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Group-Theoretical Approach to the Pairing Interaction in the Translationally Invariant Shell Model

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Summary. The group-theoretical approach to classification of shell-model states for a translationally invariant model with a $3A$-dimensional harmonic oscillator potential, where A represents the number of constituent nucleons, is extended by the consideration of an interaction which takes account of orbit pair correlation and spin-orbit pair correlation by the use of Racah and Flowers groups respectively (in place of the $SU(3)$ scheme employed hitherto), and which in consequence offers a more general and realistic treatment.

I. Introduction

Increasing interest has of late been evinced in the application of group-theoretical methods to the treatment of collective nucleon motion and quasi-collective aspects of the classification of energy states in the nuclear shell model [1, 2, 3][1]). A particularly elegant approach is that due to KRETZSCHMAR [2] in which the collective motion of the entire ensemble of the A constituent particles (rather than merely that of nucleons in unfilled shells) is taken into account within a nucleus whose potential is assumed to be that of a $3A$-dimensional harmonic oscillator. Translational invariance is introduced in order to exclude spurious states [2]. In the past, however, these considerations have been limited solely to application of the $SU(3)$ scheme for a quadrupole-quadrupole interaction, whereas, as is shown in the present paper, results (in L–S coupling) for an orbit pair correlation and (in j–j coupling) for a spin-orbit pair correlation can readily be derived by adapting the above to obtain a more general and realistic model. Section II describes how this may be accomplished in the case of L–S coupling by considering $U(2R) \supset O(2R)$ subgroups, where $R := \sum_l (l + 1/2)$, in place of $SU(3)$ subgroups, and Section III treats the case of j–j coupling by considering $U(2R) \supset \mathrm{Sp}(2R)$ subgroups, where now $R := \sum_j (j + 1/2)$, in place of $O(6)$ subgroups, attention being paid throughout to the stringent restrictions which limit the possible representations. An agreeable feature of this approach is that the calculations of Sections II and III can be taken over unchanged when *non-degenerate levels* are to be considered, and hence constitute a useful generalization in this case.

Although the treatments employed in the past have furnished correct values for the ground-state quantum numbers of light nuclei and provided an indication of

[1]) The numbers in square brackets denote references listed on page 403.

parity alternation in the excited states, they have failed to reproduce the sequence of excited states correctly and have given relative level separations which do not agree with the experimental findings. Insofar as these shortcomings may be attributed to the use of too naive a model interaction, the incorporation of spin-orbit coupling, which permits the occurrence of transitions between different spin states, offers the possibility of improving the measure of agreement. In place of the physically more realistic interaction comprising a spin-orbit force $W_{s.o.}$ and a central force between nucleon pairs, Moshinsky [4] suggested that the central force be replaced by a linear combination of a short-range pair correlation P and a quadrupole-quadrupole interaction Q^2 to take account of long-range correlations. As developed by Flores *et al.* [3], this suggestion was pursued by de Llano et al. [5] in a study of nuclear structure in the 2 s–1 d shell of light nuclei ($A = 18$ and $A = 20$), but their treatment embraces only those nucleons lying outside closed shells. Furthermore, the interactions $W_{s.o.}$ and P are not diagonal in the $SU(3)$ scheme, with the result that configuration mixing arises [6] which can adversely affect the sequence of states. The scheme outlined in the following avoids both these drawbacks in that it considers the entire nucleon ensemble and at the same time allows all the interaction potentials to be diagonalized.

II. Orbit Pair Correlation in the L–S Coupling Scheme

The model Hamiltonian H for a nucleus which contains A nucleons in a $3\,A$-dimensional harmonic oscillator potential is invariant [2] under the transformations of the product group

$$U(3\,A) \times U(4^A) \tag{1}$$

or, when the centre-of-mass motion has been separated off in order to prevent the intrusion of spurious states [2], under the group

$$U\big(3\,(A-1)\big) \times U(4^A)\,. \tag{2}$$

The total wave function then transforms [2] according to the representation

$$U\big(3\,(A-1)\big)_{[r]} \times U(4^A)_{[1]}\,, \tag{3}$$

where r stands for the number of excited oscillator quanta.

As the oscillator potential leads to a high degeneracy, it is necessary to go over to a consideration of subgroups if one is to classify the states further, and in particular to those subgroups which lend themselves to a physical interpretation and which diagonalize an interaction operator. Thus the quadrupole-quadrupole interaction can be diagonalized in an $SU(3)$ scheme [1, 2]. Not so, however, the remaining interactions $W_{s.o.}$ and P.

The orbit pair correlation described by Racah's seniority operator Q has been extended by Flowers and Szpikowski [7], as also by Ichimura [8], to handle distinguishable particles and degenerate levels. The generalized seniority operator then takes the form

$$2\,Q = \sum_{s_1, s_2} \sum_{l, m} (-)^{l-m}\, a^{+}_{l,m,s_1}\, a^{+}_{l,-m,s_2} \sum_{l', m'} (-)^{l'-m'}\, a_{l',-m',s_2}\, a_{l',m',s_1}\,, \tag{4}$$

where the a^+ and a respectively denote fermion creation and annihilation operators and each of the s betokens one of the four possible spin-isospin states of the system. The operator Q can be diagonalized [8] in accordance with the following scheme:

$$U(2\,R) \supset O(2\,R) \supset O(3)\,, \tag{5}$$

where

$$R := \sum_l \left(l + \frac{1}{2}\right). \tag{6}$$

In order to apply this model to a consideration of an orbit-pair correlation, the expression (2) has to be restricted to subgroups which inherently contain the chain (5). The orbital part of the wave function can then be classified according to the chain

$$U(3(A-1))_{[r]} \supset U(3)_{[\varrho]} \times U(A-1)_{[\varrho]} \supset U(2\,R)_{[\lambda]} \supset O(2\,R)_{[\mu]} \supset O(3)_L\,. \tag{7}$$

The group $U(3)_{[\varrho]} \times U\,(A-1)_{[\varrho]}$ has been introduced in order to distinguish between equivalent irreducible representations $I\,R(U(2\,R))$ of $U(2\,R)$, wherein $[\varrho]$ characterizes a tensor representation of rank r. For low-energy states of light nuclei with $A < 16$ this suffices, but for heavier nuclei with $A \geqslant 16$ an additional quantum number β is needed for an appropriate classification:

$$|\, r,\, [\varrho],\, [\lambda]\, \beta,\, (\mu),\, L\rangle\,. \tag{8}$$

A classification of the spin-isospin part is provided by the chain

$$U(4^{\,A})_{[1]} \supset U(4)_{[\tilde{\alpha}]} \times S(A)_{[\tilde{\alpha}]}\,, \tag{9}$$

where $U(4)$ is the group which features in the supermultiplet theory of WIGNER [9].

Instead of effecting the reduction

$$U(3)_{[\varrho]} \times U\,(A-1)_{[\varrho]} \to U(2\,R)_{[\lambda]}$$

directly, we first make use at this point of the relationship which the Pauli principle introduces between the orbital and the spin-isospin component of the wave function, viz.

(i) the irreducible representations of the permutation groups must throughout be mutually conjugate, since only if this is the case does their product yield a complete antisymmetric wave function [2]; and

(ii) the unitary groups $U(2\,R)$ and $U(4)$ are commutator groups, and their irreducible representations are characterized by conjugate Young diagrams [10].

The possible permutation symmetries of the orbital part of the wave function are given by the "inner plethysm" [2]

$$\{A-1,\,1\} \odot \{\varrho\} = \sum_\alpha V_{\varrho\alpha}\,\{\alpha\}\,, \tag{10}$$

wherein V is an integer coefficient and the curly brackets betoken the respective S-functions. This yields r, $[\varrho]$ and $[\alpha]$. All diagrams $[\varrho]$ are admitted which have at most n rows $[n = \min(3,\, A-1)]$, and all $[\alpha]$ which have at most 4 columns and for which $\sum_i \alpha_i = A$.

We see that

$$I\ R\big(S(A)^{\text{spin-isospin}}\big) := [\tilde{\alpha}] \tag{11}$$

and, in accordance with Equation (9) obtain the corresponding irreducible representation of $U(4)$ as

$$I\ R\big(U(4)\big) := [\tilde{\alpha}]\ . \tag{12}$$

Hence $I\ R(U(2\ R))$ must be characterized by the Young diagram $[\alpha]$, and it therefore follows that

$$[\lambda] \equiv [\alpha]\ . \tag{13}$$

The remaining steps in the reduction may be taken over from References [11, 12], whence the complete scheme may be derived as

$$\begin{aligned}
&U(3\ (A-1))_{[r]} \times U(4^A)_{[1]}\ ;\\
&U(3\ (A-1))_{[r]} \supset U(3)_{[\varrho]} \times U\ (A-1)_{[\varrho]} \supset U(2\ R)_{[\alpha] \equiv [\lambda]} \supset O(2\ R)_{(\mu)} \supset O(3)_L\ .\\
&U(4^A)_{[1]} \supset U(8) \times U\ (A-1) \supset U(4)_{[\tilde{\alpha}]} \times S(A)_{[\tilde{\alpha}]}\ ;\\
&U(4)_{[\tilde{\alpha}]} \supset S\ U(2)_S \times S\ U(2)_T\ .
\end{aligned} \tag{14}$$

Hitherto, $U(2\ R)$ was regarded [8] as the maximal symmetry group of the orbital part of the wave function, and the choice of the irreducible representations of $U(2\ R)$ remained arbitrary. We now proceed from the maximal symmetry group of the model Hamiltonian, and in consequence its irreducible representations $I\ R(U(2\ R))$ are, together with the corresponding multiplicity, uniquely established.

By way of illustration, we apply this classification to the specific cases $A = 14$ and $A = 16$:

(a) $A = 14$

Ground state: $r = 10$, Parity (+),

$[\varrho] = [4, 3^2]$ $\quad$ $[\alpha] = [4^2, 3^2]$;

1st Excited state: $r = 11$, Parity (−),

$[\varrho] = [6, 4, 1]$	$[\alpha] = [4^3, 2], [4^3, 1^2]$,
$[6, 3, 2]$	$[4^3, 2], [4^3, 3^2], [4^2, 3, 2, 1]$,
$[5, 4, 2]$	$[4^2, 3^2], [4^2, 3, 2, 1]$,
$[5, 3^2]$	$[4^3, 2], [4^2, 3^2], [4^2, 3, 2, 1], [4, 3^3, 1]$,
$[4^2, 3]$	$[4^3, 2], [4^3, 1^2], [4^2, 3^2], [4^2, 3, 2, 1]$;

(b) $A = 16$

Ground state: $r = 12$, Parity (+) (cf. [2]),

$[\varrho] = [4^3]$ $\quad$ $[\alpha] = [4^4]$;

1st Excited state: $r = 13$, Parity (−) (cf. [2]),

$[\varrho] = [6, 4, 3]$	$[\alpha] = [4^4], [4^3, 3, 1]$,
$[5, 4^2]$	$[4^3, 3, 1]$;

2nd Excited state: $r = 14$, Parity (+),

$[\varrho] = [8, 4, 2]$	$[\alpha] = [4^4], [4^3, 2^2], [4^3, 3, 1]$,
$[8, 3^2]$	$[4^3, 3, 1], [4^2, 3^2, 2]$,
$[7, 5, 2]$	$[4^3, 3, 1], [4^3, 2, 1^2]$,
$[6^2, 2]$	$[4^4], [4^3, 2^2]$,
$[7, 4, 3]$	$[4^4], 4\,[4^3, 3, 1], [4^3, 2, 1^2], [4^2, 3^2, 2], [4^2, 3^2, 1^2]$,
$[6, 5, 3]$	$[4^4], 3\,[4^3, 3, 1], [4^3, 2^2], [4^3, 2, 1^2], [4^2, 3^2, 2], [4^2, 3^2, 1^2]$,
$[6, 4^2]$	$4\,[4^4], 5\,[4^3, 3, 1], 2\,[4^3, 2^2], [4^3, 2, 1^2], [4^2, 3^2, 1^2]$,
$[5^2, 4]$	$[4^2, 3^2, 2]$.

The seniority operator Q can in the customary manner be expressed through the Casimir operator G of the subgroups [10],

$$2\,Q = G\big(U(2\,R)\big) - G\big(O(2\,R)\big) - A\,. \tag{15}$$

III. Spin-Orbit Pair Correlation in the j–j Coupling Scheme

The approach developed in Section II can be taken over to a consideration of the j–j coupling scheme. The maximal symmetry group is found to be

$$U\big(6\,(A-1)\big) \times U(2^A)\,, \tag{16}$$

whence the total wave function transforms according to the representation

$$U\big(6\,(A-1)\big)_{[r]} \times U(2^A)_{[1]}\,. \tag{17}$$

We now avail ourselves of the spin-orbit pair correlation operator Q as introduced by Flowers [13] and generalized by Arima and Kawarada [14],

$$2\,Q = \sum_{t_1, t_2} \sum_{j, m} (-)^{j-m}\, a^+_{j, m, t_1}\, a^+_{j, -m, t_2} \sum_{j', m'} a_{j', -m', t_2}\, a_{j', m', t_1}\,, \tag{18}$$

where the t denote one of the pair of isospin values.

The interaction matrix is diagonal if the states are classified according to the chain

$$U(2\,R) \supset \mathrm{Sp}(2\,R) \supset O(3)\,, \tag{19}$$

where now

$$R := \sum_j (j + 1/2)\,. \tag{20}$$

We then need to decompose the expression (16) into such subgroups as include the chain (19).

A possible classification of the spin-orbit part of the wave function is provided by the group chain

$$U\big(6\,(A-1)\big)_{[r]} \supset U(6)_{[\varrho]} \times U\,(A-1)_{[\varrho]} \supset U(2\,R)_{[\lambda]} \supset \mathrm{Sp}(2\,R)_{\langle\mu\rangle} \supset O(3)_J\,, \tag{21}$$

i.e., by the set of quantum numbers

$$|\, r, [\varrho], [\lambda]\, \beta, \mu, J \rangle\,. \tag{22}$$

The group $U(6) \times U(A-1)$ has been included for reasons similar to those which led to the inclusion of $U(3) \times U(A-1)$ in the L–S scheme.

For the isospin component we obtain

$$U(2^A)_{[1]} \supset U(2)_{[\tilde{\alpha}]} \times S(A)_{[\tilde{\alpha}]} \,. \tag{23}$$

The considerations which were presented in Section II for the symmetric and unitary groups of the orbital and spin-isospin component of the wave function here have their counterpart as regards the spin-orbit and isospin component. They lead to the result that irreducible representations $I\,R(U(2\,R))$ must be characterized by the identity $[\lambda] \equiv [\alpha]$. Furthermore, the requisite $[\alpha]$ can again be found on making use of the "inner plethysm"

$$\{A-1, 1\} \odot \{\varrho\} = \sum_{\alpha} V_{\varrho\alpha} \{\alpha\} \,. \tag{24}$$

In this case, all diagrams $[\varrho]$ are permitted which have at most 6 rows, and all $[\alpha]$ which have at most 2 columns. There is, once more, a sharp restriction upon the possible representations $[\alpha]$.

On carrying out the remaining reduction along the lines laid down in References [11, 12], one can deduce the complete scheme as

$$\begin{aligned}
&U(6\,(A-1))_{[r]} \times U(2^A)_{[1]} \,;\\
&U(6\,(A-1))_{[r]} \supset U(6)_{[\varrho]} \times U\,(A-1)_{[\varrho]} \supset U(2\,R)_{[\lambda]\equiv[\alpha]} \supset \mathrm{Sp}(2\,R)_{\langle\mu\rangle} \supset O(3)_J \,.\\
&U(2^A)_{[1]} \supset U(2)_{[\tilde{\alpha}]} \times S(A)_{[\tilde{\alpha}]} \,;\\
&U(2)_{[\tilde{\alpha}]} \supset S\,U(2)_T \,.
\end{aligned} \tag{25}$$

With the aid of the Casimir operators, Equation (18) can be expressed [15] in the form

$$2\,Q = G(U(2\,R)) - G(\mathrm{Sp}(2\,R)) + A \,. \tag{26}$$

IV. Concluding Remarks

This extention of KRETZSCHMAR's approach makes a sharp distinction between "physical" and "spurious" states [2], and includes a consideration of excited states, as well as of states having mutually different parity, in a much more direct manner than that to be found in earlier models [3] where only the nucleons lying outside filled shells are taken into account.

Equivalent classification schemes involving quasispin [7, 8] can readily be inserted into our model; e.g., for L–S coupling, this would be accomplished according to the chain

$$U(4^A) \supset U(8) \times U\,(A-1) \supset O(8) \times S(A) \supset U(4) \supset S\,U(2) \times S\,U(2) \,. \tag{27}$$

However, since the L-structure has to be determined by way of the chain $O(2\,R) \supset O(3)$, this can be omitted from the present treatment, as can also the corresponding considerations in j–j coupling.

Basically, our considerations offer the expectation that the incorporation of a pairing interaction to take account of short-range correlations (which have a marked influence [16] upon the relative positions of shell-model energy states) within a scheme in which the diagonality of the interaction operators avoids upsetting the level sequence could lead to quantitative results which might substantially improve upon earlier calculations which are based upon the $SU(3)$ approach summarized in Sections 32–37 of Reference [17] and which have been presented by, e.g., KOLTUN [18] for the intermediate-coupling energy spectra of 1 p-shell nuclei ($A = 6$ to $A = 11$), BRINK [19, 6] for the 0 p-shell and excited configurations of ^{16}O, HARVEY [20] for the 2 s–1 d-shells of ^{17}O, ^{18}O and ^{19}F, and the extensive project for nuclei of the 2 s–1 d shell presaged by References [3, 4, 5]. The model also, of course, lends itself directly to the group-theoretical classification of transitions [21].

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