On Two Gauge Classes and the Invariance of the S Matrix in Quantum Electro Dynamics

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Abstract. Starting with radiation- and Feynman gauge and introducing operator gradient transformations, two classes of gauges are distinguished. The operators of the two classes are not related by gradient transformations. The propagators for both classes are given. Invariance of the S Matrix ander a transformation from one class to the other is proved and it is shown that for internal lines none of the four components of the electromagnetic potential vanish for any gauge.

I. Introduction

The propagators of quantum electro dynamics are known to be affected by ambiguities originating in the invariance of the theory under gauge transformations. Whereas the form of the unrenormalized propagators in different gauges is well established [1][1]), some questions about the connexions between these various forms still remain. For example it is not possible to relate the photon propagators in radiation – and in Feynman gauge by a transformation of the same type as the one that leads e.g. from Feynman – to Landau gauge[2]). In this note we intend to investigate such questions. We distinguish two classes of gauges: the class of non manifestly covariant gauges and that of manifestly covariant gauges. Typical examples are the radiation gauge and the Feynman gauge respectively. We define the two classes by performing operator gradient transformations upon these typical examples. Similar transformations in a more restrictive sense have also been considered by Landau et al. [2]. There arises the question of how to introduce the propagators; we shall attempt to correlate the two distinct methods of Landau [2] and Zumino [1] by using Schwinger's [3] method of functional derivatives in connexion with a generating functional Z, into which we shall directly introduce the operator gradient transformation.

In Section II we first formulate the basic equations of quantum electro dynamics in radiation gauge. We include a proof in their Lorentz covariance. Section III contains the definition of the class of non manifestly covariant gauges. For the propagators we get the same result as Zumino. Section IV deals with the fundaments of Feynman gauge which serves to define the class of manifestly covariant gauges in Section V.

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[1]) Numbers in brackets refer to References, page 450.

[2]) We use the nomenclature of Reference [1].

In the second part of this note, the question is treated as to why, if there is no simple gradient connexion between radiation and Feynman gauge, the S matrix is identical for both cases. The answer is that the so called Coulomb term in the interaction Lagrangian in radiation gauge is essentially a constituent of a relativistic propagator and so brings back the unphysical degrees of freedom which were omited. We give an explicite proof of the invariance of the S matrix under the transition from radiation to Feynman gauge. This proof is valid for a more general case, viz. the invariance under the transition from any gauge of the first class to any of the second, just because each gauge can be transformed by a gradient transformation either into radiation or Feynman gauge. The surface terms appearing in the S matrix for these gradient transformations give the correct generators for the gauge change for in and out fields.

In Section VI we give the S matrix perturbation theory in radiation gauge which, in Section VII, will be used to prove the invariance of the S matrix.

II. The Radiation Gauge

We define the class of non manifestly covariant gauges by the properties found for a special example, the radiation gauge, and by operator gradient transformations which bring us to all other gauges of the class.

A. Fundamental Equations

The radiation gauge is fixed by the non renormalized field equation[3])

$$\Box A_k - \partial_k \dot{A}_0 = - j_k \tag{1.a}$$

$$\triangle A_0 = j_0 \tag{1.b}$$

$$[i \gamma^\mu (\partial_\mu - i e A_\mu) - m] \psi = 0 \tag{1.c}$$

$$\partial_k A^k = 0, \quad j_\mu = \frac{e}{2} [\overline{\psi}, \gamma_\mu \psi], \; \partial^\mu j_\mu = 0 . \tag{1.d}$$

For A_0, there is no dynamical equation; A_0 is totally and for all times prescribed by the charge density of the ψ-field. Integration of (1.b) with the help of the Laplace Green function $\hat{G}(\boldsymbol{x})$.

$$\triangle \hat{G}(\boldsymbol{x}) = - \delta(\boldsymbol{x}) \tag{2.a}$$

$$G(x) = \hat{G}(\boldsymbol{x}) \, \delta(x^0) \tag{2.b}$$

yields the following equation for the photon field[4])

$$\Box A_k = - j_k - \partial_k \partial_l G * j^l \tag{3.a}$$

$$A_0 = - G * j^0 . \tag{3.b}$$

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[3]) Metric: $g_{\mu\nu}$, $g_{00} = -g_{kk} = 1$; latin indices run from 1 to 3, greek ones from 0 to 3. The space components of the vector x^μ are denoted by $\boldsymbol{x}$.

[4]) The star (*) will always stand as a short hand notation for a 4-dimensional integral. For (3.b) e.g.

$$A_0 = - \int dy \, G(x-y) \, j^0(y) .$$

Symbolically these two equations can be united in a single integral equation:

$$A_\mu = A_\mu^{in} + G_{\mu\nu}^{ret} * j^\nu \tag{4.a}$$

$$A_0^{in} = 0, \quad \partial_k A_{in}^k = 0, \quad \Box A_k^{in} = 0 . \tag{4.b}$$

Here the retarded integral kernel is constructed in the following way:

$$G_{00}^{ret} = - G, \qquad G_{0k}^{ret} = G_{k0}^{ret} = 0, \qquad G_{kl}^{ret} = \delta_{kl} D^{ret} + D^{ret} * \partial_k \partial_l G \tag{5}$$

D^{ret} is the usual retarded d'Alembert Green function: $\Box D^{ret}(x) = - \delta(x)$.

A similar integral equation holds for the electron field:

$$\psi = \psi^{in} + S^{ret} * (\gamma^\mu A_\mu \psi) \tag{6}$$

In this expression, A_0 must be thought of as defined by (3.6).

B. Lorentz Convariance

In view of the space-time asymmetry of these equations doubts about their Lorentz covariance might arise. Indeed, they are not covariant under Lorentz transformations, yet they are covariant under combined Lorentz and gauge transformations. Consider the following infinitesimal transformations[5]):

Time translations:

Equations (1.a) and (1.c)

Space translations with the parameter a_k:

$$\delta A_k = a_l \partial^l A_k , \qquad \delta \psi = a_l \partial^l \psi$$

Lorentz rotations with the angle $\omega_{\mu\nu}$:

$$\delta A_k = \omega_{\mu\nu} (x^\mu \partial^\nu A_k - x^\nu \partial^\mu A_k + g_k^\mu A^\nu - g_k^\nu A^\mu) - \partial_k \Lambda$$

$$\delta \psi = \omega_{\mu\nu} \left(x^\mu \partial^\nu \psi - x^\nu \partial^\mu \psi + \frac{1}{2} \gamma^\mu \gamma^\nu \psi\right) + i e \Lambda \psi .$$

We adjust the gauge function $\Lambda(x)$ in these equations so that for all these transformations

$$\delta(\partial_l A^l) = \delta(\partial_l \dot{A}^l) = 0 .$$

Now the generators for the transformation listed above can be found easily with the help of the equal time commutation relations of the system:

$$[\dot{A}_k(x), A_l(y)]_{x^0=y^0} = i \left(\delta_{kl} \delta (\boldsymbol{x} - \boldsymbol{y}) - \partial_k \partial_l \hat{G} (\boldsymbol{x} - \boldsymbol{y})\right) \tag{7.a}$$

$$\{\psi^+(x), \psi(y)\}_{x^0=y^0} = - \delta (\boldsymbol{x} - \boldsymbol{y}) . \tag{7.b}$$

The generators are uniquely determined and one can check that i they fulfil the correct structure relations of the Lorentz group, ii they fix the gauge function in dependence on the rotation angle $\omega_{\mu\nu}$:

$$\Lambda = - \omega_{\mu\nu} G * F^{\mu\nu} , \quad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

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[5]) Whenever A_0 appears in these and the following equations, it is always just an abbreviation for (3.b).

and iii they belong to the right Lagrangian of the system, viz.

$$\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}'$$

$$\mathfrak{L}_0 = -\frac{1}{2}\left(\dot{A}_k \dot{A}^k + \partial_k A_l\, \partial^k A^l\right) + \overline{\psi}\left(\frac{i}{2}\gamma_\mu \overleftrightarrow{\partial^\mu} + m\right)\psi \tag{8.a}$$

$$\mathfrak{L}' = A_k j^k + \frac{1}{2} j_0 G * j_0 . \tag{8.b}$$

By reversing the argument, we can say that if we start with the Lagrangian (8) of our system and deduce from it the generators of the Lorentz group, these generators produce, apart from Lorentz rotations, the necessary gauge transformations such that in each coordinate system $\partial_k A^k = 0$.

Thus the radiation gauge is a Lorentz covariant gauge in the sense that it automatically provides us with the gauge transformation needed to have a transversal field again.

C. *Propagators in Radiation Gauge*

Schwingers [3] generating functional for the propagators in radiation gauge is given by

$$Z[J, \eta, \overline{\eta}] = \langle 0 \mid T \exp\{i \int dz\, (A_k J^k + \overline{\eta}\, \psi + \overline{\psi}\, \eta)\} \mid 0\rangle \tag{9}$$

J_k, η and $\overline{\eta}$ are external c-number currents with respect to which one can build derivatives. Because of (1) and (7) Z satisfies

$$\left\{\Box \frac{1}{i} \frac{\delta}{\delta J} i - (\delta_{ij} - \partial_i \partial_j G *)\left(J^j - \frac{e}{2i}\left[\frac{\delta}{\delta \eta}, \gamma^j \frac{\delta}{\delta \overline{\eta}}\right]\right)\right\} Z = 0 . \tag{10}$$

This equation can be integrated with the help of the causal d'Alembert Green function D^c and the electron propagator $\tilde{G}^c$ for an external c-number field B_μ:

$$D^c(x) : \Box D^c(x) = -\delta(x) \qquad \tilde{G}^c[B_{\mu j x}] : \quad [i \gamma^\mu (\partial_\mu - i e B_\mu) + m]\, \tilde{G}^c[B_\mu; x] = \delta(x) .$$

The index c means that the 'causal' path of integration has to be used. The vacuum polarisation induced by the external field B_μ is now given by

$$F[B_\mu] = \exp\{- \mathrm{Tr} \int dx\, dy \log \tilde{G}^c [B_\mu; x - y]\, \tilde{G}^{c-1}[0; y - x]\} .$$

With this the solution of (10) is given by

$$\begin{aligned} Z[J, \eta, \overline{\eta}] = \exp\Big\{ &- \frac{i}{2} \int dx\, J^i(x) \int dy\, D^c(x - y)\, \Delta_{ij}\, J^j(y) \\ &+ i \int dx\, \overline{\eta}(x) \int dy\, \tilde{G}^c\left[\frac{1}{i} \frac{\delta}{\delta J_i}; x - y\right] \eta(y)\, F\left[\frac{1}{i} \frac{\delta}{\delta J_i}\right] . \end{aligned} \tag{11}$$

For abbreviation we have put

$$\Delta_{ij} = (\delta_{ij} - \partial_i \partial_j G^*) . \tag{12}$$

The non renormalized propagators can now be calculated by

$$\begin{aligned} D^c_{ij}(x - y) &= \frac{1}{i} \frac{\delta}{\delta J^i(x)} \frac{\delta}{\delta J^j(y)} Z[0, 0, 0] \\ G^c(x - y) &= \frac{1}{i} \frac{\delta}{\delta \eta(y)} \frac{\delta}{\delta \overline{\eta}(x)} Z[0, 0, 0] \end{aligned} \tag{13}$$

and are given by

$$D^c_{ij}(x) = \Delta_{ij}\, D^c(x) \quad \text{and} \quad \begin{aligned} G(x) &= \frac{1}{i\,\gamma^\mu\,\partial_\mu - m - i\,\varepsilon}\,\delta(x) \\ &= (i\,\gamma^\mu\,\partial_\mu + m)\,\Delta^c(x)\,. \end{aligned} \tag{14a, b}$$

III. The Class of Non Manifestly Covariant Gauges

In order to distinguish the fields in radiation gauge from those in new gauges to be discussed now we designate the former ones by a subscript 'R'. The new class of fields is defined by the following operator gradient transformations

$$A'_\mu = \underset{R}{A}_\mu + \partial_\mu (c_\nu * \underset{R}{A}^\nu)\,, \quad \psi' = \psi + i\,e(c_\nu * \underset{R}{A}^\nu)\,\psi \quad (\underset{R}{A}_0 = -\,G * j_0\,,\ \partial_k\,\underset{R}{A}^k = 0)\,. \tag{15}$$

These transformations are only well defined when all the fields entering them carry the same time argument. The kernel is therefore of the form

$$c_\nu(x) = \hat{c}_\nu(\boldsymbol{x})\,\delta(x^0)\,.$$

(Further below, the index ν will often be omitted.) We assume $c(x)$ to be a symmetric function of the argument.

It should be stressed that (15) is not a symmetry transformation of the photon-electron system. The new fields A', ψ' will in general obey other commutation relations and other field equations. But since the radiation field $\underset{R}{A}_\mu$ is purely transversal, the new photon field A'_μ also will possess only two degrees of freedom. With (15) and some reasonable assumptions about $c(x)$, the class of non manifestly covariant gauges is introduced. Clearly for a specific kernel c and the appropriate Lagrangian for the new field, the Lorentz covariance can be proved in the same manner as in Section II.

The generating functional for the propagators of the new field is expected to be of the form

$$\begin{aligned} Z'[J, \eta, \bar{\eta}] = \langle 0 \mid T \exp \{ i \textstyle\int dz\, [(\underset{R}{A}_\mu + \partial_\mu\, c * \underset{R}{A})\, J^\mu \\ + \bar{\eta}\,(\psi + i\,e\,c * \underset{R}{A}\,\psi) + (\bar{\psi} - i\,e\,c * \underset{R}{A}\,\bar{\psi})\,\eta] \} \mid 0 \rangle\,. \end{aligned} \tag{16}$$

Using (1) and (7) we get for it the following equation:

$$\begin{aligned} \Big\{ \Box\, \frac{1}{i}\, \frac{\delta}{\delta J^\mu} - (g^i_\mu\, g^j_\lambda\, \Delta_{ij} - \partial_\mu\, g^i_\tau\, g^j_\lambda\, \Delta_{ij}\, c^\tau * - \partial_\lambda\, g^i_\mu\, g^j_\tau\, c^\tau *) \\ \times \Big(J^\lambda - \frac{e}{2\,i} \Big[\frac{\delta}{\delta \eta}\,,\ \gamma^\lambda\, \frac{\delta}{\delta \bar{\eta}} \Big] \Big) \Big\}\, Z' = 0\,. \end{aligned} \tag{17}$$

The first order solution in c reads

$$\begin{aligned} Z'[J, \eta, \bar{\eta}] = \exp \Big\{ \frac{i}{2} \int dx\, (J^\mu + c^\mu * \partial^\lambda\, J_\lambda) \int dy\, D^c\,(x - y)\, g^i_\mu\, g^j_\nu\, \Delta_{ij}\, (J^\nu + c^\nu * \partial^\sigma\, J^\sigma) \\ + i \int dx\, \bar{\eta}(x) \int dy\, \tilde{G}^c \Big[\frac{1}{i}\, \frac{\delta}{\delta J_\mu}\,,\ x - y \Big]\, i\,e\, \Big[(c * \underset{R}{A})\,(x) - (c * \underset{R}{A})\,(y) \Big] \\ \times\, \eta(y)\, F \Big[\frac{1}{i}\, \frac{\delta}{\delta J^\mu} \Big] \Big\}. \end{aligned} \tag{18}$$

For the propagators we establish

$$D^{c'}_{\mu\nu} = (\delta^\sigma_\mu + \partial_\mu c^\sigma *)\,(\delta^\tau_\nu + \partial_\nu c^\tau *)\,\underset{R}{D}{}^c_{\sigma\tau} \tag{19.a}$$

$$G^{c'} = \exp\{(c * \underset{R}{A})\,(x) - (c * \underset{R}{A})\,(y)\}\,\underset{R}{G}{}^c \tag{19.b}$$

Here, $\underset{R}{D}{}^c_{\sigma\tau}$ and $\underset{R}{G}{}^c$ stand for the propagators of Equation (14), with the obvious meaning $\underset{R}{D}{}^c_{\sigma\tau} = g^i_\sigma\, g^j_\tau\, D^c_{ij}$.

Equation (19.a) was first given by Zumino [1].

IV. Feynman Gauge

In a similar way as in Section III we intend to introduce the class of manifestly covariant gauges by a special example, the Feynman gauge, and by operator gradient transformations which define all other gauges of this class.

A. *Feynman Gauge*

The basic equations are

$$\Box\,\underset{F}{A}{}_\mu = -\,j_\mu \tag{20.a}$$

$$[i\,\gamma^\mu\,(\partial_\mu - i\,e\,\underset{F}{A}{}_\mu) - m]\,\psi = 0 \tag{20.b}$$

$$[\dot{\underset{F}{A}}{}_\mu(x),\,\underset{F}{A}{}_\nu(y)]_{x^0=y^0} = i\,g_{\mu\nu}\,\delta\,(\boldsymbol{x} - \boldsymbol{y}) \tag{20.c}$$

$$\{\psi^+(x),\,\psi(y)\}_{x^0=y^0} = \delta\,(\boldsymbol{x} - \boldsymbol{y})\,. \tag{20.d}$$

The relativistic covariance of this gauge is obvious. In contradistinction to the class of gauges considered until now, $\underset{F}{A}{}_\mu$ describes 4 degrees of freedom. In momentum space this requires, apart from the creation and destruction operators for transversal photons, the introduction of longitudinal and timelike photons which should compensate each other. It is therefore clear that the $\underset{F}{A}{}_\mu$ cannot be constructed from the $\underset{R}{A}{}_\mu$ by a simple gradient transformation of the type (15), since at least the polarisation of the timelike photons parallel to the timelike unit propagation vector n_μ ($n_\mu\, n^\mu = 1$) cannot be written in gradient form.

B. *Propagators in Feynman Gauge*

For the generating functional we try again

$$Z\,[J,\eta,\overline{\eta}] = \langle 0 \mid T \exp\{i \int dz\,(\underset{F}{A}{}_\mu\, J^\mu + \eta\,\overline{\psi} + \psi\,\overline{\eta})\} \mid 0\rangle \tag{21}$$

Equations (20) give for it

$$\left\{\Box\,\frac{1}{i}\,\frac{\delta}{\delta J^\mu} - \left(J_\mu - \frac{e}{2\,i}\left[\frac{\delta}{\delta\eta},\,\gamma_\mu\,\frac{\delta}{\delta\overline{\eta}}\right]\right)\right\} Z = 0\,. \tag{22}$$

The solution is

$$Z[J,\eta,\overline{\eta}] = \exp \frac{i}{2} \int dx\, J_\mu(x) \int dy D^c(x-y)\, J^\mu(y)$$
$$+ i \int dx\, \overline{\eta}(x) \int dy\, \tilde{G}^c \left[\frac{1}{i} \frac{\delta}{\delta J_\mu}; x-y\right] \eta(x)\, F\left[\frac{1}{i} \frac{\delta}{\delta J^\mu}\right] \tag{23}$$

and for the photon propagator we have

$$\underset{F}{D^c_{\mu\nu}} = - g_{\mu\nu}\, D^c . \tag{24}$$

V. The Class of Manifestly Covariant Gauges

We consider again operator gradient transformations which we expect to lead to all gauges of the manifestly covariant class:

$$A'_\mu = \underset{F}{A}_\mu + \partial_\mu (d_\nu * \underset{F}{A}^\nu), \quad \psi' = \psi + i\, e\, (d_\nu * \underset{F}{A}^\nu)\, \psi . \tag{25}$$

For the sake of definiteness, $d_\nu(x)$ contains again a δ-function in the time argument, and we furthermore assume it to be symmetric.

The transformation (25) will in general change the field equations as well as the commutation relations of the system. But it will always yield a field with 4 degrees of freedom. Also, for reasonable kernels $d(x)$, the new field will satisfy again Lorentz covariant equations. Some possible choices for $d(x)$ which have been used in the literature are summarized by PERÈS [4].

For the generating functional of the propagators in the class of manifestly covariant gauges we put

$$Z'[J,\eta,\overline{\eta}] = \langle 0 \mid T \exp \{ i \int dz\, [(\underset{F}{A}_\mu + \partial_\mu\, d * \underset{F}{A})\, J^\mu$$
$$+ \overline{\eta}\, (\psi + i\, e\, d * \underset{F}{A}\, \psi) + (\overline{\psi} - i\, e\, d * \underset{F}{A}\, \overline{\psi})\, \eta]\} \mid 0 \rangle . \tag{26}$$

It obeys the equation

$$\left\{ \Box \frac{1}{i} \frac{\delta}{\delta J^\mu} - (g_{\mu\nu} + \partial_\mu d_\nu * + \partial_\nu d_\mu *) \left(J^\nu + \frac{e}{2i} \left[\frac{\delta}{\delta \eta}, \gamma^\nu \frac{\delta}{\delta \overline{\eta}} \right] \right) \right\} Z' = 0 . \tag{27}$$

The solution is

$$Z'[J,\eta,\overline{\eta}] = \exp \left\{ \frac{i}{2} \int dx\, J_\mu(x) \int dy\, D^c(x-y)\, J^\mu(y) \right.$$
$$+ \frac{i}{2} \int dx\, J^\mu(x) \int dy\, (\partial_\mu d_\nu (x-y) + \partial_\nu d_\mu (x-y))\, J^\nu(y)$$
$$+ i \int dx\, \overline{\eta}(x) \int dy\, \tilde{G}^c \left[\frac{1}{i} \frac{\delta}{\delta J_\mu}; x-y \right] \eta(y) . \tag{28}$$
$$F\left[\frac{1}{i} \frac{\delta}{\delta J^\mu}\right] i\, e\, \big((d * \underset{F}{A})\, (x) - (d * \underset{F}{A})\, (y) \big) .$$

By developing for small d we find for the propagators:

$$D^{c'}_{\mu\nu}(x) = \underset{F}{D^c_{\mu\nu}}(x) + \partial_\mu d_\nu(x) + \partial_\nu d_\mu(x) \tag{29.a}$$

$$G^{c'}(x) = \exp \{ i\, e\, \big((c^* \underset{F}{A})\, (x) - (c^* \underset{F}{A})\, (y) \big) \}\, \underset{R}{G^c}(x) . \tag{29.b}$$

These transformation properties of the propagators were first given by Landau. Zumino's result follows from the special case

$$d_\nu(x) = \frac{1}{2}\,\partial_\nu\, M(x)\,.$$

VI. S-Matrix in Radiation Gauge

Going back to Equations (4) and (6) and similar equations for the advanced boundary condition we define the S matrix in the usual way by solving for the in and out fields

$$A_k^{out} = S^+ A_k^{in}\, S\,, \qquad \psi^{out} = S^+ \psi^{in}\, S\,. \tag{30}$$

In perturbation theory S for radiation gauge is given by

$$S = T\, e^{i \int dx\, \mathfrak{L}'_{in}(x)} \tag{31.a}$$

$$\mathfrak{L}'_{in} = A_k^{in}\, j_{in}^k + \frac{1}{2}\, j_{in}^0\, G * j_{in}^0\,. \tag{31.b}$$

The A_k^{in} are the free fields obeying (4.b). For them we shall use the commutation relations

$$[A_k^{in}(x),\, A_l^{in}(y)] = \frac{1}{i}\, D_{kl}\,(x-y) = \frac{1}{i}\, \Delta_{kl}\, D\,(x-y) \tag{32.a}$$

$$\{\psi^{in}(x),\, \overline{\psi}^{in}(y)\} = \frac{1}{i}\, S\,(x-y)\,. \tag{32.b}$$

With the help of these equations we are going to calculate the S matrix in radiation gauge. For (31.a) we get in a somewhat abbreviated, but self-explanatory notation

$$S = \sum_{n=0}^{\infty} \int (dx)^n\, \frac{i^n}{n\,!} \binom{n}{m}\, T\left[(A_k^{in}\, j_{in}^k)^m \left(\frac{1}{2}\, j_{in}^0\, G * j_{in}^0\right)^{n-m}\right]. \tag{33}$$

For a fixed value of $n > 0$ each term in the sum over m in (33) contains still various powers of the coupling constant e ($j_\mu^{in} \sim e$). It is therefore convenient to reorder the sums and to substitute $p = m + 2\,(n-m)$ so that only terms with the same power e^p correspond to a fixed value of p.

Furthermore, to see the effects of the radiation gauge, it is only necessary to consider in p^{-th} order those processes which have $p - 2\,q$ external photon lines. Using Wick's theorem (some details of this calculation are given in appendix) we get[6])

$$S^{(p)} = \sum_m{}^{+} \int (dx)^p\, \frac{i^{p-q}}{(p-m/2)\,!}\; \frac{1}{(p-2\,q)\,!\,\mu\,!\,2^\mu}\, (D_{k\,k}^c)^\mu$$
$$: A_i^{in}(x_1)\,\ldots\, A_i^{in}\,(x_p - 2\,q);\; T\left[\left(\frac{1}{2}\, j_{in}^0\, G * j_{in}^0\right)^{p-m/2} (j_{in}^k)^m\right], \tag{34}$$

where $\mu = p - m/2 + q$ and $+$ means that the sum runs over the values $m = p,\, p-2,\, p-4,\, \ldots$ down to 1 or 0 depending on whether p is even or odd. Discussion of (34):

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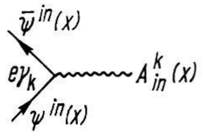
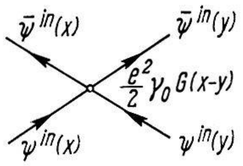
[6]) The double k for the photon propagator index is not meant to be summed over, but stands as an abbreviation for a series of space indices to be contracted with j_{in}^k.

For a fixed value of p and for $m = p$ we get the usual graphs of the Feynman-theory which are only modified in that i $A_k^{in}(x)$ describes free photons in transversal external states, ii the photon propagator to be used for internal lines is (14.a), and iii each vertex contains factor $e\,\gamma_k$;

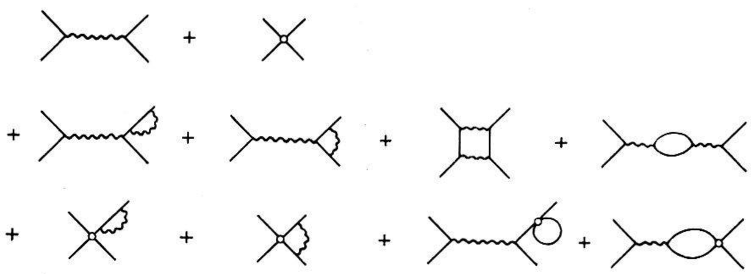


For $m = p - 2$ new graphs are obtained from the above ones by replacing one internal photon line at a time by a vertex of the following type:



The internal photon line is replaced by a vertex function $e^2\,\gamma_0\,G\,(x - y)/2$.

For $m = p - 4$ we have to replace 2 internal photon lines at a time by the new vertex, and so on. External photon lines and electron lines are always the same as in the original graphs. As an example we write down the topologically different diagrams of Møller scattering in 4th order:



VII. General Invariance of the S-Matrix

We shall prove the invariance of the S matrix under a transformation from one class of gauges to the other in the following steps: We show first that the formulation of the theory in radiation gauge can be replaced by an equivalent intermediate theory which makes use of the unphysical photons by means of A_0. Then we show that the propagators of this new theory are related to those of Feynman gauge by a gradient transformation (25), under which the S matrix is invariant. Since each gauge can be transformed by the aid of a gradient transformation into either radiation or Feynman gauge, the proof for the general invariance is completed.

The intermediate theory mentioned above is defined by the equations

$$\Box\, A_\mu - \partial_\mu\, \dot{A}_0 = -\, j_\mu\,, \tag{35.a}$$

$$[i\,\gamma^\mu\,(\partial_\mu - i\,e\,A_\mu) - m]\,\psi = 0\,, \tag{35.b}$$

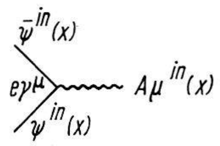
$$\mathcal{L}'_{in} = A^{in}_\mu\, j^\mu_{in} \tag{35.c}$$

and the photon propagator (compare with (5))

$$G^c_{\mu\nu}(x) = g^k_\mu \, g^l_\nu \, D^c_{kl}(x) - g^0_\mu \, g^0_\nu \, G(x) \tag{36}$$

with $D^c_{kl}(x)$ from (14.a) and $G(x)$ from (2.b).

The vertex to be used here is



Notice that longitudinal as well as timelike photons appear.

Now the equivalence of this formulation with the original radiation gauge is seen directly by computing the S matrix in p^{-th} order and concentrating again upon processes with $p - 2q$ external photon lines. From (35.c) we get

$$S^{(p)} = \int (dx)^p \, \frac{i^p}{p!} \, \frac{p!}{(p-2q)!} \, \frac{1}{q!\,2^q} \left(\frac{1}{i} \, G^c_{\mu\mu}\right)^q : A^{in}(x_1) \ldots A^{in}(x_{p-2q}) \colon T(j^\mu_{in})^p .$$

In a symbolic notation we write $G^c_{\mu\mu} = D^c_{kk} + G$ and put G with a factor $1/2$ under the T product:

$$S^{(p)} = \sum_{\nu=1}^{q} \int (dx)^p \, \frac{i^{p-q}}{(\nu-q)!} \, \frac{1}{(p-2q)!\,\nu!\,2^\nu} \, (D^c_{kk})^\nu$$

$$: A^{in}_\lambda(x_1) \ldots A^{in}_\lambda(x_{p-2q}) : T\left[\left(\frac{1}{2} \, j^0_{in} \, G * j^0_{in}\right)^{q-\nu} (j^k_{in})^\nu \, j^\lambda_{in}(x_1) \ldots j^\lambda_{in}(x_{p-2q})\right] . \tag{37}$$

Comparison of (37) with (34) and identification of γ with $\mu = p - m/2 + q$, $\nu = q, q-1, q-2, \ldots$; $m = p, p-2, p-4, \ldots$ completes the proof of the first step.

From these considerations the meaning of the Coulomb term in the Lagrangian (31.b) becomes clear: it is part of a relativistic propagator and, in the development of the S matrix, it gives all those contributions which in the intermediate theory are provided by the operators of the unphysical photons. Though radiation gauge contains only transversal photons in the external lines, all 4 components of the electromagnetic potential are used in the internal lines.

The last step is to show that $G^c_{\mu\nu}(x)$ is correlated to $\underset{F}{D^c_{\mu\nu}}(x) = -g_{\mu\nu} \, D^c$ by a transformation (25). This is indeed the case, if we put

$$d_\nu(x) = \partial_\nu \, d(x)$$

$$d(x) = -\frac{1}{2} \int dz \; G(x-z) \, D^c(z) . \tag{38}$$

So the operators of the intermediate theory (we denote them by $\underset{I}{A}_\mu$) are given by

$$\underset{I}{A}_\mu = \underset{F}{A}_\mu + \partial_\mu \, \partial_\nu \, d * \underset{F}{A}^\nu . \tag{39}$$

Constructing the S matrix in the intermediate theory according to

$$\underset{I}{S} = T \exp \left\{i \int dz \; j^\mu_{in}(z) \, (\underset{F}{A}^{in}_\mu + \partial_\mu \, \partial_\nu \, d * \underset{F}{A}^\nu_{in})\right\} , \tag{40}$$

we can integrate partially and use the continuity equation for j^μ. As a consequence, S matrix in Feynman gauge will appear, but since $d(x)$ does not vanish for infinite times, we get surface terms:

$$\underset{I}{S} = \exp \{i \int_{t\to-\infty} d^3x\, j^0_{in}(x)\, \partial_\nu\, d * \underset{F}{A^\nu_{in}}\}\, \underset{F}{S} \exp \{i \int_{t\to+\infty} d^3x\, j^0_{in}(x)\, \partial_\nu\, d * \underset{F}{A^\nu_{in}}\}\,.$$

With the aid of the field equations, we have

$$\int_{t\to\pm\infty} d^3x\, j^0_{in}\, \partial_\nu\, d * \underset{F}{A^\nu_{in}} = \int_{t\to\pm\infty} d^3x\, \Box\, \underset{F}{A^{in}_0}\, \partial_\nu\, d * \underset{F}{A^\nu_{in}} = C_\pm\,. \tag{41}$$

These are just the generators for the transformation (25). So for

$$\underset{I}{A^{out}_\mu} = \underset{I}{S^+}\, \underset{I}{A^{in}_\mu}\, \underset{I}{S}$$

follows with

$$\underset{I}{S} = e^{-iC_-}\, \underset{F}{S}\, e^{iC_+} \quad : \quad \underset{F}{A^{out}_\mu} = \underset{F}{S^+}\, \underset{F}{A^{in}_\mu}\, \underset{F}{S}\,.$$

This completes the proof of the invariance of the S matrix.

VIII. Conclusions

Starting with electrodynamics in radiation and in Feynman gauge, we have first defined by the use of gradient transformations two classes of gauges: the non manifestly covariant class which describes only transversal photons in a positive definite Hilbert space, and the manifestly covariant class which makes use of four photon states in an indefinite Hilbert space. Field operators and propagators belonging to different classes cannot be connected by simple gradient transformations since time-like photons are polarized along the unit time-like vector. We have then treated the S matrix in perturbation theory. That S matrix is invariant under transformations of a single gauge class can be seen easily since in the integral $\int A^{in}_\mu j^\mu_{in}\, dx$ the gradient can be transferred to j^μ_{in} by partial integration and the continuity equation can be used. The surface terms for infinite time values are the generators of the gradient transformations which relate the two in fields. Finally, the invariance of the S matrix was proved under transformations leading from one class to the other. This was possible through the Coulomb term in the Lagrangian of a non manifestly covariant gauge which reintroduces the effects of the unphysical photons for internal photon lines. In the virtual states the electromagnetic field always enters with all four degrees of freedom.

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Appendix

We show here some details of the calculation leading from (34) to (37).

Reordering the sums in (34) to equal powers of e by the substitution of $p = m + 2\,(n - m)$ we find

$$S = \sum_{p=0}^{\infty} \sum_m{}^* \int (dx)^p \frac{i^{p+m/2}}{m!\,(p-m/2)!}\, T\left[\left(\frac{1}{2}\, j^0_{in}\, G * j^0_{in}\right)^{p-m/2} (j^k_{in})^m\right] T(A^{in}_k)^m\,,$$

where the star * for the m sum means that m assumes the values $m = p, p-2, p-4, \ldots,$ 1 or 0 depending on whether p is even or odd.

We then restrict ourselves to processes in p^{-th} order which contain $p - 2q$ external photon lines. Consider only the part with the photon operators:

$$S^{(p)} = \sum_m{}^* \int (dx)^p \ldots T\,[A_i^{in}(x_1)\, A_j^{in}(x_2) \ldots A_l^{in}(x_m)]\,.$$

For each value of m we use Wick's theorem and make all possible contractions until only

$$: A_i^{in}(x_1) \ldots A_k^{in}(x_{p-2q}) :$$

remains. We get first, contracting in $S^{(p)}$ only from the right hand side over neighbouring A^{in}, for $m = p$ the q functions

$$\frac{1}{i}\, D_{lr}^c\,(x_p - x_{p-1})\, \frac{1}{i}\, D_{rs}^c\,(x_{p-2} - x_{p-3}) \ldots \frac{1}{i}\, D_{tk}^c\,(x_{p-2q+2} - x_{p-2q+1})\,,$$

for $m = p - 2$ only $q - 1$ such functions (the first one is missing), etc. In the general case for arbitrary m we get μ contraction functions D_{rs}^c where μ is fixed by $m - 2\mu = p - 2q$ or $\mu = (p - m)/2 + q$. Up till now, $S^{(p)}$ has the form

$$S^{(p)} = \sum_m{}^* \int (dx)^p \ldots \left(\frac{1}{i}\, D_{ll}^c\right)^\mu : A_i^{in}(x_1) \ldots A_k^{in}(x_{p-2q}):$$

Now we also take contractions over non neighbouring photon operators into account. For the arguments $x_m \ldots x_{p-2+1}$ there is a bigger number of contractions than those considered above. The factor is

$$\frac{m\,(m-1) \ldots m - 2\mu + 1}{\mu\,!\; 2^\mu}\,.$$

With this, we get finally

$$S^{(p)} = \sum_m{}^* \int (dx)^p\, \frac{i^{p+m/2}}{m\,!\,(p - m/2)\,!}\, \frac{m\,!}{(m-2\mu)\,!}\, \frac{1}{\mu\,!\; 2^\mu} \left(\frac{1}{i}\, D_{kk}^c\right)^\mu$$

$$: A_i^{in}(x_1) \ldots A_i^{in}(x_{p-2q}) : T\left[\left(\frac{1}{2}\, j_{in}^0\, G * j_{in}^0\right)^{p-m/2} (j_{in}^k)^m\right]$$

or, with $m - 2\mu = p - 2q$, the result (37).

Note that in our abbreviated notation repeated indices $k\,k$ in the propagator stand for the space type of different summation indices to be contracted with the different currents j_{in}^k.

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