Generalized localizability

Generalized Localizability

by **J. M. Jauch** and **C. Piron**

Institute of Theoretical Physics, University of Geneva, Geneva, Switzerland

(18. II. 67)

Abstract: It is well known that particles of restmass zero with spin $\neq 0$ belonging to an irreducible representation of the Lorentzgroup do not admit position operators. Yet such particles exist in nature (for instance the neutrino and the photon) and they are localizable in the experimental sense of this term.

The mathematical description of localizability is generalized in this paper so as to be applicable in cases where the conventional position operator does not exist. The generalization consists in omitting the hypothesis that all observations of position measurements are compatible with one another. Compatibility is maintained only for space domains which do not overlap or for which one is contained inside the other. For the other cases of overlapping domains compatibility cannot be justified on empirical grounds and it can be dropped. The resulting mathematical object is a *generalized system of imprimitivities* and it is the appropriate concept for the mathematical description of certain localizable systems.

We give a standard method for constructing such systems based on a theorem of Neumark. A particle which is localizable in this sense is called *weakly localizable.*

A further weakening of the conditions leads us to the notion of *nearly localizable* systems. In this case there exists no states which localize the particle exactly in a given space domain, but only states which approximate this property to an arbitrary degree of accuracy (in the topology induced by the states). We have verified that particles of mass $m = 0$ and spin $^1/_2$ (neutrinos) are nearly localizable. In addition we have verified that the photons are in fact also weakly localizable.

I. Introduction

The purpose of this paper is to suggest and discuss some ideas about a generalized notion of localizability within the conceptual frame of quantum mechanics. The preliminary results obtained with the use of this notion justify our hope that it might become a fruitful subject for further investigations.

The proposal that we offer is a radical departure from the conventional theory and we give therefore a rather more elaborate motivation than it would be usual for such a preliminary communication.

From the very beginning in the history of quantum mechanics the notion of localizability has been intimately interwoven with the conceptual frame of general quantum mechanics. In fact quantum mechanics of the realy days was essentially the theory of localizable systems [1]. Yet it is important to realize that the logical structure of quantum mechanics is independent of that of localizability. The latter is intimately connected with the mathematical representation of space and time by a continuum of four dimensions. A notion which has been transferred without change from classical into quantum mechanics.

Such a representation of space and time constitutes an abstraction from a collection of rather general experiences, none of them detailed enough to really justify such an abstraction into the vast domain of the continuum where no empirical foundation is available. We feel therefore that a study of the notion of localizability might be a fruitful undertaking in preparation for a general theory of elementary particles.

We are reinforced in this believe by the remarkable stability of general quantum mechanics under the onslaught of new experimental facts in elementary particle physics. Whatever facts became known in the course of recent years, no matter how startling and unexpected they may have been, they did not lead to a basic modification of quantum mechanics.

For instance the discovery of isotopic spin, strangeness, baryonic and leptonic numbers etc. require for their descriptions merely additional dimensions of the underlying Hilbert space and the various multiplets are adequately described by the irreducible representations of certain symmetry groups in that space. Similarly the violation of parity in weak interactions merely requires for its quantitative description a modification of the Hamiltonian. The various restmasses of the resonances are parameters in the theory for which the general frame of quantum mechanics gives no restrictions, so that this frame is in fact compatible with any value of these parameters.

But this very insensitivity of the general concept structure is just one of the unsatisfactory features of the present theoretical situation. Very few of these properties which are being discovered can be related in any natural way to each other or to a more fundamental principle from which they could be derived.

In order to illustrate this it is convenient to recall that the structure of the quantum mechanics of elementary particles presents itself on three levels of increasing specialization.

1) *General quantum mechanics*, described by a certain lattice of elementary (yes-no) experiments. The structure of this lattice is determined by the nature of the system alone and it is independent of the state of the system [2] [3].

2) *Localizability* in a homogeneous and isotropic space described mathematically by a system of imprimitivities with respect to the Euclidean group of motions [6].

3) *The dynamical law* described by a one parameter group of evolution of states [7].

The three items of this schema are not independent of one another. For instance not every lattice will admit localizability. In fact the latter property is one of the strongest arguments against the modular lattices as has been pointed out by one of us [2]. On the other hand the dynamical structure is not independent of localizability either. This was illustrated for instance by Inönü and Wigner [4] who showed that the vectorrepresentation of the Galileo group is incompatible with localizability. In a relativistic theory on the other hand localizability imposes certain restrictions on the representation of the Poincaré group which unites the dynamical law with the space-time symmetry. This restriction is such that localizability excludes the value of spin $s \neq 0$ as well as continuous spin for particles of mass $m = 0$ [5] [6].

This last point alone would suffice to show that something is amiss with our present notion of localizability because according to the above result both neutrinos and photons would fall outside the category of localizable particles if their restmass is indeed exactly zero and not only unobservably small. This result is in such flagrant contradiction with the usual physical picture of a particle that the status of a photon as a particle, for instance, is seriously challenged. Yet for an experimental physicist it seems meaningful and consistent to speak of the localization of an individual photon in a finite region of space and time as demonstrated for instance in the beautiful experiments of JANOSSY et al. [8]. We encounter here a situation where the mathematical structure is not conform to the physical reality. This seems therefore quite a convincing reason to change this mathematical structure.

We might perhaps mention here that the nonlocalizability of the photon was known to PAULI as early as 1932 [9]. PAULI identified localizability with the existence of a positiv definite probability density which satisfies a differential conservation law and which has the correct transformation properties under the Lorentz transformation. Pauli was most explicit in his affirmation that such an object does not exist for photons.

A different aspect of the problem of localizability is brought to light in the various attemps of defining a position operator in a relativistic theory [10, 11, 12]. In this form there seems to remain an ambiguity even for restmass $m \neq 0$ which is difficult to reconcile with the unique definition of the probability density for such particles with spin. The resolution of this paradox was given by GALINDO [13] who showed that the regularity condition of NEWTON and WIGNER [5] admits different interpretations and that only one of them is compatible with the explicit form of the position operator as given by NEWTON and WIGNER.

These experiences indicate that the best way to study the notion of localizability is by studying the concept of the "systems of imprimitivities". This concept, introduced by MACKEY [14] [15] in its modern form, is the adequate transcription into mathematical language of the physical concept of localizability. It has the advantage that with it one can express this concept with relatively modest effort in full mathematical rigor and that there exist important representation theorems which permit an exhaustive classification of all the representations of such systems.

We shall therefore try to generalize the notion of localizability by defining generalized systems of imprimitivities. The nature of this generalization will be motivated by physical considerations.

One of the advantages of systems of imprimitivities is precisely that this notion is still meaningful in situations where a position operator no longer exists for instance in the case of a system subject to certain constraints.

In the next section we give a brief description of localizability in term of systems of imprimitivities for the reader who is not familiar with this notion and who does not have the time to work through the difficult mathematical literature.

II. Localizability

The notion of localizability comprises two different aspects, one which expresses the fact that there exists a family of observables each of which corresponds to the question whether the system is contained within a certain region in space, and another which expresses the homogeneity and isotropy of space.

In order to translate the first of these properties into an appropriate mathematical language we adopt the following notation. We denote by $M = R^3$ the Euclidean space of three dimensions, by $\mathfrak{B}$ the Borel sets of this space, $\Delta \in \mathfrak{B}$ a general Borel set, Δ' the complementary set and ϕ the nul set. Two sets Δ_1, Δ_2 are *disjoint* (denoted by $\Delta_1 \perp \Delta_2$) if their intersection is the nul set.

We assume that to each Borel set Δ is associated a projection E_Δ in a complex Hilbert space $\mathfrak{H}$. The function $\Delta \to E_\Delta$ shall satisfy the following properties

$$E_\phi = 0 , \qquad E_M = I . \tag{1}$$

For any sequence Δ_i such that $\Delta_i \perp \Delta_j$ for $i \neq j$

$$E_{\cup \Delta_i} = \sum_i E_{\Delta_i} . \tag{2}$$

Condition (2) then implies that for any pair Δ_1, Δ_2

$$E_{\Delta_1 \cap \Delta_2} = E_{\Delta_1} E_{\Delta_2} \equiv E_{\Delta_1} \cap E_{\Delta_2} \tag{3}$$

and that all the projections E_Δ commute with each other. A function from the Borel sets to the projections in a Hilbertspace which satisfies conditions (1) and (2) is called a *spectral measure*. Localizability is thus always represented by a spectral measure.

The second of the above mentioned properties, viz. homogeneity and isotropy of space, is described mathematically in the following manner: We are given a group G acting transitively on the points $q \in M$. Thus to each pair $q \in M$ and $x \in G$ there exists an element $[q]x \in M$ called the image of q under the action of the group element. Furthermore for any pair q_1, $q_2 \in M$ there exists at least one $x \in G$ such that $q_2 = [q_1]x$. In the case of physical particles in a free space the group G is the six-parameter Lie group of Euclidean motions.

The requirement that these motions are induced by symmetry transformations is expressed in the following way. There exist a representation $x \to U_x$ of the group G by unitary operators acting in the Hilbertspace $\mathfrak{H}$ and satisfying the fundamental property

$$E_{[\Delta]x} = U_x^{-1} E_\Delta U_x \qquad \forall \; x \in G . \tag{4}$$

Where $[\Delta]x = \{q \mid [q]x^{-1} \in \Delta\}$ is the set of all $q \in M$ which are image points of the points in Δ.

The Equ. (1), (2), and (4) define a *transitive system of imprimitivities.*

We shall add some sundry remarks to this formal definition in order to clarify some points which might appear puzzling to a physicist.

1) We have intentionally used a notation of sufficient generality which can accomodate more general situations than the one considered here. In the particular case of a single particle the space M is of course to be identified with the Euclidean space R^3 of three dimensions and the group G will then be the 6-parameter Lie group of Euclidean motions. This case can always serve as example for the illustration of general theorems.

2) We have reduced the conditions imposed on the function $\Delta \to E_\Delta$ to the minimum. In particular we did not specify whether the sequences of disjoint sets Δ_i are finite or infinite. If we required condition (2) only for finite sequences (which would be physically more desirable) we could always, by a standard procedure, extend the

function in a unique manner to one for which (2) is true for infinite sequences. (cf. ref. [7] appendix I).

3) The commutativity of all the projections which is affirmed after Equ. (3) is an immediate consequence of the following two facts

a) Every pair of Borel subsets Δ_1 and Δ_2 can be decomposed in a unique manner into three mutually disjoint Borel subsets A_1, A_2, and B such that $\Delta_1 = A_1 \cup B$, $\Delta_2 = A_2 \cup B$.

b) The sum of two projections $E_1 + E_2$ is a projection only if they commute. Thus let $E_{\Delta 1}$ and $E_{\Delta 2}$ for any pair Δ_1 and Δ_2. We write then $\Delta_1 = A_1 \cup B$, $\Delta_2 = A_2 \cup B$ and denote by $E_1 = E_{A_1}$, $E_2 = E_{A_2}$, $E_3 = E_B$. From property (2) follows then that $E_{\Delta_1} = E_1 + E_3$, $E_{\Delta_2} = E_2 + E_3$ and $E_1 + E_2$ are projections. Thus the three projections E_1, E_2, E_3 commute pairwise. Consequently so do E_{Δ_1} and E_{Δ_2}.

4) We have adopted a convention which results in an antihomomorphism of the group G into the functions $[q]x$ on the space M. This has certain advantages and it is in agreement with the convention adopted by Mackay. It would be very easy to change the convention and the notation so as to obtain a homomorphism instead.

5) We have not specified that the representation shall be a vectorrepresentation. It could be a rayrepresentation. For the Euclidean group this distinction is not important.

6) It is useful to restrict ones attention to so called irreducible systems of imprimitivities. With this we mean if Q is a projection which commutes with both the set of all E_Δ and the set of all the U_x then Q is trivial.

The so called imprimitivity theorem (15) gives a complete characterization of all the irreducible systems of imprimitivities and thereby a determination of all localizable systems. The result is as follows: Denote by G_0 the subgroup of G consisting of these elements x which leave an arbitrary but fixed point $q_0 \in M$ invariant. Thus

$$G_0 = \{x \mid [q_0]x = q_0\}\,. \tag{5}$$

To every irreducible representation of G_0 there corresponds exactly one equivalence class of irreducible representations of the system (1), (2), and (4).

For the Euclidean group which interests us especially the subgroup G_0 consists of all rotations around a fixed point. The irreducible representations of this group are all known. Each of them is finite-dimensional of dimension $2s + 1$ where s assumes one of the values $s = 0, {}^1/_2, 1, {}^3/_2, \ldots$ The number s is the spin of the particle.

The cases of integer spin s give true vector representations of the rotation group G_0, the cases of half integer spin are true vectorrepresentations of the covering group of G_0 but they are representations up to a factor ± 1 of G_0.

This theorem tells us therefore, that localizability as formulated here leaves no room for any generalizations of the well-known cases of particles of rest mass $m \neq 0$ with spin s. Thus particles with spin different from zero and rest mass zero are not localizable in this sense. Neither are the irreducible relativistic systems with continuous spin discovered by Wigner.

III. Weak localizability

We turn now to the question how best to generalize the notion of localizability. In this endeavour we let ourselves be guided by the physical interpretation of the various conditions (1) ... (4) which we have formulated for a system of imprimitivities.

We have remarked that (2) alone suffices to affirm that all projections E_Δ commute with one another. This would mean physically that every position measurement is compatible with every other one. We have reasons to believe that this compatibility is verified for certain pairs of measurements, but these reasons are not equally compelling for all of them. Let us for instance consider the situation of Figure 1 (a).

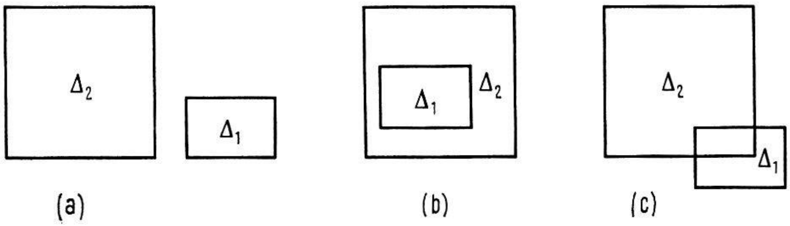

Figure 1

Three situations for pairs of spatial regions. Compatibility of corresponding measurements is abandoned for case (c).

In that case compatibility of simultaneous measurements of the projections F_{Δ_1} and F_{Δ_2} is the adequate expression of the notion of microcausality. In a relativistic theory this property is often extended to the measurement of all local observables and it is then justified by the fact that no physical signals can be propagated faster than the speed of light. But even in a non-relativistic theory without limiting velocity it is not unreasonable to suppose that a measurement of F_{Δ_1} which gives the result yes is equivalent with a measurement of F_{Δ_2} with the result no, and vice versa. The reasonableness is due to the fact that such measurements can actually be performed and are in fact being performed daily in many laboratories (although not for the purpose of proving compatibility). All the known results are in perfect agreement with the above mentioned relation between the two measurements. This suffices to affirm the compatibility of these two measurements.

In the case of Figure 1 (b) we can use a similar argument insofar as a measurement of the projection F_{Δ_1} with positive outcome implies a measurement of F_{Δ_2} with the same result. This means that the two projections must stand in the relationship $F_{\Delta_1} \leq F_{\Delta_2}$ and it is well known that this implies that they commute.

In the case of Figure 1 (c) we have no such arguments to affirm that the two measurements are compatible. We shall therefore admit that they may not be.

From the remarks which we have made in connection with properties (1), (2), and (3) of a spectral measure it follows that we have to abandon the property (2) which implies commutability of all projections. We are therefore lead to the following definition for a generalized spectral measure.

To each Borel set $\Delta \subset M$ there is associated a projection F_Δ in a Hilbertspace $\mathfrak{H}$. The function $\Delta \to F_\Delta$ satisfies

$$F_\phi = 0, \qquad F_M = I \tag{1)'$$

$$\Delta_1 \perp \Delta_2 \Rightarrow F_{\Delta_1} \perp F_{\Delta_2} \tag{2)'$$

$$F_{\Delta_1 \cap \Delta_2} = F_{\Delta_1} \cap F_{\Delta_2} . \tag{3)'$$

Since not all projections commute with each other we can no longer express $F_{\Delta_1} \cap F_{\Delta_2}$ in the simple manner as the product of the two projections. Instead $F_{\Delta_1} \cap F_{\Delta_2}$ is the greatest lower bound which can be defined only by the formula

$$F_{\Delta_1} \cap F_{\Delta_2} = \operatorname*{s-lim}_{n \to \infty} (F_{\Delta_1} F_{\Delta_2})^n . \tag{6}$$

It may be interpreted geometrically as the projection onto the intersection of the two ranges of F_{Δ_1} and F_{Δ_2}.

The condition (2)′ still assures that the operator $F_{\Delta_1} + F_{\Delta_2}$ is a projection and it is in fact equal to $F_{\Delta_1} \cup F_{\Delta_2}$. But in general it is smaller than $F_{\Delta_1 \cup \Delta_2}$:

$$F_{\Delta_1} + F_{\Delta_2} \leq F_{\Delta_1 \cup \Delta_2} \quad \text{for} \quad \Delta_1 \perp \Delta_2 .$$

In particular if Δ' is the complementary set of Δ then according to (2)′ the two projections F_Δ and $F_{\Delta'}$ still commute, but the projection $F_\Delta + F_{\Delta'}$ is in general no longer equal to the unit operator. Instead we have $F_\Delta + F_{\Delta'} < I$.

In fact we can prove that

$$F_\Delta + F_{\Delta'} = I \qquad \forall \Delta \in \mathfrak{B} \tag{7}$$

is the necessary and sufficient condition that the generalized spectral measure is an ordinary spectral measure, that is, that all projections commute and condition (2) is satisfied. Since the proof is easy, we shall give it here.

Necessity of (7) is obvious, so we prove sufficiency. Let us consider two disjoint sets $\Delta_1 \perp \Delta_2$. Then we can obviously write $\Delta_2 = (\Delta_1 \cup \Delta_2) \cap \Delta_1'$. Consequently $F_{\Delta_2} = F_{(\Delta_1 \cup \Delta_2) \cap \Delta_1'} = F_{\Delta_1 \cup \Delta_2} \cap F_{\Delta'}$, by property (3)′. Now using (7), we can write $F_{\Delta'} = F_{\Delta_1}' \equiv I - F_{\Delta_1}$, hence

$$F_{\Delta_2} = F_{\Delta_1 \cup \Delta_2} \cap F_{\Delta_1}' .$$

This equation shows first of all that $F_{\Delta_2} \leq F_{\Delta_1}'$. Thus F_{Δ_2} commutes with F_{Δ_1}', consequently also with F_{Δ_1}, so that

$$F_{\Delta_1 \cup \Delta_2} = F_{\Delta_1} + F_{\Delta_2} .$$

By induction one proves then finite additivity and finally by the standard extension procedure, mentioned before, one arrives at (2). Since (2) implies commutativity for all projections. The proof is complete.

In the general case, that is if (7) is not satisfied then one has instead of (2) only the weaker relation

$$\Sigma F_{\Delta_i} \leq F_{\cup \Delta_i} \tag{8}$$

for any sequence of pairwise disjoint sets Δ_i.

The occurrence of inequalities in these relations is very suggestive from a physical point of view.

For instance the relation $F_\Delta + F_{\Delta'} < I$ implies that it is not possible to give a complete description of all the states of the system in terms of superpositions of state vectors which are localized in Δ and in Δ'. In other words there are additional degrees of freedom present which are superimposed on those which are associated with localization.

In such a system pure states can in general no longer be described by Schrödinger functions as complex valued functions of position. That this is not possible for photons has been known for a long time [16] but until now a clear understanding of the origin this fact has been missing.

In our view, one of the basic problems of elementary particle physics is this: to find a natural expression for the degrees of freedom which manifest themselves in conservation laws and symmetry principles for the known elementary particles. The fact that generalized localizability as introduced here implies additional degrees of freedom seems to open up new and encouraging perspectives in this direction.

We now give the following formal definition of localizability, which includes the ordinary one as a special case:

To each Borel set $\Delta \subset M$ there is associated a projection F_Δ in a Hilbert space $\mathfrak{H}$. The function $\Delta \to F_\Delta$ satisfies

$$F_\phi = 0, \qquad F_M = I \tag{1'}$$

$$\Delta_1 \perp \Delta_2 \Rightarrow F_{\Delta_1} \perp F_{\Delta_2} \tag{2'}$$

$$F_{\Delta_1 \cap \Delta_2} = F_{\Delta_1} \cap F_{\Delta_2} . \tag{3'}$$

There exists a representation $x \to U_x$ of a group G of motions in M such that

$$F_{[\Delta]x} = U_x^{-1} F_\Delta U_x . \tag{4'}$$

A trivial realisation of these axioms are the systems of imprimitivities which satisfy the stronger properties (1), (2), (3), and (4). In the next section we shall show that there exist generalized systems of imprimitivities which satisfy only (1)', ..., (4)' but *not* (1), ... ,(4).

Such generalized systems can be constructed with the help of another generalized measure, the so called *positive operator valued measure* (hence forth called POV-measure) which we shall describe in the next section.

IV. The POV-measures

A selfadjoint operator T is called positive if $(f, Tf) > 0$ for all $f \in \mathfrak{H}$. It is called bounded if there exists a constant $C < \infty$ such that $\| Tf \| \leq C \| f \|$.

A POV-measure is a function $\Delta \to T_\Delta$ from the Borel sets $\mathfrak{B}$ of M to the bounded positive selfadjoint operators which satisfies the following properties

$$T_\phi = 0 \qquad T_M = I . \tag{8}$$

For every sequence Δ_i such that $\Delta_i \perp \Delta_j$

$$T_{\cup_i \Delta_i} = \sum_i T_{\Delta i} . \tag{9}$$

If the operators T_Δ are projections we obtain the PV-measures (projection-valued measures). For such measures we have the characteristic properties $T_\Delta^2 = T_\Delta$ for all $\Delta \in \mathfrak{B}$. For the POV-measures this property is not assumed [17].

POV-measures can be constructed very easily out of PV-measures. Indeed let $E_\Delta^{(1)}$ and $E_\Delta^{(2)}$ be two different PV-measures then

$$T_\Delta \equiv \lambda_1 E_\Delta^{(1)} + \lambda_2 E_\Delta^{(2)}$$

with $\lambda_1 + \lambda_2 = 1$, $\lambda_i > 0$ is a POV-measure.

One sees immediately from this remark that the POV-measures are a convex set with the PV-measures as extremal points.

The POV-measures arise naturally in the spectral theory of normal operators. For us the following representation of POV-measures is of great importance:

Let E_Δ be a PV-measure in a Hilbert space $\mathfrak{H}^+$, denote by P a fixed non-trivial projection in $\mathfrak{H}^+$ and by $\mathfrak{H} \subset \mathfrak{H}^+$ the range of P. Assume furthermore that P does not commute with all the operators E_Δ. Then the operators

$$T_\Delta = P\, E_\Delta\, P \tag{10}$$

are defined in $\mathfrak{H}$ and they are zero on $\mathfrak{H}^\perp$. It is easy to verify that the T_Δ are a POV-measure in $\mathfrak{H}$.

More remarkable is the converse of this: For every POV-measure T_Δ in a Hilbert-space $\mathfrak{H}$ one can construct an extension $\mathfrak{H}^+$ of $\mathfrak{H}$ and a PV-measure E_Δ in $\mathfrak{H}^+$ such that $T_\Delta = P\, E_\Delta\, P$, where P is the projection from $\mathfrak{H}^+$ to $\mathfrak{H}$. This is a theorem due to NEUMARK [18] [19].

One can furthermore prove that there exists a unique minimal extension in the sense of isomorphism [19].

From the point of view of this paper the interest of the POV-measure lies in the following fact: If P and E_Δ are as in the theorem of Neumark, then the projections

$$F_\Delta = P \cap E_\Delta$$

are a generalized spectral measure which satisfies relations (1)′, (2)′, and (3)′ of the preceding section. Thus we see to every POV-measure corresponds such a generalized spectral measure. We state a converse of this with the following

Conjecture: For any generalized spectral measure in a Hilbert space $\mathfrak{H}$ which satisfies (1)′, (2)′, and (3)′ there exists a minimal extension $\mathfrak{H}^+$ of the Hilbertspace $\mathfrak{H}$ such that $F_\Delta = P \cap E_\Delta$, where P is the projection from $\mathfrak{H}^+$ to $\mathfrak{H}$.

Even without this conjectured theorem the preceding remarks show that we have a vast store of generalized spectral measures which we can construct from POV-measures.

With this technique we can now easily also construct generalized imprimitivity systems. It suffices indeed to consider an ordinary such system under the assumption that the representation U_x^+ of the group G is not irreducible. There exists then a non-trivial projection P which commutes with all the U_x^+ but which need not commute with every E_Δ. The following formulae define then a generalized system of imprimitivities:

$$F_\Delta = P \cap E_\Delta \qquad U_x = P\, U_x^+ \tag{11}$$

as one verifies easily.

V. Localizability of the photon

In this section we shall now show, how the notion of weak localizability can be applied to the photon.

We describe the photon by a fourcomponent function $\varphi^\mu(p)$ ($\mu = 0, \ldots, 3$) define on the lightcone

$$p^\mu p_\mu = p_0^2 - \boldsymbol{p}^2 = 0 , \qquad p^0 \geq 0 . \tag{12}$$

The wave function $\varphi^\mu(p)$ is to satisfy the Lorentz condition

$$p_\mu \varphi^\mu(p) = 0 . \tag{13}$$

We define a scalar product by setting

$$(\varphi, \psi) = \int \frac{d^3p}{p_0} \varphi_\mu^*(p) \psi^\mu(p) . \tag{14}$$

The functions for which this product is finite form a linear vector space $\mathfrak{G}$. Because of the Lorentz condition the scalar product in $\mathfrak{G}$ is non negative. But there are in fact functions $\varphi^\mu(p)$, namely those of the form $\varphi^\mu(p) = p^\mu \Lambda(p)$ with $\Lambda(p)$ an arbitrary scalar function, for which

$$(\varphi, \psi) = \int \frac{d^3p}{p_0} p_\mu p^\mu \Lambda^*(p) \Lambda(p) = 0 .$$

The set of all functions of this kind are a linear subspace $\mathfrak{G}_0$. We define the factorspace $\mathfrak{H} = \mathfrak{G}/\mathfrak{G}_0$ as the set of all equivalence classes of functions $\varphi^\mu(p)$ modulo the subspace $\mathfrak{G}_0$. Two functions φ_1^μ and φ_2^μ are said to be equivalent if they differ by an element in $\mathfrak{G}_0$:

$$\varphi_1^\mu \sim \varphi_2^\mu \quad \text{if} \quad \varphi_1^\mu - \varphi_2^\mu = p^\mu \Lambda(p) . \tag{15}$$

We designate by $\hat{\varphi}$ the equivalence class containing one particular function $\varphi^\mu(p)$. The scalar product for the equivalence classes is then defined by

$$(\hat{\varphi}, \hat{\psi}) = \int \frac{d^3p}{p_0} \varphi_\mu^*(p) \psi^\mu(p) . \tag{16}$$

It is positive definite by construction.

The space $\mathfrak{H}$ thus constructed induces an irreducible unitary representation of the Poincaré group $\mathfrak{P}$. Let a, Λ be a general element of this group consisting of the translation a and the pure Lorentztransformation Λ then we define

$$[U(a, \Lambda)\varphi]^\mu(p) = e^{-i a p} \Lambda_v^\mu \varphi^v(\Lambda^{-1} p) . \tag{17}$$

It is easily verified that the transformation $U(a, \Lambda)$ thus defined is really a transformation of the equivalence classes, that is it leaves the subspace $\mathfrak{G}_0$ invariant.

In order to construct the generalized system of imprimitivities for the photon according to formula (11) we must identify the extended space $\mathfrak{H}^+$, the projection P from $\mathfrak{H}^+$ onto $\mathfrak{H}$, the representation $U^+(a, R)$ of the Euclidean group which commutes with P and the system of imprimitivities E_Δ in $\mathfrak{H}$ which must not commute with P.

We begin with the construction of the extended space $\mathfrak{H}^+$. To this end we choose in each equivalence class of $\mathfrak{G}$ a particular representative as follows: Let $\varphi^\mu(p)$ be any element in $\hat{\varphi}$. We define

$$f^\mu(p) = \varphi^\mu(p) - \frac{p^\mu}{p^0} \varphi^0(p) . \tag{18}$$

Since the passage from φ^μ to f^μ is a gaugetransformation $f^\mu(p)$ is also in the class $\hat{\varphi}$. Furthermore it has the property $f^0(p) = 0$. We shall therefore denote it by $\boldsymbol{f}(p)$ instead of $f^\mu(p)$.

The Lorentzcondition and the scalar product can be expressed in terms of these functions in the form

$$p \cdot \boldsymbol{f}(p) = 0 \tag{19}$$

$$(\boldsymbol{f}, \boldsymbol{g}) = \int \frac{d^3p}{p^0} \boldsymbol{f}^*(p) \cdot \boldsymbol{g}(p) . \tag{20}$$

The Hilbertspace $\mathfrak{H}^+$ consists of all the vectorvalued functions for which (20) is finite, but they need not satisfy (18). The projection operator P from $\mathfrak{H}^+$ to $\mathfrak{H}$ is defined by

$$[P \boldsymbol{f}] (p) = f(p) - \frac{p}{p_0} (\boldsymbol{p} \cdot \boldsymbol{f}(p)) . \tag{21}$$

The condition (19) is invariant under the representation $U^+(a, R)$ which is an extension of $U(a, R)$ (the representation of the Euclidean group) to $\mathfrak{H}^+$ in an obvious sense. Hence this projection P commutes with $U^+(a, R)$.

In order to define the system of imprimitivities $E_\varDelta$ we carry out two transformations as follows. The first denoted by K transforms $f(p)$ according to the formula

$$K: \boldsymbol{f}(p) \to \frac{1}{\sqrt{p^0}} \boldsymbol{f}(p) . \tag{22}$$

It is an isomorphism with the scalar product of $\boldsymbol{L}^2(p)$. The second is the Fourier-transformation F defined by

$$F: \boldsymbol{f}(p) \to f(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \, e^{ipx} \boldsymbol{f}(p) . \tag{23}$$

The transformation FK maps the space $\mathfrak{H}^+$ onto the space $\boldsymbol{L}^2(x)$ consisting of Lebesgue square integrable vectorvalued functions $\boldsymbol{f}(x)$ over the 3-dimensional Euclidean space. Likewise, the space $\mathfrak{H}$ is mapped onto the subspace $\boldsymbol{L}_0^2(y)$ consisting of similarfunctions which satisfy in addition the divergence condition

$$\varDelta_x \cdot \boldsymbol{f}(x) = 0 . \tag{24}$$

We define the projections $\tilde{E}_\varDelta$ for each Borel set $\varDelta$ by the formula

$$(\tilde{E}_\varDelta \boldsymbol{f}) (x) = \chi_\varDelta(x) \boldsymbol{f}(x) \tag{25}$$

where $\chi_\varDelta(x)$ is the characteristic function of the Borel set $\varDelta$.

The functions (25) in general do not satisfy condition (24) even if $\boldsymbol{f}(x)$ does. This shows that the projections $\tilde{E}_\varDelta$ do not commute with the projection $\tilde{P}$ from $\boldsymbol{L}^2(x)$ onto $\boldsymbol{L}_0^2(x)$.

We now return to the spaces $\mathfrak{H}^+$ and $\mathfrak{H}$ by defining

$$E_\varDelta = (F K)^{-1} \tilde{E}_\varDelta F K . \tag{26}$$

It is now easy to verify that this is a system of imprimitivities with respect to the representation $U^+(a, R)$ of the group of Euclidean motions.

Let us now define the projections

$$F_\Delta = E_\Delta \cap P \equiv \lim_{n\to\infty} (E_\Delta P)^n . \tag{27}$$

This system of projections defines then a generalized system of imprimitivities in accordance with the definition given in the preceding section.

In order to verify that this system F_Δ is not trivial (that is $F_\Delta = 0$ for all compact $\Delta \in \mathfrak{B}$) it must be shown that the POV-measure $T_\Delta = P E_\Delta P$ has eigenvectors with eigenvalues 1, since F_Δ is simply the projection onto the subspace spanned by these eigenvectors. This can in fact be shown. We shall not do this here. This and other aspects of the problem will be the subject of a forthcoming thesis by Mr. Amrein.

In conclusion we summarize that the foregoing calculations show that the photons are indeed weakly localizable, but not localizable in the ordinary sense.

Acknowledgements

The mathematical background of this paper was discussed in a joint seminar devoted to extensions of operators with N. Bazeley and B. Zwahlen of the Batelle Memorial Institute. The physical applications of this material evolved gradually in numerous discussion with B. Misra, J.-P. Marchand, W. Amrein and J. Oberholzer. To all of them we wish to thank here for their respective contributions.

This work was financially supported by the Swiss National Science Foundation, Batelle and CERN.

References

[1] Max Jammer, *The Conceptual Development of Quantum Mechanics*, Mc Graw-Hill, New York 1966.
[2] C. Piron, Helv. phys. Acta *37*, 439 (1964).
[3] G. Birkhoff and J. v. Neumann, Ann. Math. *37*, 823 (1936).
[4] Inönü and E. P. Wigner, Nuovo Cim. *9*, 705 (1952).
[5] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. *21*, 400 (1949).
[6] A. S. Wightman, Rev. Mod. Phys. *34*, 845 (1962).
[7] J. M. Jauch, *The existence of the Hamiltonian for causal systems* in Dispersion Relations, Oliver and Boyd, Edinburghsh, 1960.
[8] L. Janossy, Z. Nagy, and P. Varga, cf. Phys. Abstr. *65*, 1108 (1962).
[9] W. Pauli, Collected Scientific Papers, Vol. 2, p. 608, Interscience Publishers 1964.
[10] M. H. L. Pryce, Proc. Roy. Soc. *195*,62 (1948).
[11] T. F. Jordan and N. Mukunda, Phys. Rev. *132*, 1842 (1963).
[12] R. A. Berg, J. Math. Phys. *6*, 34 (1965).
[13] A. Galindo, Nuovo Cim. *37*, 413 (1965).
[14] G. W. Mackey, Proc. Nath. Acad. Sci. U.S. *35*, 537 (1949).
[15] G. W. Mackey, Acta Math. *99*, 265 (1958).
[16] L. Landau and R. Peirerts, Z. Phys. *62*, 188 (1930); *ibid. 69*, 56 (1931).
[17] For a brief and very clear introduction to the theory of POV-measures we have found the following reference very useful S. K. Berberian, *Notes on Spectral Theory*, D. Van Nostrand, Princeton (1966).
[18] M. A. Neumark, Doklady Acad. Sci. URSS *41*, 359 (1961).
[19] B. St. Nagy, Appendix to Functional Analysis, F. Ungar. Publ. Co. New York (1960).