathrm{tr}\,(b_\mu \overline{\psi} \gamma^\mu \delta \psi)$$
$$\mathrm{tr}\,(\overline{\psi} \gamma^\mu \psi \delta b_\mu) = \mathrm{tr}\,(\delta b_\mu \overline{\psi} \gamma^\mu \psi)$$
$$\mathrm{tr}\,(F_{\mu\nu} \delta b^\mu b^\nu) = \mathrm{tr}\,(b^\nu F_{\mu\nu} \delta b^\mu) \tag{4.3}$$

and others closely related to these. With the help of (4.3) we then obtain

$$\delta \mathfrak{L} = \mathrm{tr}\Big(i \{\delta \overline{\psi} (\gamma^\mu \partial_\mu \psi - i \gamma^\mu \psi b_\mu + i \varkappa \psi) - (\partial_\mu \overline{\psi} \gamma^\mu + i b_\mu \overline{\psi} \gamma^\mu - i \varkappa \overline{\psi}) \delta \psi\}$$
$$+ \frac{1}{2} (F_{\mu\nu} - (\partial_\mu b_\nu - \partial_\nu b_\mu + i [b_\mu, b_\nu])) \delta F^{\mu\nu} + \frac{1}{2} \{\overline{\psi} \gamma^\mu \psi \delta b_\mu - \gamma^\mu \psi \delta b_\mu \overline{\psi}\}$$
$$+ \{\partial^\mu F_{\mu\nu} \delta b^\nu - i [b^\nu, F_{\mu\nu}] \delta b^\mu\}\Big) + \partial_\mu \mathrm{tr}\left(\frac{i}{2} \{\overline{\psi} \gamma^\mu \delta \psi - \delta \overline{\psi} \gamma^\mu \psi\} - F^{\mu\nu} \delta b_\nu\right). \tag{4.4}$$

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11) This is, as in the case of electrodynamics, not the only gauge invariant Lagrangian, nor are our Euler-Lagrange equations, consequently, uniquely determined by gauge invariance. They are, however, 'minimal' in the sense argued in Reference [3], and provide adequate illustration of the structure of gauge fields in an algebraic Hilbert space for our present purposes.

The first bracketed term of (4.4) provides the Equation (2.8) for ψ and the adjoint equation for $\bar{\psi}$, and the second of (4.4) provides the relation (2.10) between $F_{\mu\nu}$ and the b_μ-field. The third term contains the variation δb_μ in its second part in the way anticipated by our discussion above of the bilinear gauge properties of the commutator form of j^μ; we understand

$$j^\mu \delta b_\mu = \frac{1}{2} (\bar{\psi} \gamma^\mu \psi - \gamma^\mu \psi \bar{\psi}) \delta b_\mu = \frac{1}{2} (\bar{\psi} \gamma^\mu \psi \delta b_\mu - \gamma^\mu \psi \delta b_\mu \bar{\psi}) .$$

Consequently, the third and fourth terms of (4.4) provide precisely (3.11), where (4.1) is taken for j^μ.

The last of (4.4) is a boundary term and will be of no further interest for our present purposes.

So far, the current j^μ has entered as a formal definition. It also follows in the usual way from the variation of the b_μ-field in the part of the Lagrangian referring only to the spinor fields, with gauge invariant derivatives. However, although $\mathfrak{L}$ is invariant to total gauge transformations, it is not invariant to (constant) gauge transformations of the spinor fields alone because the b_μ-field does not commute with the gauge function. Hence j^μ, as pointed out in (3.5), is not conserved[12]).

It is clear from (3.11) and the fact that the divergence of $\partial_\nu F^{\mu\nu}$ vanishes identically that the quantity

$$\mathfrak{J}^\mu = j^\mu + i [F^{\mu\nu}, b_\nu] \tag{4.5}$$

is divergenceless, and is in fact the total source for the b_μ-field. We may follow Yang and Mills [1] in identifying $\mathfrak{J}^\mu$ with the *total* algebraic current (in their case the total isotopic spin current), where the second term contributes the algebraic current carried by the b_μ-field itself. The particle density current (3.1) is given by the trace of $\mathfrak{J}^\mu$, i.e.

$$J^\mu = \mathrm{tr}\, \mathfrak{J}^\mu = \mathrm{tr}\, j^\mu, \tag{4.6}$$

since the trace of the commutator vanishes in a finite algebra (tr $[F^{\mu\nu}, b_\nu] = 0$ for the quantized variables as well, according to our assumptions).

As in the case of isotopic spin, $\mathfrak{J}^\mu$ does not transform like j^μ under gauge transformations, i.e., under the transformation $\psi = \psi' s$,

$$\mathfrak{J}^\mu = s^{-1} [\mathfrak{J}^{\mu'} - [(\partial_\nu s) s^{-1}, F^{\mu\nu'}]] s ,$$

i.e.,

$$\mathfrak{J}^{\mu'} = s [\mathfrak{J}^\mu + [s^{-1}(\partial_\nu s), F^{\mu\nu}]] s^{-1} , \tag{4.7}$$

where $\mathfrak{J}^{\mu'}$ is defined by (4.5) with all quantities primed, and is, of course, still divergenceless. However, the time independent quantity [1]

$$A = \int \mathfrak{J}^0 d^3x = \int \frac{\partial}{\partial x^i} F^{0i} d^3x , \tag{4.8}$$

analogous to total isotopic spin, transforms under a general gauge transformation, with $s \to s_0$ on an infinitely large sphere, according to the inner automorphism

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[12]) See, for example, Reference [11] for the usual connection between gauge invariance and conserved currents.

$A' = s_0 A s_0^{-1}$. We note in this connection that the integral of the fourth component of the particle current,

$$N = \int J^0 d^3x = \text{tr} A \tag{4.9}$$

is conserved and is identically gauge invariant.

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